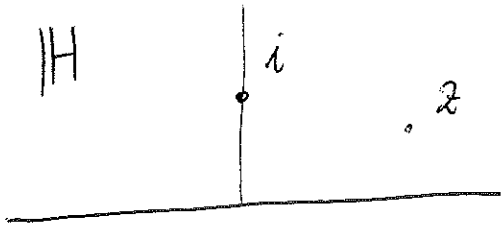
☆ Il Piano iperbolico come spazio omogeneo
(simmetrico)

$\mathbb{H}$



$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

$z = x + iy$

in forma complessa

$$dz = dx + i\,dy$$
$$d\bar{z} = dx - i\,dy$$

$$ds^2 = \frac{dz\,d\bar{z}}{(\mathrm{Im}\, z)^2}$$

Sia dato $G = \underset{\text{gruppo lineare speciale}}{SL(2,\mathbb{R})} = \left\{ \overset{A}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \Big/ \underbrace{ad-bc}_{\det A} = 1 \right\}$ $\quad a,b,c,d \in \mathbb{R}$

☆ G manda $\mathbb{H}$ in sé: azione:

$$z \longmapsto z' \equiv A \cdot z := \frac{az+b}{cz+d}$$

$\boxed{\text{☆ trasformazione di Möbius}}$

$$\frac{az+b}{cz+d} \cdot \frac{c\bar{z}+d}{c\bar{z}+d} = \frac{ac|z|^2 + bd + ad\overset{x+iy}{\overset{\shortparallel}{z}} + bc\overset{x-iy}{\overset{\shortparallel}{\bar{z}}}}{|cz+d|^2}$$

$$\Rightarrow \underbrace{\mathrm{Im}\, z'}_{y'} = \frac{\overbrace{ad-bc}^{1}}{|cz+d|^2}\,\overbrace{\mathrm{Im}\, z}^{y}$$

$z \neq \frac{d}{c}$ & $c \neq 0$ (nessun problema per $z \in \mathbb{H}$)

$$= \frac{\overbrace{\mathrm{Im}\, z}^{y}}{|cz+d|^2} > 0$$

Dato, ad es. $z_0 = i$, $\forall z \in \mathbb{H}$ $\exists$

$A \in SL(2,\mathbb{R})$ tale che $z = A z_0$;

infatti
$$\underset{\underset{x+iv}{\parallel}}{z} = \frac{a \cdot i + b}{c \cdot i + d}$$

$$(x+iv)(ci+d) = ai+b$$

$$(xc+vd)\,i + (xd - vc) = ai + b$$

$$b = xd - vc$$
$$a = xc + vd$$

Poniamo $c = 0$
$$b = xd$$
$$a = vd$$

$$ad - b\overset{0}{\cancel{c}} = 1$$

$$\Rightarrow \quad \underset{\underset{0}{\vee}}{v}\, d^2 = 1 \quad \Rightarrow \quad d = \frac{1}{\sqrt{v}}$$

$$a = vd = \frac{v}{\sqrt{v}} = \sqrt{v}$$

$$\begin{pmatrix} \sqrt{v} & \frac{x}{\sqrt{v}} \\ 0 & \frac{1}{\sqrt{v}} \end{pmatrix} \Leftarrow \quad b = xd = \frac{x}{\sqrt{v}}$$

Check:
$$\frac{\sqrt{v}\, i + \frac{x}{\sqrt{v}}}{\frac{1}{\sqrt{v}}} = vi + x = z$$

✓

$\mathbb{H}$ è G- omogeneo

G - homogeneous

gruppo di
isotropia di $z_0 = i$ - $A: \ A i = i$
isotropy group

$$\frac{ai+b}{ci+d} = i$$

$$ai + b = -c + id$$

$$\begin{cases} a = d \\ b = -c \end{cases} \qquad \begin{array}{l} ad - bc = 1 \\ d^2 + c^2 = 1 \end{array}$$

$$\Rightarrow \quad A = \begin{pmatrix} \overset{a}{\overbrace{\cos\varphi}} & \overset{b}{\overbrace{-\sin\varphi}} \\ \underset{c}{\underbrace{\sin\varphi}} & \underset{d}{\underbrace{\cos\varphi}} \end{pmatrix}$$ per φ opportuno

i.e $A \in SO(2, \mathbb{R})$

$$\boxed{\mathbb{H} \cong SL(2,\mathbb{R}) / SO(2,\mathbb{R})}$$

4 ds^2 è G-invariante:

$f = f(z) = \dfrac{az+b}{cz+d}$ è olomorfa

Da
$$dz' = \frac{a(cz+d) - (az+b)c}{(cz+d)^2}\, dz$$
$$= \frac{\cancel{(ac-ac)z} + \overbrace{ad-bc}^{=1}}{(cz+d)^2}\, dz$$
$$= \frac{dz}{(cz+d)^2}, \quad d\bar{z}' = \frac{d\bar{z}}{\overline{(cz+d)}^2}$$

Si ha
$$\frac{dz'\,d\bar{z}'}{(\operatorname{Im} z')^2} = \frac{dz\,d\bar{z}}{\cancel{|cz+d|^4}} \cdot \frac{\cancel{|cz+d|^4}}{(\operatorname{Im} z)^2}$$
$$= \frac{dz\,d\bar{z}}{(\operatorname{Im} z)^2} \qquad \square$$