Zerosum-sets in Design theory

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Inspired by the point-flat design of an affine geometry over GF(q), in 2015, together with A. Caggegi and M. Pavone, we introduced the definition of "additive" block design [2], as an embedded v-subset P of a commutative group where the "blocks" are k-subsets of P whose points sum up to zero (and form a BIBD, that is, any two points of P are contained in exactly λ blocks). Surprisingly, it turned out that, together with affine resolvable designs, symmetric designs are additive, as well, (also, the proof for affine resolvable designs is much more involved [3]). A property which, in my opinion, makes this embedding interesting is that, for the forementioned classes of designs, blocks are "the only" k-subsets of P adding up to zero in a suitable group (possibly much larger than P). In such a case, we say that the design is "strongly additive".

Point-line designs of either an affine geometry or a projective plane are additive Steiner designs, that is, additive designs with $\lambda = 1$, but recently M. Buratti and A. Nakić produced new infinite families of additive Steiner designs [1], which, furthermore, admit a regular group of automorphisms, in which P is additively embedded (and which is as small as P). Even more recently, with the powerful tool of difference sets, they settled, with a beautiful proof, the question of the size of the group where a projective plane over GF(q) can be additively embedded. More generally, they proved that, for any dimension d, the point-flat design of the projective geometry PG(d, q) over a prime power q is additive in a "very small" group, and also that the point-line design of PG(d, p) over a prime p is strongly additive.

In order to give an example of a non-strongly additive design, M. Pavone [5] used two different representations in $\mathrm{GF}(2)^4$ of the 20 lines of the affine plane $\mathrm{GF}(4)^2$. Rationality questions arise also when considering permutations of a field/vector space which map zerosum k-sets onto zerosum k-sets [4]. Any linear map of course is such a permutation. Are there any others? Finally, some connections to Hamming codes have been considered [6], as one can easily imagine.

References

[1] Buratti, Marco; Nakić, Anamari, Super-regular Steiner 2-designs. Finite Fields Appl. 85, Article ID 102116, 29 p. (2023).

[2] Caggegi, Andrea; Falcone, Giovanni; Pavone, Marco, On the additivity of block designs. J. Algebr. Comb. 45, No. 1, 271-294 (2017).

*Jointly with M. Pavone.

[3] Caggegi, Andrea; Falcone, Giovanni; Pavone, Marco, Additivity of affine designs. J. Algebr. Comb. 53, No. 3, 755-770 (2021).

[4] Falcone, Giovanni; Pavone, Marco, Permutations of zero-sumsets in a finite vector space. Forum Math. 33, No. 2, 349-359 (2021).

[5] Pavone, Marco, A quasidouble of the affine plane of order 4 and the solution of a problem on additive designs, (under review).

[6] Falcone, Giovanni; Pavone, Marco, Binary Hamming codes and Boolean designs. Des. Codes Cryptography 89, No. 6, 1261-1277 (2021).