Università degli Studi di Verona
Corso di Laurea Magistrale in Matematica Applicata

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Some exercises of functional analysis - A.A. 2013/14 - N.3

Pb 1. Prove that $C^1([a,b])$ endowed with the norm

$$\|f\|_{C^1} := \|f\|_\infty + \|f'\|_\infty$$

is a real Banach space. Furthermore, show that its unit ball is not compact.

Pb 2. Let X be a linear space and $\|\cdot\|_1$ and $\|\cdot\|_2$ two norms on X. Assuming that they are equivalent, prove that $(X, \|\cdot\|_1)$ is a Banach space if and only if $(X, \|\cdot\|_2)$ is a Banach space. Give an example of X, $\|\cdot\|_1$ and $\|\cdot\|_2$ such that $(X, \|\cdot\|_1)$ is a Banach space but $(X, \|\cdot\|_2)$ is not. Prove that the function $\|\cdot\|_1 : X \to \mathbb{R}$ is continuous on $(X, \|\cdot\|_1)$. Must it be continuous also on $(X, \|\cdot\|_2)$?

Pb 3. Prove that the functional $T : C([0,b]) \to C([0,b])$ defined by

$$(Tf)(t) := \int_0^t e^{t-\sigma} f(\sigma) d\sigma$$

is continuous. Prove also that for suitable $b > 0$, the norm $\|T\|_{\mathcal{L}(C([0,b]),C([0,b]))}$ is less than 1.

Pb 4. If $z \in X$ and $\varphi \in X^*$, show that $T : X \to X$ defined by $Tx := \langle \varphi, x \rangle z$ is compact.

Pb 5. Show that $T : \ell^p \to \ell^p$ defined by

$$(Tx)_j := \frac{x_j}{j}, \quad j \geq 1, \qquad 1 < p < \infty,$$

is a bounded linear operator. Furthermore, show that it can be approximated in the norm of $\mathcal{L}(\ell^p, \ell^p)$ by a sequence of linear operators with finite dimensional image.

Pb 6. Prove that the set

$$A = \Big\{ f \in L^2(0,1) : \int_0^1 x f^2(x) dx < 1 \Big\}$$

is open, convex and not bounded in $L^2(0,1)$ when endowed with the canonical norm.

Pb 7. Prove that the set

$$B = \Big\{ f \in L^2(0,1) : \int_0^1 (1+x) f^2(x) dx < 1 \Big\}$$

is bounded in $L^2(0,1)$ when endowed with the canonical norm.

Pb 8. Consider the vector space

$$X = \Big\{ f : (0,+\infty) \to \mathbb{R} \text{ measurable such that } \int_0^{+\infty} x f^2(x) dx < +\infty \Big\}.$$

Prove that $X \setminus L^2(0,+\infty) \neq \emptyset$ as well as $L^2(0,+\infty) \setminus X \neq \emptyset$.

Pb 9. Consider, for any $f \in L^2(0,+\infty)$, the sequence of reals

$$T_j(f) = \int_j^{j+1} f(x)dx, \quad j \in \mathbb{N}.$$

Prove that $T : L^2(0,+\infty) \to \ell^2(\mathbb{N})$ such that $(T(f))_j = T_j(f)$ defines a bounded linear operator which is surjective but not injective.

Pb 10. For $1 \leq p \leq \infty$ consider the set

$$X = \{ f \in C(-1,1) : f \in L^p(-1,1) \},$$

and define the operator $\varphi : X \to \mathbb{R}$ by setting $\varphi(f) = f(0)$, for every $f \in X$. Prove that φ is discontinuous for every $p < \infty$ while it is continuous for $p = \infty$, in which case there exists a bounded linear operator $\Phi : L^2(-1,1) \to \mathbb{R}$ having the same norm of φ and with $\Phi|_X = \varphi$.

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