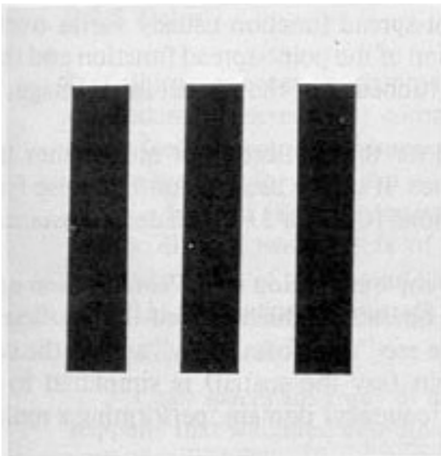
2D Continuous FT

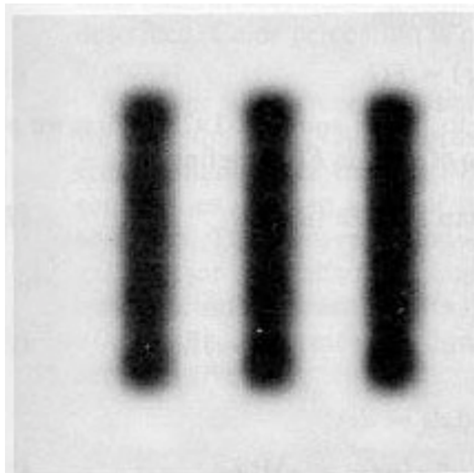
How do frequencies show up in an image?

- Low frequencies correspond to slowly varying information (e.g., continuous surface).
- High frequencies correspond to quickly varying information (e.g., edges)

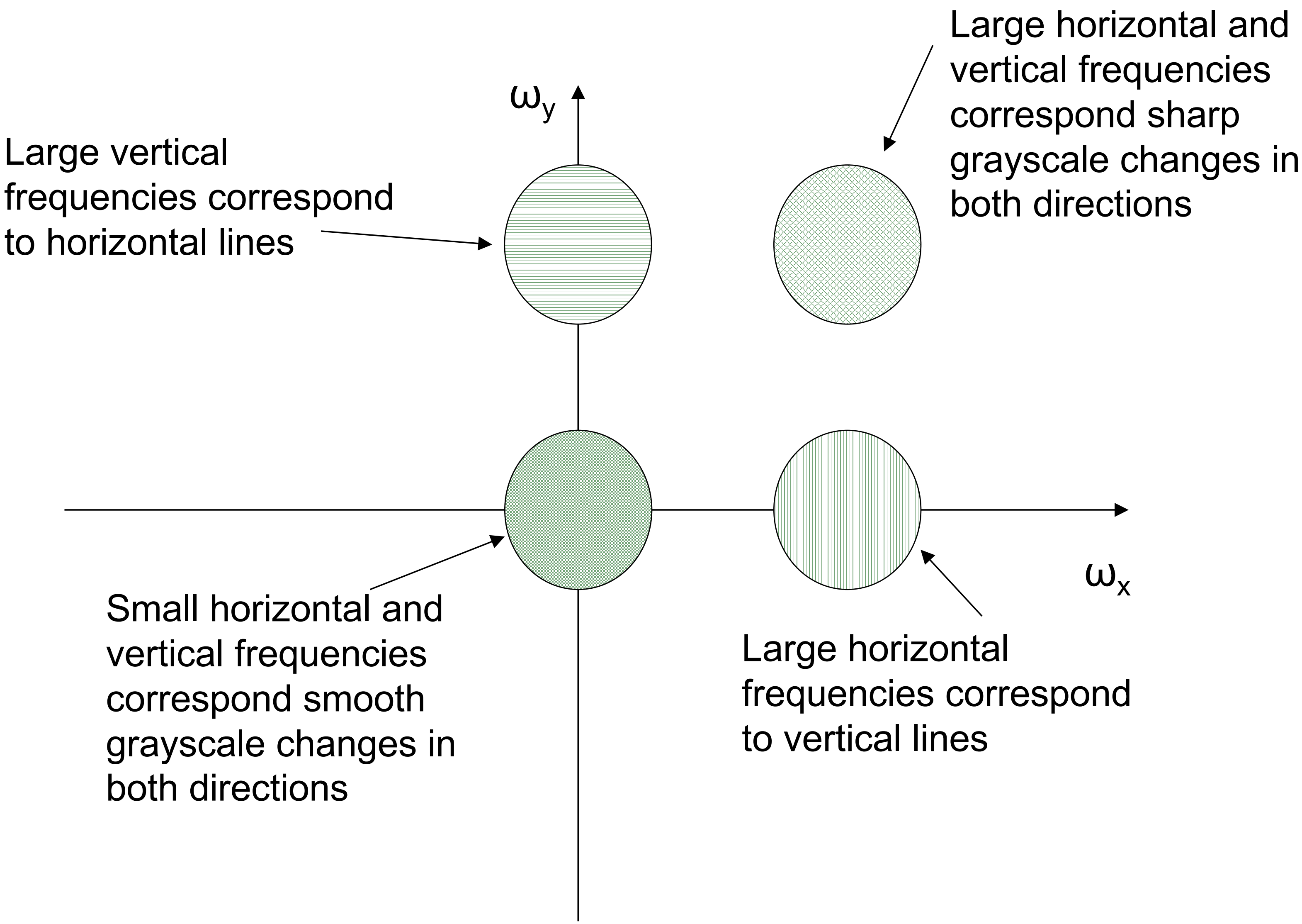
Original Image



Low-passed



2D Frequency domain

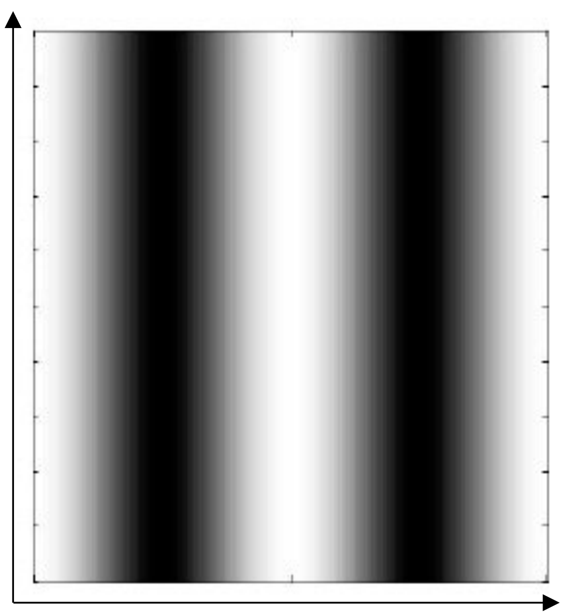


2D spatial frequencies

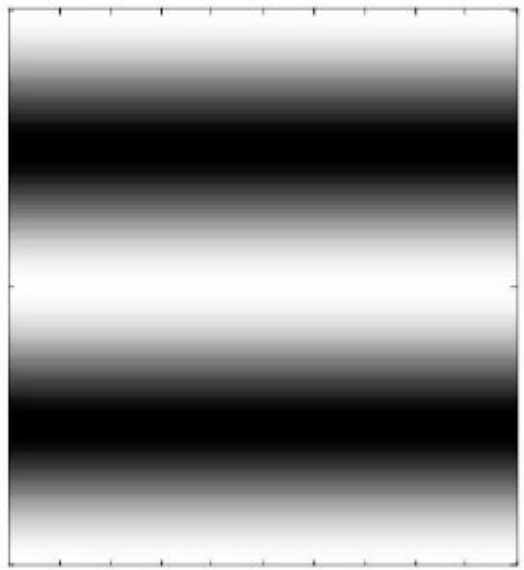
- 2D spatial frequencies characterize the image spatial changes in the horizontal (x) and vertical (y) directions
  - Smooth variations -> low frequencies
  - Sharp variations -> high frequencies

y

$\omega_x=1$
$\omega_y=0$



x



$\omega_x=0$
$\omega_y=1$

2D Continuous Fourier Transform

- 2D Continuous Fourier Transform (notation 2)

$$\hat{f}(u,v) = \int_{-\infty}^{+\infty} f(x,y)\, e^{-j2\pi(ux+vy)}\, dxdy$$

$$f(x,y) = \int_{-\infty}^{+\infty} \hat{f}(u,v)\, e^{j2\pi(ux+vy)}\, dudv =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|f(x,y)\right|^2 dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\hat{f}(u,v)\right|^2 dudv$$

Plancherel’s equality

Delta

- Sampling property of the 2D-delta function (Dirac’s delta)

$$\int_{-\infty}^{\infty} \delta(x-x_0, y-y_0) f(x,y)\,dxdy = f(x_0, y_0)$$

- Transform of the delta function

$$F\{\delta(x,y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,y) e^{-j2\pi(ux+vy)}\,dxdy = 1$$

$$F\{\delta(x-x_0, y-y_0)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x-x_0, y-y_0) e^{-j2\pi(ux+vy)}\,dxdy = e^{-j2\pi(ux_0+vy_0)}$$

shifting property

Constant functions

- Inverse transform of the impulse function

$$F^{-1}\left\{\delta(u,v)\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(u,v) e^{j2\pi(ux+vy)} dudv = e^{j2\pi(0x+v0)} = 1$$

- Fourier Transform of the constant (=1 for all x and y)

$$k(x,y) = 1 \qquad \forall x, y$$

$$F\left(u,v\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi(ux+vy)} dxdy = \delta(u,v)$$

Trigonometric functions

- Cosine function oscillating along the x axis
  - Constant along the y axis

$$s(x,y)=\cos(2\pi fx)$$

$$F\left\{\cos(2\pi fx)\right\}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cos(2\pi fx)e^{-j2\pi(ux+vy)}dxdy=$$

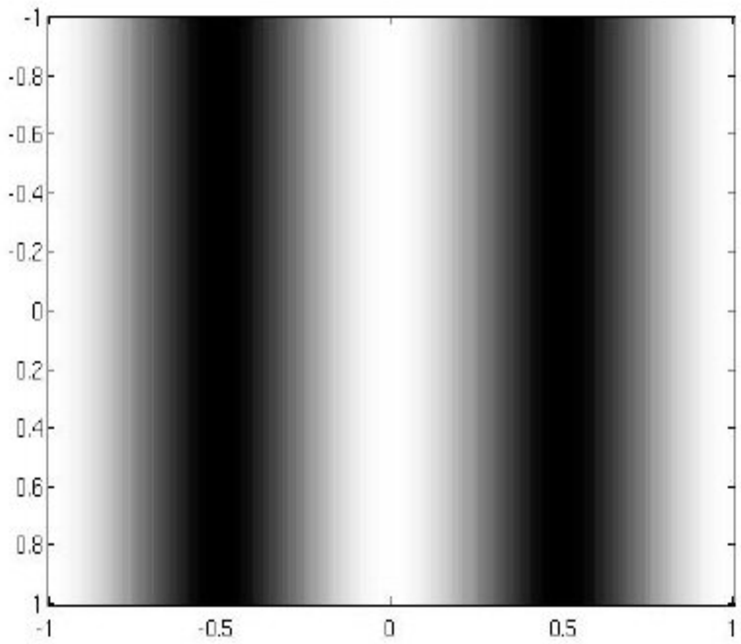
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{e^{j2\pi(fx)}+e^{-j2\pi(fx)}}{2}\right]e^{-j2\pi(ux+vy)}dxdy$$

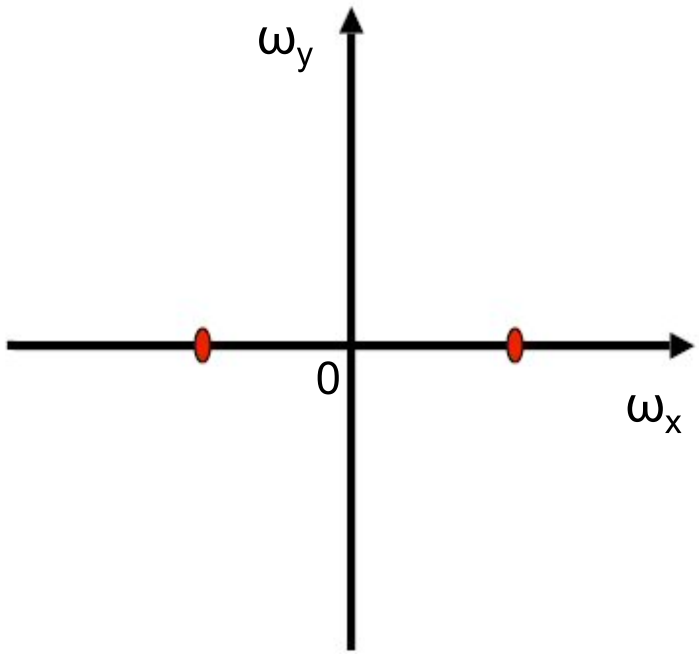
$$=\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[e^{-j2\pi(u-f)x}+e^{-j2\pi(u+f)x}\right]e^{-j2\pi vy}dxdy=$$

$$=\frac{1}{2}\int_{-\infty}^{\infty}e^{-j2\pi vy}dy\int_{-\infty}^{\infty}\left[e^{-j2\pi(u-f)x}+e^{-j2\pi(u+f)x}\right]dx=\frac{1}{2}1\int_{-\infty}^{\infty}\left[e^{-j2\pi(u-f)x}+e^{-j2\pi(u+f)x}\right]dx=$$

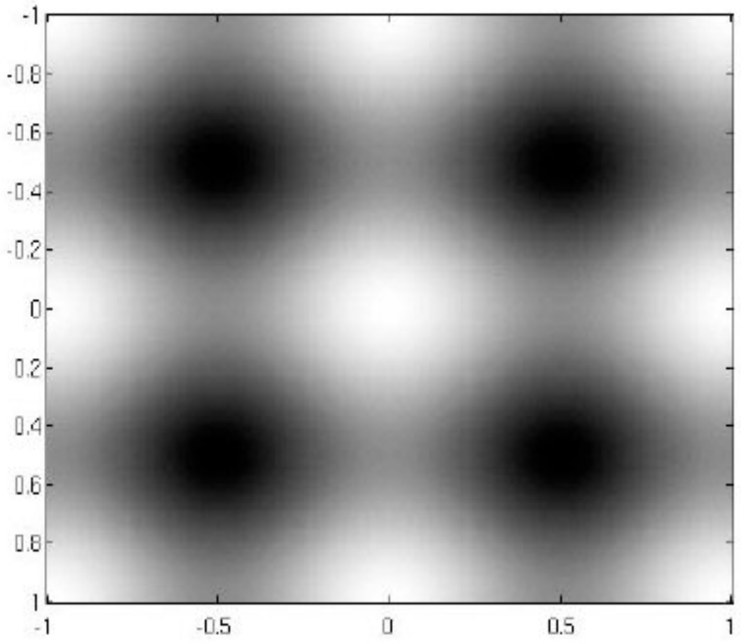
$$\frac{1}{2}\left[\delta(u-f)+\delta(u+f)\right]$$

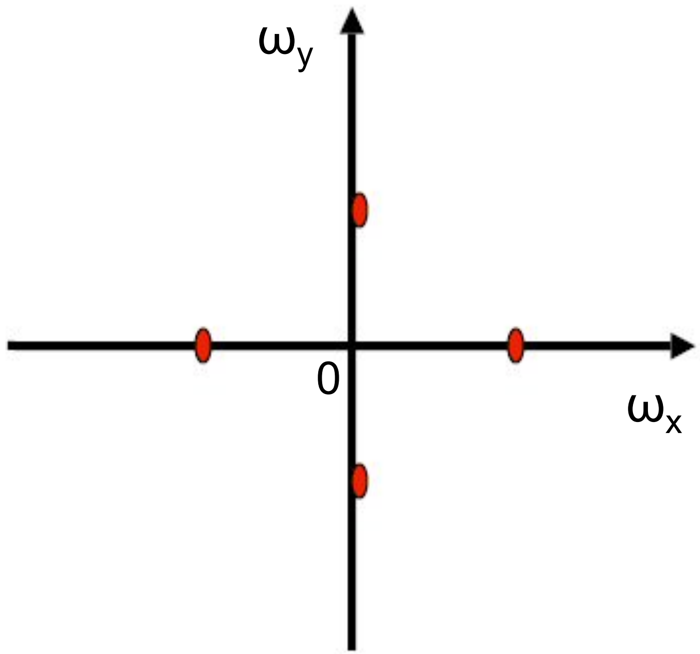
Vertical grating



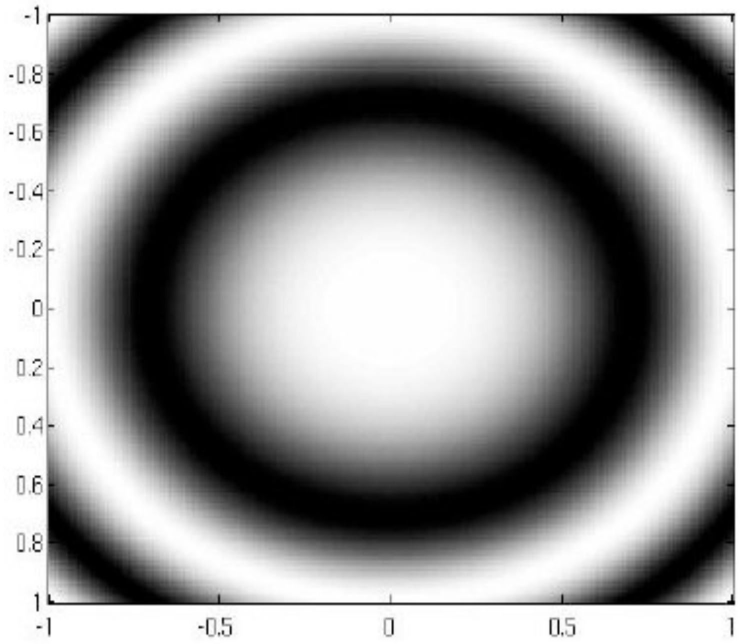


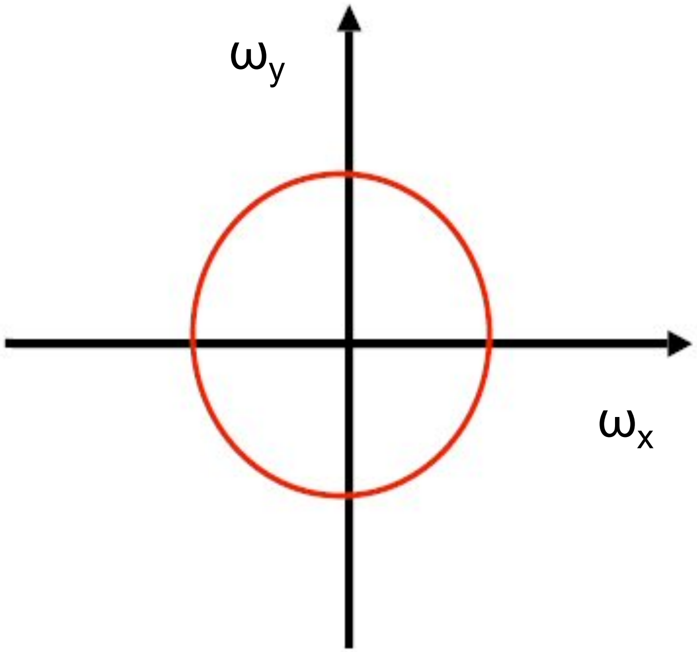
Double grating



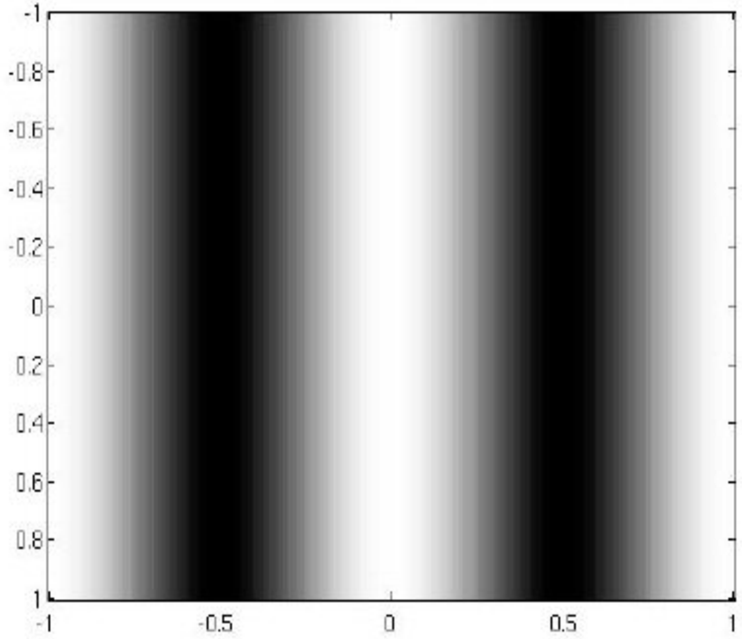


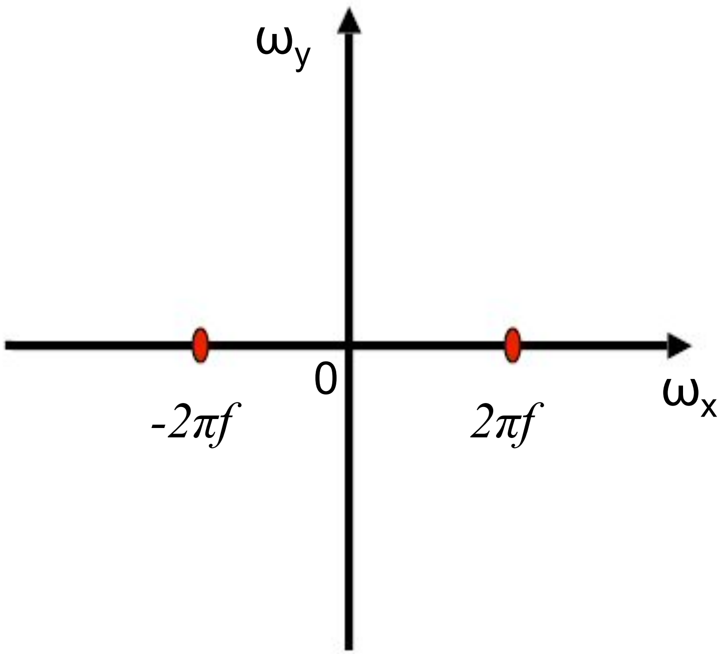
Smooth rings





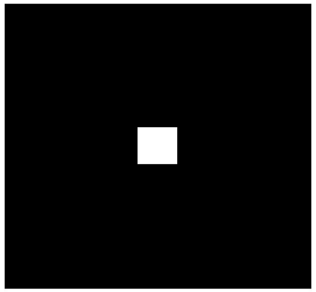
Vertical grating

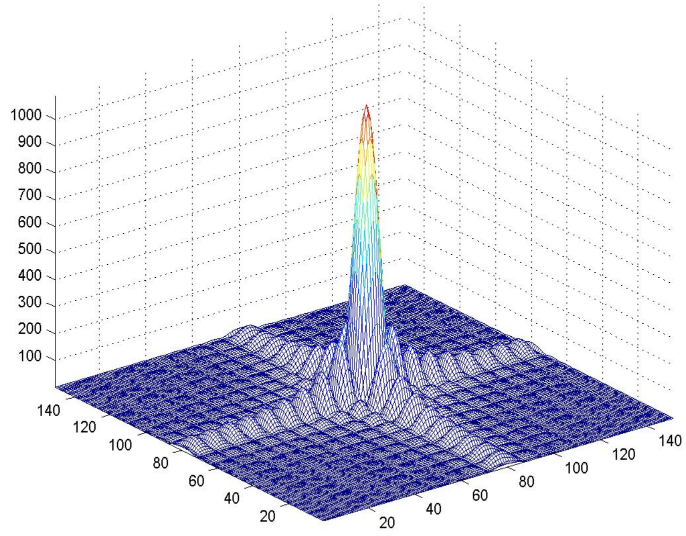


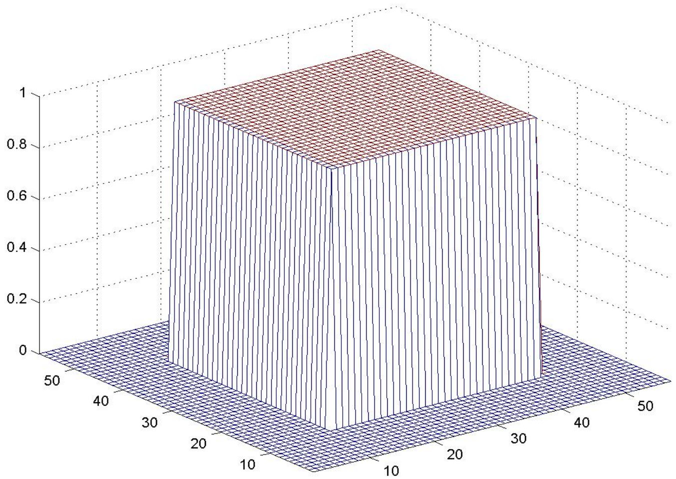


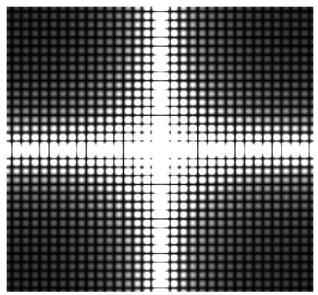
2D box

2D sinc









2D CTFT of the box

$$f(x,y)=\left\{\begin{array}{cc} 1 & -X\le x\le X, -Y\le y\le Y \\ 0 & otherwise \end{array}\right\}$$

$$F(\omega_x,\omega_y)=\int_{-X}^{X}\int_{-Y}^{Y} f(x,y)\exp\left\{-j2\pi\left(f_x x+f_y y\right)\right\}dxdy$$

$$=\int_{-X}^{X}\exp\left\{-j2\pi\left(f_x x\right)\right\}dx\cdot\int_{-Y}^{Y}\exp\left\{-j2\pi\left(f_y y\right)\right\}dy=$$

$$\left[\frac{1}{-j2\pi f_x}\exp\left(-j2\pi f_x x\right)\right]_{-X}^{X}+\left[\frac{1}{-j2\pi f_y}\exp\left(-j2\pi f_y y\right)\right]_{-Y}^{Y}=$$

$$\frac{1}{2}\left(\frac{\exp\left(-j2\pi f_x X\right)-\exp\left(j2\pi f_x X\right)}{-j\pi f_x}\right)=\left(\frac{\exp\left(j2\pi f_x X\right)-\exp\left(-j2\pi f_x X\right)}{j2\pi f_x}\right)=-$$

$$=\frac{1}{\pi f_x}\left(\frac{\exp\left(j2\pi f_x X\right)-\exp\left(-j2\pi f_x X\right)}{2j}\right)=\frac{1}{\pi f_x}\sin\left(2\pi f_x X\right)=$$

$$=2X\frac{\sin\left(2\pi f_x X\right)}{2\pi f_x X}=2X\sin c\left(2\pi f_x X\right)$$

$$2\pi f_x X=2k\pi\rightarrow f_x=\frac{k}{X}$$

Zeros of the sinc at multiples of k/X

CTFT properties

- Linearity $$af(x,y)+bg(x,y) \Leftrightarrow aF(u,v)+bG(u,v)$$
- Shifting $$f(x-x_0, y-x_0) \Leftrightarrow e^{-j2\pi(ux_0+vy_0)}F(u,v)$$
- Modulation $$e^{j2\pi(u_0x+v_0y)}f(x,y) \Leftrightarrow F(u-u_0, v-v_0)$$
- Convolution $$f(x,y)*g(x,y) \Leftrightarrow F(u,v)G(u,v)$$
- Multiplication $$f(x,y)g(x,y) \Leftrightarrow F(u,v)*G(u,v)$$
- Separability $$f(x,y)=f(x)f(y) \Leftrightarrow F(u,v)=F(u)F(v)$$

Separability

1. Separability of the 2D Fourier transform
   - 2D Fourier Transforms can be implemented as a sequence of 1D Fourier Transform operations performed *independently* along the two axis

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi ux}e^{-j2\pi vy}dxdy = \int_{-\infty}^{\infty} e^{-j2\pi vy}dy \int_{-\infty}^{\infty} f(x,y)e^{-j2\pi ux}dx =$$

$$= \int_{-\infty}^{\infty} F(u,y)e^{-j2\pi vy}dy = F\left(u,v\right)$$

2D FT $\Rightarrow$ 1D FT along the rows $\rightarrow$ 1D FT along the cols

Separability

- Separable functions can be written as

2. The FT of a separable function is the product of the FTs of the two functions

$$f(x,y) = f(x)g(y)$$

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x)g(y)e^{-j2\pi ux}e^{-j2\pi vy}dxdy = \int_{-\infty}^{\infty} g(y)e^{-j2\pi vy}dy\int_{-\infty}^{\infty} h(x)e^{-j2\pi ux}dx =$$

$$= H(u)G(v)$$

$$f(x,y) = h(x)g(y) \Rightarrow F(u,v) = H(u)G(v)$$

Discrete Time Fourier Transform (DTFT)

Applies to Discrete domain signals and time series - 2D

Fourier Transform: 2D Discrete Signals

- Fourier Transform of a 2D discrete signal is defined as

$$F(u,v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f[m,n] e^{-j2\pi(um+vn)}$$

where $\frac{-1}{2} \leq u,v < \frac{1}{2}$

- Inverse Fourier Transform

$$f[m,n] = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} F(u,v) e^{j2\pi(um+vn)} du dv$$

Properties

- Periodicity: 2D Fourier Transform of a *discrete* a-periodic signal is *periodic*
  - *The period is 1 for the unitary frequency notations and 2π for normalized frequency notations*.
  - Proof (referring to the firsts case)

$$F(u+k,v+l)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f[m,n]e^{-j2\pi((u+k)m+(v+l)n)}$$

$$=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f[m,n]e^{-j2\pi(um+vn)}\overbrace{e^{-j2\pi km}}^{1}\,\overbrace{e^{-j2\pi ln}}^{1}$$

Arbitrary integers

$$=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f[m,n]e^{-j2\pi(um+vn)}$$

$$=F(u,v)$$

Fourier Transform: Properties

- Linearity, shifting, modulation, convolution, multiplication, separability, energy conservation properties also exist for the 2D Fourier Transform of discrete signals.

DTFT Properties

- Linearity $af[m,n]+bg[m,n] \Leftrightarrow aF(u,v)+bG(u,v)$
- Shifting $f[m-m_0,n-n_0] \Leftrightarrow e^{-j2\pi(um_0+vn_0)}F(u,v)$
- Modulation $e^{j2\pi(u_0m+v_0n)}f[m,n] \Leftrightarrow F(u-u_0,v-v_0)$
- Convolution $f[m,n]*g[m,n] \Leftrightarrow F(u,v)G(u,v)$
- Multiplication $f[m,n]g[m,n] \Leftrightarrow F(u,v)*G(u,v)$
- Separable functions $f[m,n]=f[m]f[n] \Leftrightarrow F(u,v)=F(u)F(v)$
- Energy conservation $\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\left|f[m,n]\right|^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|F(u,v)\right|^2 du dv$

Fourier Transform: Properties

- Define *Kronecker delta function*

$$\delta[m,n] = \begin{cases} 1, & \text{for } m = 0 \text{ and } n = 0 \\ 0, & \text{otherwise} \end{cases}$$

- Fourier Transform of the Kronecker delta function

$$F(u,v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left[\delta[m,n] e^{-j2\pi(um+vn)} \right] = e^{-j2\pi(u0+v0)} = 1$$

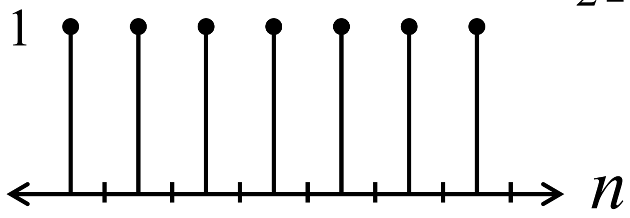
Impulse Train

- Define a *comb* function (impulse train) as follows

$$comb_{M,N}[m,n] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \delta[m-kM, n-lN]$$

where M and N are integers

$comb_2[n]$



2D Discrete Fourier Transform (DFT)

Outline

- Circular and linear convolutions
- 2D DFT
- 2D DCT
- Properties
- Other formulations
- Examples

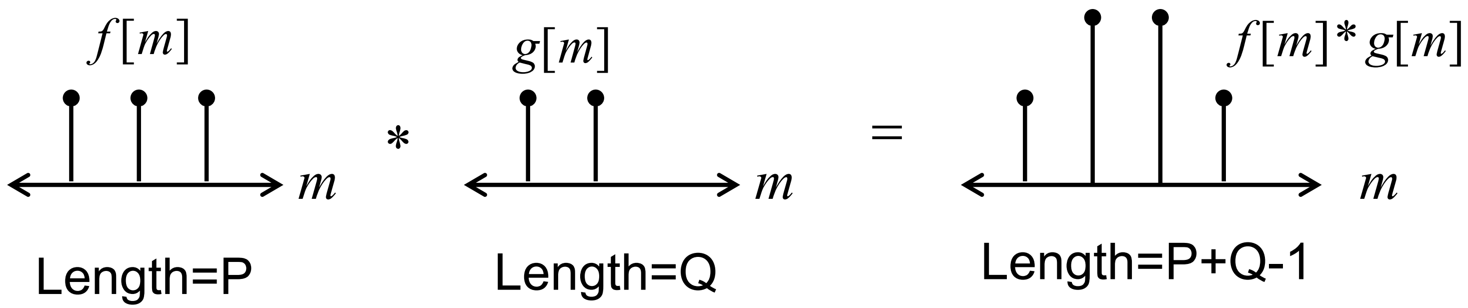
Circular convolution

- Finite length signals (N_0 samples) → circular or periodic convolution
  - the summation is over 1 period
  - the result is a N_0 period sequence

$$\boxed{c[k] = f[k] \otimes g[k] = \sum_{n=0}^{N_0-1} f[n]g[k-n]}$$

- The circular convolution is equivalent to the linear convolution of the zero-padded equal length sequences

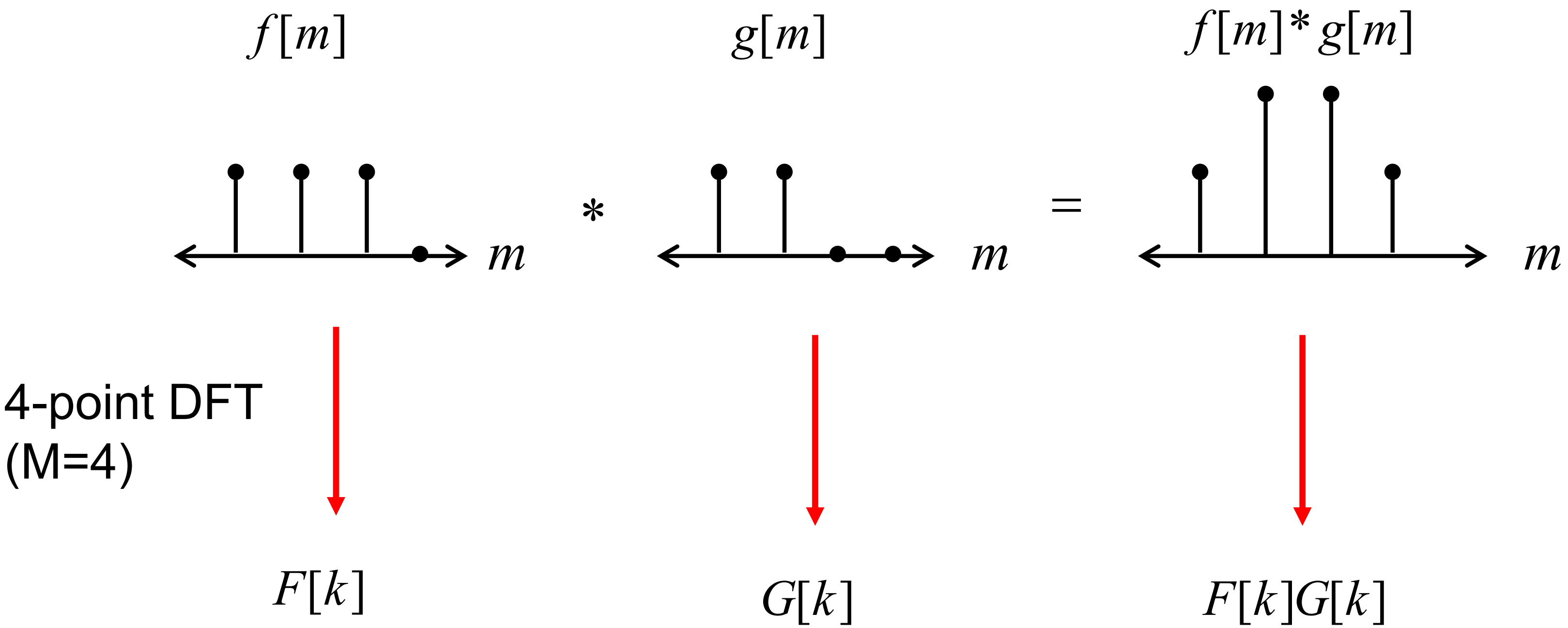
$$f[m] * g[m] \Leftrightarrow F[k]G[k]$$



For the convolution property to hold, M must be *greater than or equal* to P+Q-1.

Convolution

- Zero padding $f[m] * g[m] \Leftrightarrow F[k]G[k]$



In words

- Given 2 sequences of length N and M, let y[k] be their linear convolution

$$y[k] = f[k] * h[k] = \sum_{n=-\infty}^{+\infty} f[n]h[k-n]$$

- y[k] is also equal to the circular convolution of the two suitably zero padded sequences making them consist of the same number of samples

$$c[k] = f[k] \otimes h[k] = \sum_{n=0}^{N_0-1} f[n]h[k-n]$$

$$N_0 = N_f + N_h - 1: \text{ length of the zero-padded seq}$$

- In this way, the linear convolution between two sequences having a different length (filtering) can be computed by the DFT (which rests on the circular convolution)
  - The procedure is the following
    - Pad f[n] with N_h-1 zeros and h[n] with N_f-1 zeros
    - Find Y[r] as the product of F[r] and H[r] (which are the DFTs of the corresponding zero-padded signals)
    - Find the inverse DFT of Y[r]
- **Allows to perform linear filtering using DFT**

2D Discrete Fourier Transform

- Fourier transform of a 2D signal defined over a discrete finite 2D grid of size MxN

or equivalently

- Fourier transform of a 2D set of samples forming a bidimensional sequence
- As in the 1D case, 2D-DFT, though a self-consistent transform, can be considered as a mean of calculating the transform of a 2D sampled signal defined over a discrete grid
- The signal is periodized along both dimensions and the 2D-DFT can be regarded as a *sampled version of the 2D DTFT*

2D Discrete Fourier Transform (2D DFT)

- 2D Fourier (discrete time) Transform (DTFT) [Gonzalez]

$$F(u,v)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f[m,n]e^{-j2\pi(um+vn)}$$

a-periodic signal
periodic transform

- 2D Discrete Fourier Transform (DFT)

$$F[k,l]=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n]e^{-j2\pi\left(\frac{k}{M}m+\frac{l}{N}n\right)}$$

periodized signal
periodic and
sampled transform

2D DFT can be regarded as a sampled version of 2D DTFT.

2D DFT: Periodicity

- A [M,N] point DFT is periodic with period [M,N]
  - Proof

$$F[k,l]=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f[m,n]e^{-j2\pi\left(\frac{k}{M}m+\frac{l}{N}n\right)}$$

$$F[k+M,l+N]=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f[m,n]e^{-j2\pi\left(\frac{k+M}{M}m+\frac{l+N}{N}n\right)}$$

$$=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f[m,n]e^{-j2\pi\left(\frac{k}{M}m+\frac{l}{N}n\right)}\cancelto{1}{e^{-j2\pi\left(\frac{M}{M}m+\frac{N}{N}n\right)}}$$

$$=F[k,l]$$

(In what follows: spatial coordinates=k,l, frequency coordinates: u,v)

DFT: Periodicity

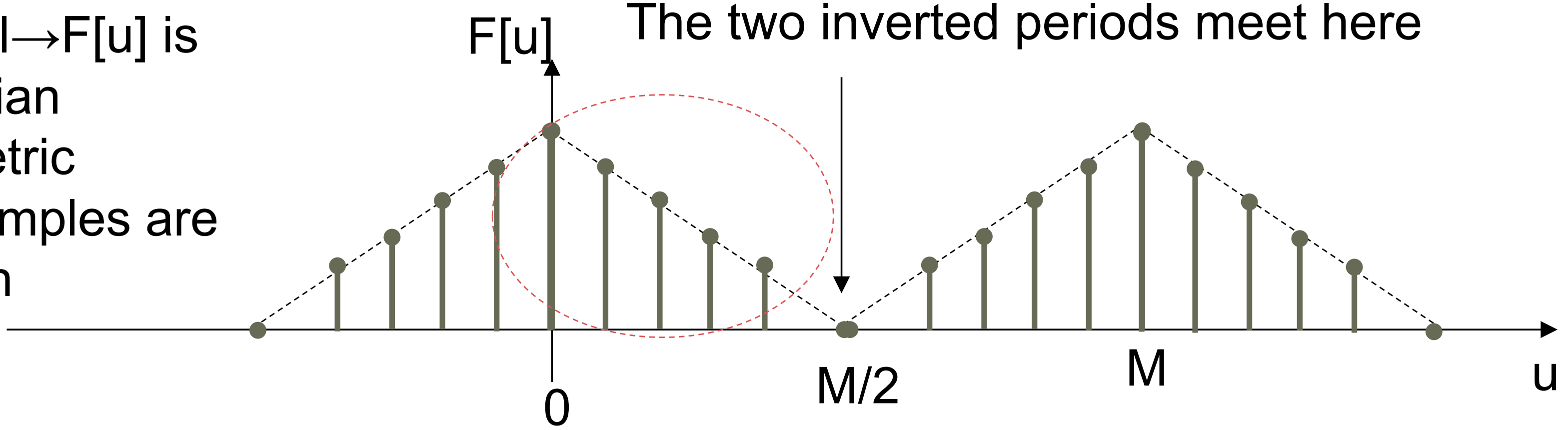
- Periodicity

$$F[u,v]=F[u+mM,v]=F[u,v+nN]=F[u+mM,v+nN]$$

$$f[k,l]=f[k+mM,l]=f[k,l+nN]=f[k+mM,l+nN]$$

- This has important consequences on the implementation and energy compaction property
  - 1D $\qquad F[N-u]=F^*[u]$

f[k] real→F[u] is Hermitian symmetric
M/2 samples are enough



Periodicity: 1D

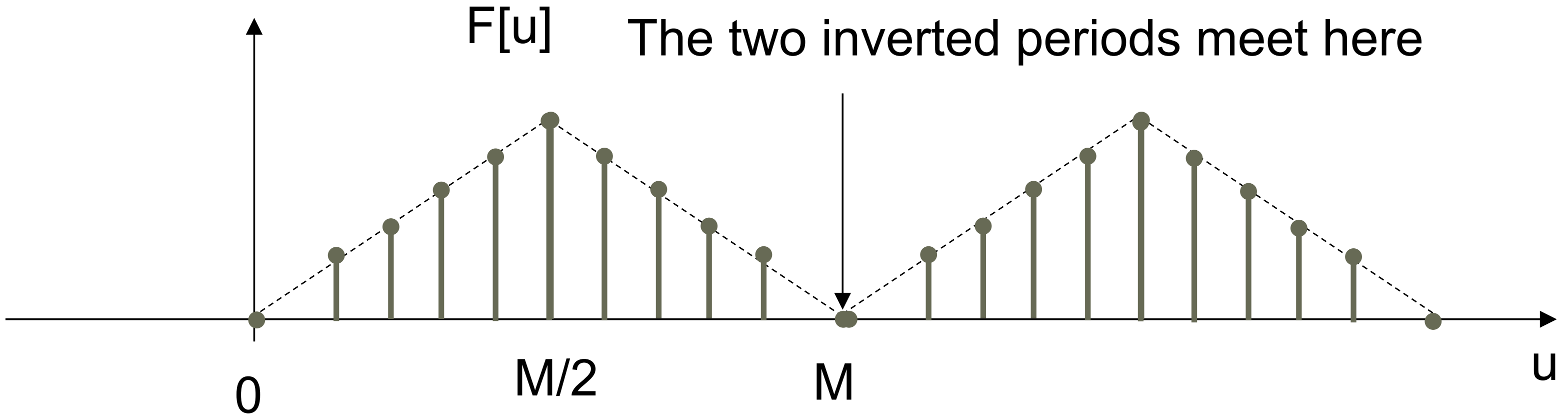
$$f[k] \leftrightarrow F[u]$$

$$f[k]\mathrm{e}^{j2\pi\frac{u_0 k}{M}} \leftrightarrow F[u-u_0]$$

$$u_0 = \frac{M}{2} \rightarrow \mathrm{e}^{j2\pi\frac{u_0 k}{M}} = \mathrm{e}^{j2\pi\frac{Mk}{2M}} = \mathrm{e}^{j\pi k} = (-1)^k$$

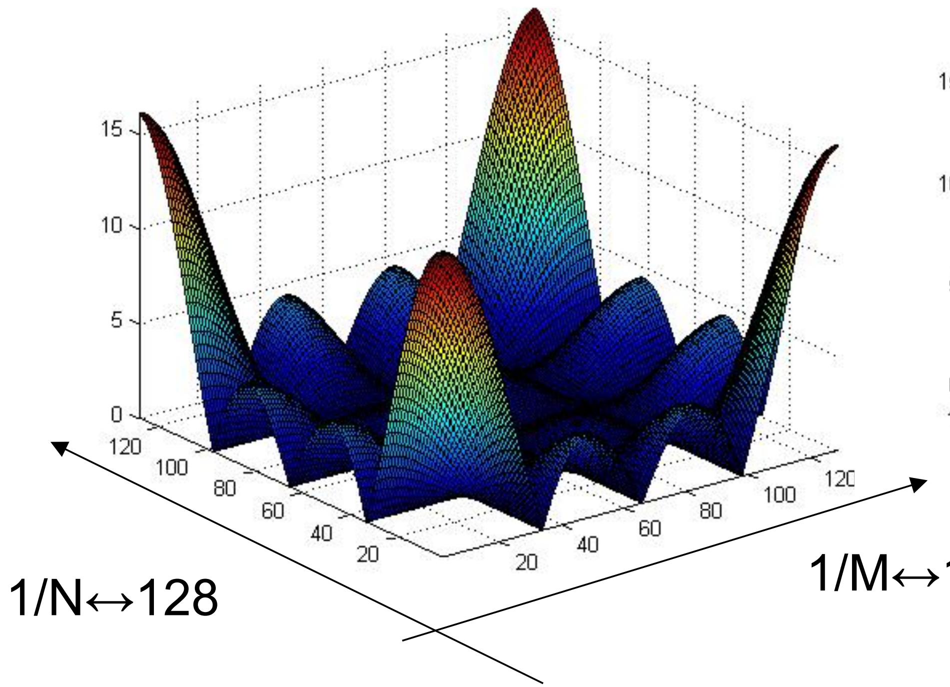
$$(-1)^k f[k] \leftrightarrow F[u-\frac{M}{2}]$$

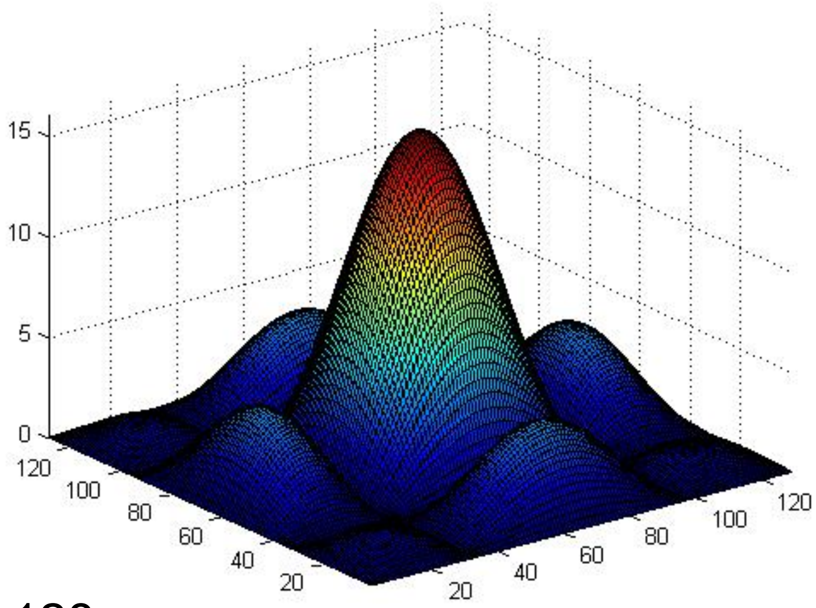
changing the sign of every other sample of the DFT puts F[0] at the center of the interval [0,M]



It is more practical to have one complete period positioned in [0, M-1]

Periodicity in 2D



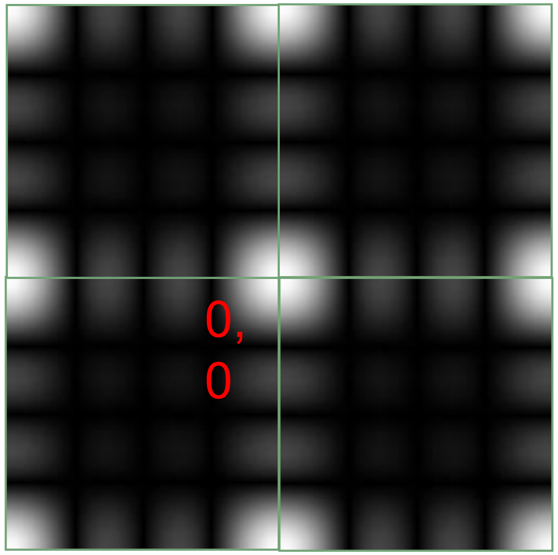


I 4 semiperiodi si incontrano ai vertici

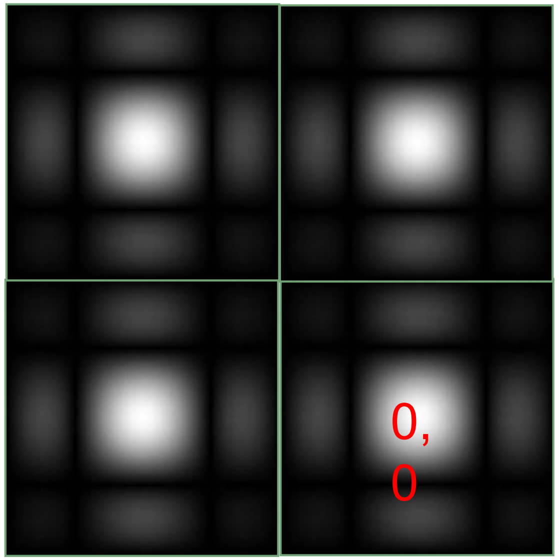
I 4 semiperiodi si incontrano al centro

Periodicity

fft2

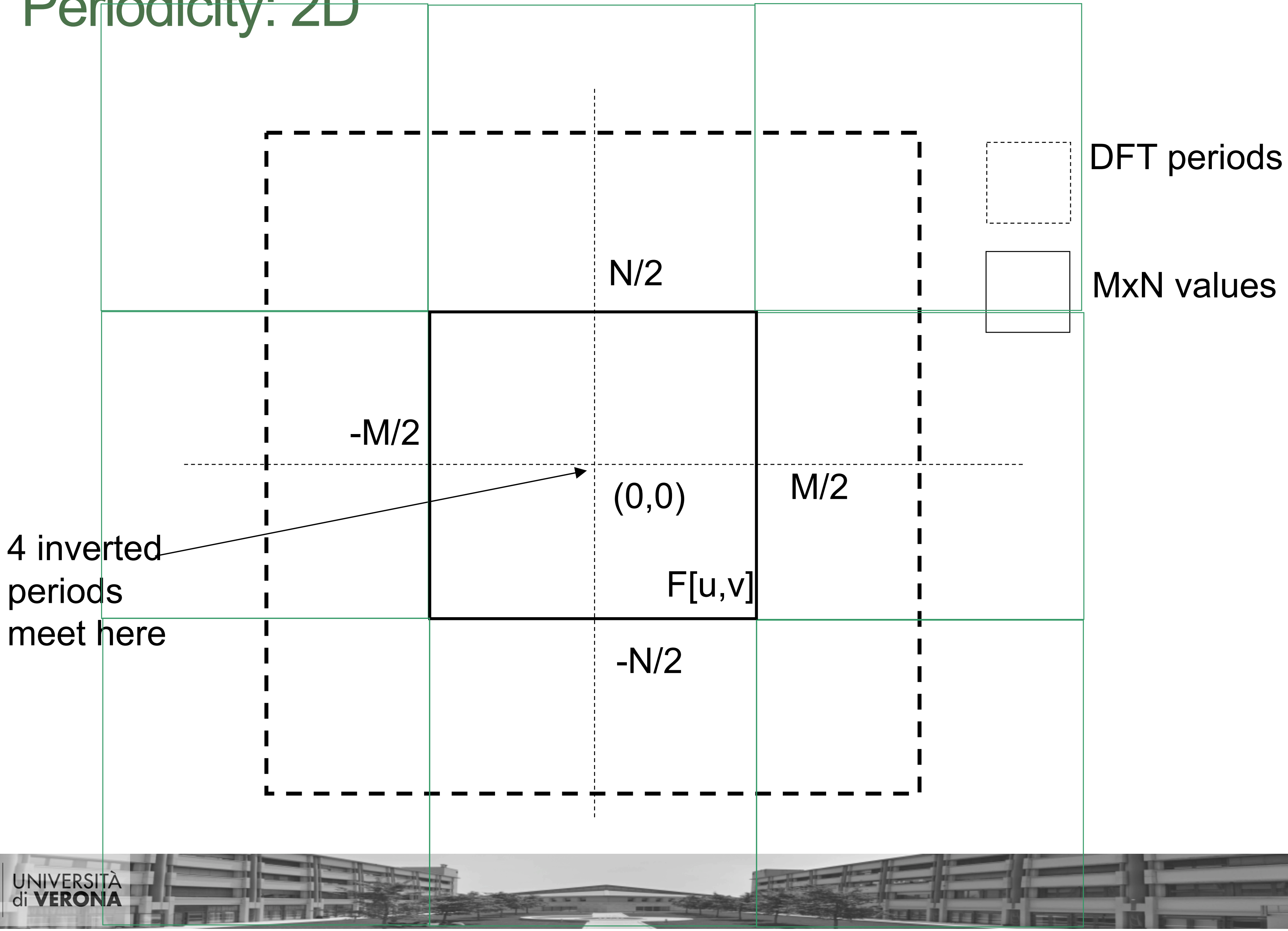


fftshift(fft2)



0,127=1/M,1/N

Periodicity: 2D



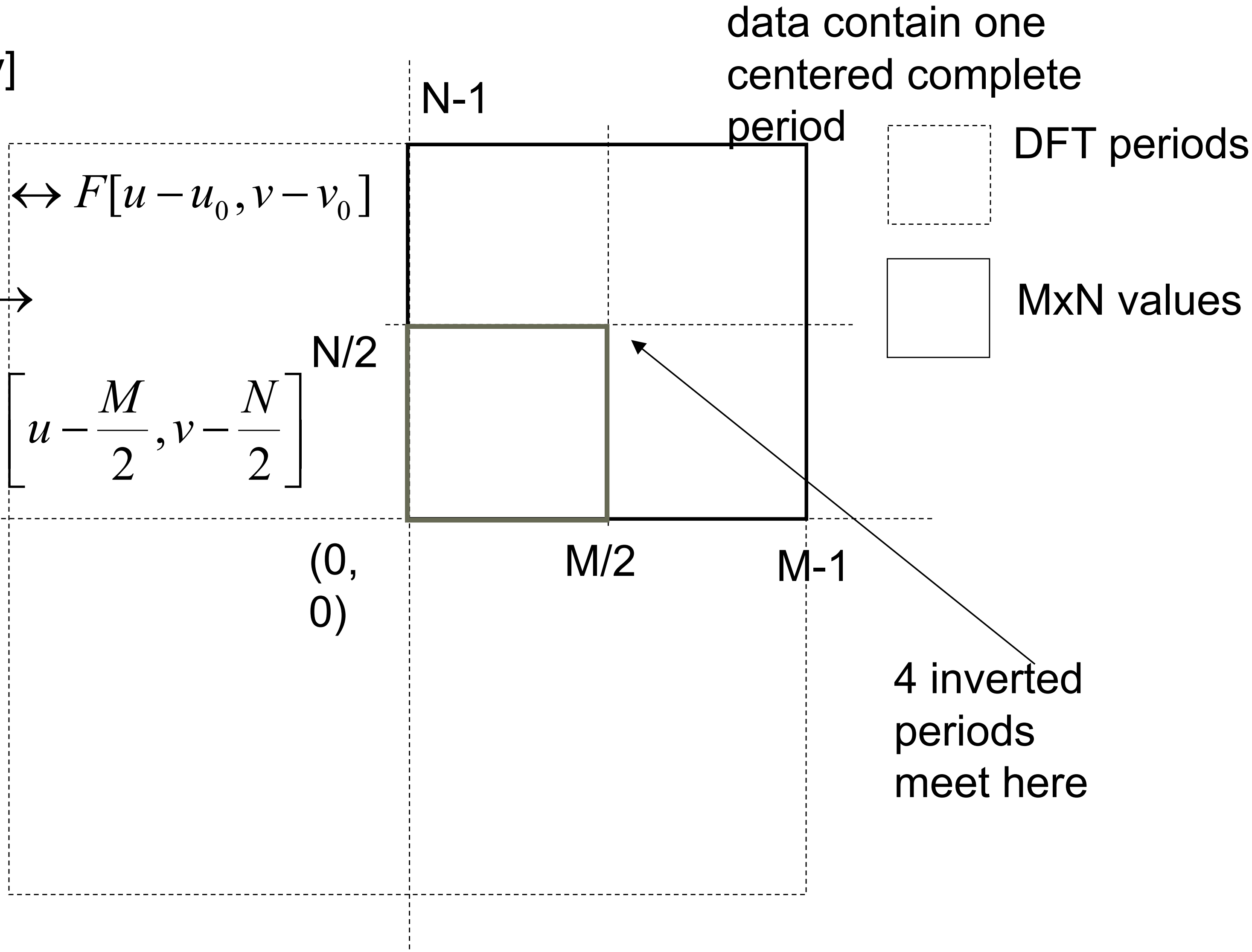
Periodicity: 2D

F[u,v]

$$f[k,l]e^{j2\pi(\frac{u_0k}{M}+\frac{v_0l}{N})} \leftrightarrow F[u-u_0,v-v_0]$$

$$u_0=\frac{M}{2}, v_0=\frac{N}{2} \rightarrow$$

$$(-1)^{k+l} f[k] \leftrightarrow F\left[u-\frac{M}{2}, v-\frac{N}{2}\right]$$



Angle and phase spectra

$$F[u,v] = |F[u,v]| e^{j\Phi[u,v]}$$

$$|F[u,v]| = \left[\mathrm{Re}\{F[u,v]\}^2 + \mathrm{Im}\{F[u,v]\}^2 \right]^{1/2}$$

modulus (amplitude spectrum)

$$\Phi[u,v] = \arctan\left[\frac{\mathrm{Im}\{F[u,v]\}}{\mathrm{Re}\{F[u,v]\}} \right]$$

phase

$$P[u,v] = |F[u,v]|^2$$

power spectrum

For a real function

$$F[-u,-v] = F^*[u,v]$$

conjugate symmetric with respect to the origin

$$|F[-u,-v]| = |F[u,v]|$$

$$\Phi[-u,-v] = -\Phi[u,v]$$

Translation and rotation

$$f[k,l]e^{j2\pi\left(\frac{m}{M}k+\frac{n}{N}l\right)} \leftrightarrow F\left[u-m, v-l\right]$$

$$f\left[k-m, l-n\right] \leftrightarrow F\left[u,v\right]^{-j2\pi\left(\frac{m}{M}k+\frac{n}{N}l\right)}$$

$$\begin{cases} k = r\cos\vartheta \\ l = r\sin\vartheta \end{cases} \qquad \begin{cases} u = \omega\cos\varphi \\ l = \omega\sin\varphi \end{cases}$$

$$f\left[r, \vartheta+\vartheta_0\right] \leftrightarrow F\left[\omega, \varphi+\vartheta_0\right]$$

Rotations in spatial domain correspond equal rotations in Fourier domain