Bi-matrix games

A finite two-person general-sum game can be expressed as a pair of $m \times n$ matrices, $A = (a_{ij})_{m \times n}$. The numbers a_{ij} and b_{ij} are the payoffs of I and II, respectively, corresponding to their i-th and j-th pure strategies. A game in this form is called a bi-matrix game.

Definition

A pair $(\bar{x}, \bar{y})$ is said to be a Nash equilibrium if

$$P(x, \bar{y}) \leq P(\bar{x}, \bar{y}) \text{ and } Q(\bar{x}, y) \leq Q(\bar{x}, \bar{y}),$$

where $P(x, y) = x^T A y =$

$$= \sum_{i=1}^{m} x_i \sum_{j=1}^{n} a_{ij} y_j, \;\; Q(x, y) = x^T B y = \sum_{j=1}^{n} y_j \sum_{i=1}^{m} a_{ij} x_i.$$

Bi-matrix noncooperative games

Theorem

Every bi-matrix game has at least one Nash equilibrium.

Proof.

Let x and y be arbitrary pair of mixed strategies. We set

$$c_i := \max(0, P(i,y) - P(x,y)), d_j = \max(0, Q(x,j) - Q(x,y)),$$

$i = 1, 2, \dots, m,\ j = 1, 2, \dots, n$, and define the map $F : X \times Y \to X \times Y$ as follows: $(x', y') = F(x, y)$, where

$$x_i' := \frac{x_i + c_i}{1 + \sum_{k=1}^{m} c_k}, y_j' := \frac{y_j + d_j}{1 + \sum_{l=1}^{n} d_l},$$

$i = 1, 2, \dots, m,\ j = 1, 2, \dots, n.$

Bi-matrix noncooperative games

Brower fixed point theorem

Let S be a convex compact set and $F : S \to S$ be a continuous map. Then there exists a point $\bar{x} \in S$ so that $F(\bar{x}) = \bar{x}$.

Theorem

Let S be a convex compact set, $F : S \to P$ is a continuous bijection and $H : P \to P$ be a continuous map. Then there exists a point $\bar{x} \in P$ so that $H(\bar{x}) = \bar{x}$.

Proof.

The continuous map $F^{-1} \circ H \circ F : S \to S$ has a fixed point y, i.e. $F^{-1}(H(F(y))) = y$. By setting $\bar{x} = F(y)$ complete the proof.

Proof (continuation).

Clearly, the map F is continuous. Also, x' and y' are mixed strategies. Applying the Brower fixed point theorem, we obtain the existence of a pair of mixed strategies $(\bar{x}, \bar{y})$ such that $F(\bar{x}, \bar{y}) = (\bar{x}, \bar{y})$.

If all $c_i \leq 0$, $i = 1, \ldots, m$, and all $d_j \leq 0$, $j = 1, \ldots, n$, then $P(i, \bar{y}) \leq P(\bar{x}, \bar{y})$, $i = 1, \ldots, m$, and $Q(\bar{x}, j) \leq Q(\bar{x}, \bar{y})$, $j = 1, \ldots, n$. Let $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$ be arbitrary pair of mixed strategies. The previous inequalities imply that $x_i P(i, \bar{y}) \leq x_i P(\bar{x}, \bar{y})$, $i = 1, \ldots, m$, and $y_j Q(\bar{x}, j) \leq y_j Q(\bar{x}, \bar{y})$, $j = 1, \ldots, n$. Summing these inequalities, we obtain that $P(x, \bar{y}) \leq P(\bar{x}, \bar{y})$ and $Q(\bar{x}, y) \leq Q(\bar{x}, \bar{y})$, i.e. the pair $(\bar{x}, \bar{y})$ is a Nash equilibrium.

Proof (continuation).

Let us assume that there exists an index i_0 so that $c_{i_0} >$. Let us assume that whenever $\bar{x}_i > 0$, then $c_i > 0$, i.e.

$$P(i, \bar{y}) > P(\bar{x}, \bar{y}) \text{ whenever } \bar{x}_i > 0.$$

These inequalities imply that

$$\bar{x}_i P(i, \bar{y}) > \bar{x}_i P(\bar{x}, \bar{y}) \text{ whenever } \bar{x}_i > 0.$$

Summing these inequalities, we obtain

$$P(\bar{x}, \bar{y}) = \sum_{x_i > 0} x_i P(i, \bar{y}) > \sum_{i=1}^{m} \bar{x}_i P(\bar{x}, \bar{y}) = P(\bar{x}, \bar{y}).$$

Bi-matrix noncooperative games

Proof (continuation).

The obtained contradiction shows that there exists an index i_1 so that $x_{i_1} > 0$ and $c_{i_1} = 0$. But then the equality

$$0 < \bar{x}_{i_1} = \frac{\bar{x}_{i_1} + c_{i_1}}{1 + \sum_{k=1}^{m} c_k} = \frac{\bar{x}_{i_1}}{1 + \sum_{k=1}^{m} c_k} < \bar{x}_{i_1}$$

is impossible. Hence the assumption that there exists an index i_0 so that $c_{i_0} >$ is wrong. This complete the proof of the theorem.

Lemma 1.

Let $(\bar{x}, \bar{y}) \in X \times Y$ be a Nash equilibrium of the bi-matrix game determined by the payoff-functions P and Q. If $\bar{x}_{i_0} > 0$, then $P(i_0, \bar{y}) = P(\bar{x}, \bar{y})$. Also, if $\bar{y}_{j_0} > 0$, then $Q(\bar{x}, j_0) = Q(\bar{x}, \bar{y})$.

Proof.

Let us assume the contrary, i.e. $P(i_0, \bar{y}) < P(\bar{x}, \bar{y})$. Since $(\bar{x}, \bar{y}) \in X \times Y$ is a Nash equilibrium of the payoff-function P, we have that

$$P(i, \bar{y}) \leq P(\bar{x}, \bar{y}) \text{ for each } i = 1, 2, \ldots, m, \text{ with } i \neq i_0.$$

Multiplying the both sides of these inequalities by $\bar{x}_i$, we obtain that

$$\bar{x}_i P(i, \bar{y}) \leq \bar{x}_i P(\bar{x}, \bar{y}).$$

Proof of Lemma 1. (continuation)

After adding of all these m inequalities, we obtain that

$$\sum_{i=1}^{m} \bar{x}_i P(i, \bar{y}) < \sum_{i=1}^{m} \bar{x}_i P(\bar{x}, \bar{y}), \text{ i.e. } P(\bar{x}, \bar{y}) < P(\bar{x}, \bar{y}).$$

The obtained contradiction shows that our assumption is wrong, and hence $P(i_0, \bar{y}) = P(\bar{x}, \bar{y})$. Analogously, one can prove that $Q(\bar{x}, j_0) = Q(\bar{x}, \bar{y})$. This completes the proof.

Lemma 2.

Let $(\bar{x}, \bar{y}) \in X \times Y$ be a Nash equilibrium of the bi-matrix game determined by the payoff-functions P and Q. If $P(i_0, \bar{y}) < P(\bar{x}, \bar{y})$, then $\bar{x}_{i_0} = 0$. Also, if $Q(\bar{x}, j_0) < Q(\bar{x}, \bar{y})$, then $\bar{y}_{j_0} = 0$.

Proof.

Let us assume that $\bar{x}_{i_0} > 0$. According to Lemma 4 we obtain that $P(i_0, \bar{y}) = P(\bar{x}, \bar{y})$. This contradiction shows that $\bar{x}_{i_0} = 0$. Analogously, one can prove that $\bar{y}_{j_0} = 0$.

Example

Let $(\bar{x}, \bar{y}) \in X \times Y$ be a Nash equilibrium of the bi-matrix game determined by the matrices

$$A = \begin{pmatrix} 2 & 4 & 5 \\ 4 & 2 & 1 \end{pmatrix} \; B = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 2 & 3 \end{pmatrix}$$

Applying Lemma 1 and Lemma 2, one can that the only possibilities for $\bar{x}$ are $\bar{x} = (1/3, 2/3)$ and $\bar{x} = (2/3, 1/3)$. The case $\bar{x} = (1/3, 2/3)$ is impossible. Solving the linear system

$$\left(\begin{array}{l} 2\bar{y}_1 + 4\bar{y}_2 = P(\bar{x}, \bar{y}) \\ 4\bar{y}_1 + 2\bar{y}_2 = P(\bar{x}, \bar{y}) \\ \bar{y}_1 + \bar{y}_2 = 1 \end{array} \right.$$

one can obtain that $\bar{x} = (2/3, 1/3)$ and $\bar{y} = (1/2, 1/2, 0)$ is the unique Nash equilibrium of this game.

Two person cooperative games: the bargaining problem

Motivation

Let us consider the following bi-matrix games

$$\begin{pmatrix} (1,4) & (0,0) \\ (0,0) & (4,1) \end{pmatrix} \text{ and } \begin{pmatrix} (5,5) & (0,10) \\ (10,0) & (1,1) \end{pmatrix}$$

The first game has two Nash equilibriums ((1,1) and (2,2) in pure strategies) and it is not clear how to choose "an optimal strategy". The second game has one Nash equilibrium ((2,2) in pure strategies), but clearly the pure strategies (1,0) for both players ensure bigger payoffs for them. For that reason, the cooperation between the two players is more reasonable.

Two person cooperative games: the bargaining problem

Motivation (continuation)

Let us consider the following bi-matrix game

$$\begin{pmatrix} (2,1) & (-1,-1) \\ (-1,-1) & (1,2) \end{pmatrix}$$

This game has again two Nash equilibriums ((1,1) and (2,2) in pure strategies) and it is not clear how to choose "an optimal strategy". Let us denote by p_{ij} the probability the I player to choose his i-th pure strategy and the II player to choose his j-th pure strategy. If the two players determine together the probabilities $p_{11} = p_{22} = 1/2$ and $p_{12} = p_{21} = 0$, then the expected payoff for the both players will be $3/2$. One can check that $(3/2, 3/2)$ is not expected payoff for the two players, corresponding to suitable mixed strategies.

Motivation (continuation)

Let us consider the following bi-matrix game

$$\begin{pmatrix} (a_{11}, b_{11}) & (a_{12}, b_{12}) & \dots & (a_{1n}, b_{1n}) \\ (a_{21}, b_{21}) & (a_{22}, b_{22}) & \dots & (a_{2n}, b_{2n}) \\ \dots & \dots & \dots & \dots \\ (a_{m1}, b_{m1}) & (a_{m2}, b_{m2}) & \dots & (a_{mn}, b_{mn}) \end{pmatrix}$$

If denote by $p_{ij} \geq 0$ the probability the I player to choose his i-th pure strategy and the II player to choose his j-th pure strategy, then the corresponding payoff for the two players is

$$(P(\vec{p}), Q(\vec{p}))^T = \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij}(a_{ij}, b_{ij})^T, \quad \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} = 1.$$

Here $\vec{p} = (p_{11}, p_{12}, \dots, p_{mn})$.

Motivation (continuation)

Let X and Y be the sets of all mixed strategies for the first end for the second player, respectively. Then the guaranteed payoff u_0 for the first player is $u_0 := max_{x \in X} min_{y \in Y} P(x, y)$, and for the second player is $v_0 := max_{y \in Y} min_{x \in X} Q(x, y)$. Let us denote by S the convex hull of the points $(a_{ij}, b_{ij})^T$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. Then we can formulate the following problem: It is given a bounded closed convex subset S of R^2 and a point $(u_0, v_0)^T \in R^2$. Here, $(u_0, v_0)^T$ is the point whose components are the guaranteed payoffs for the two players and S is the feasible set, i.e. given any point $(x, y)^T \in S$, then it is possible for the two players acting together to obtain payoff u for the first player and payoff v for the second player. We want to assign to each triple (S, u_0, v_0) a bargaining solution $(\bar{u}, \bar{v})^T$.

Two person cooperative games: the bargaining problem

Properties of $(\bar{u}, \bar{v})^T$.

The following axioms (given by John Nash) seem reasonable conditions that the solution $(\bar{u}, \bar{v})^T$ of the problem (S, u_0, v_0) have to satisfy:

1. Individual rationality: $(\bar{u}, \bar{v})^T \geq (u_0, v_0)^T$;
2. Feasibility: $(\bar{u}, \bar{v})^T \in S$;
3. Pareto-optimality: If $(u, v)^T \in S$ and $(u, v)^T \geq (\bar{u}, \bar{v})^T$, then $(u, v)^T = (\bar{u}, \bar{v})^T$;
4. Independence on irrelevant alternatives: if $(\bar{u}, \bar{v})^T \in T \subset S$ and $(\bar{u}, \bar{v})^T$ is a solution of the problem (S, u_0, v_0), then $(\bar{u}, \bar{v})^T$ is a solution of the problem (T, u_0, v_0);

Properties of $(\bar{u}, \bar{v})^T$ (continuation).

5. Independence on linear transformations: Let S', $(u_0', v_0')^T$ and $(\bar{u}', \bar{v}')^T$ be obtained from S, $(u_0, v_0)^T$ and $(\bar{u}, \bar{v})^T$, respectively, by the linear transformation $u' = \alpha_1 u + \beta_1$, $v' = \alpha_2 v + \beta_2$, where $\alpha_1 > 0$, $\alpha_2 > 0$, β_1 and β_2 are fixed constants. If $(\bar{u}, \bar{v})^T$ is a solution of the problem (S, u_0, v_0), then $(\bar{u}', \bar{v}')^T$ is a solution of the problem (S', u_0', v_0');
6. Symmetry: Let S be such that if $(u, v)^T \in S$, then $(v, u)^T \in S$. Suppose that $(\bar{u}', \bar{v}')^T$ is a solution of the problem (S, w_0, w_0). Then $\bar{u} = \bar{v}$.

Remark.

The first three axioms are obvious and need no justification. The 4-th states that, if a point is a solution of the bargaining problem, and then the feasible set is enlarged, the solution of the new problem will be the same point or one of the new points of the enlarged set, but not a point in the old, smaller set. The 5-th is natural enough, especially if we think of a set S coming from a bi-matrix game. The 6-th states that if the two players have equal capacities.

Theorem 1.

The problem determined by (S, u, v) has a unique solution $(\bar{u}, \bar{v})^T$ satisfying the axioms 1-6.

The proof is based on the following lemmas:

Lemma 1.

If there is a point $(u, v)^T \in S$, satisfying $u > u_0$ and $v > v_0$, then the problem

$$(u - u_0)(v - v_0) \to \max$$

subject to

$$u \geq u_0, v \geq v_0,$$
$$(u, v) \in S$$

has a unique solution $(\bar{u}, \bar{v})$.

Proof of Lemma 1.

By hypothesis, the set S is compact. The function $g(u,v) := (u-u_0)(v-v_0)$ is continuous, and hence it reaches its maximum on the considered set. Also, by hypothesis, this maximum M is positive. Let us assume that there are two points $(u_1, v_1)^T$ and $(u_2, v_2)^T$ such that $M = g(u_1, v_1) = g(u_2, v_2)$. As $M > 0$, it is not possible that $u_1 = v_1$ (this equality implies that $v_1 = v_2$. Let us assume that $u_1 < u_2$. Then this inequality implies that $v_1 > v_2$. As S is a convex set, the point $(\bar{u}, \bar{v}) \in S$, where $\bar{u} := (u_1 + u_2)/2$ and $\bar{v} := (v_1 + v_2)/2$. But the we obtain that

Proof of Lemma 1 (continuation).

$$g(\bar{u},\bar{v}) = \frac{(u_1-u_0)+(u_2-u_0)}{2}\cdot\frac{(v_1-v_0)+(v_2-v_0)}{2} =$$

$$\frac{(u_1-u_0)(v_1-v_0)}{2}+\frac{(u_2-u_0)(v_2-v_0)}{2}+\frac{(u_1-u_2)(v_2-v_1)}{4} > M.$$

because the first two terms are equal to $M/2$ and the third term is positive. This completes the proof.

Lemma 2.

Let S, $(u_0,v_0)^T$ and $(\bar{u},\bar{v})^T$ are as in Lemma 1. We set

$$h(u,v) = (\bar{v}-v_0)u + (\bar{u}-u_0)v.$$

Then $h(u,v) \leq h(\bar{u},\bar{v})$ for each $(u,v)^T \in S$.

Proof of Lemma 2.

Let us assume that there exists a point $(u, v)^T \in S$ such that $h(u, v) > h(\bar{u}, \bar{v})$. Let us choose an arbitrary $\varepsilon \in (0, 1)$. We set $u_\varepsilon := \bar{u} + \varepsilon(u - \bar{u})$ and $v_\varepsilon := \bar{v} + \varepsilon(v - \bar{v})$. Because h is a linear function, $h(u - \bar{u}, v - \bar{v}) > 0$. Since

$$g(u_\varepsilon, v_\varepsilon) = g(\bar{u}, \bar{v}) + \varepsilon[h(u - \bar{u}, v - \bar{v}) + \varepsilon(u - \bar{u})(v - \bar{v})] > g(\bar{u}, \bar{v}),$$

we obtain a contradiction with the maximality of $g(\bar{u}, \bar{v})$. This completes the proof.

Proof of Theorem 1

• Let the assumptions of Lemma 1 hold true. Then there exists a unique point $(\bar{u}, \bar{v})$ that maximizes the function g. By definition the point $(\bar{u}, \bar{v})^T$ satisfies axioms 1 and 2.
• We set $S_0 := \{(u, v)^T \in S : \ u \geq u_0, v \geq v_0\}$. Let the point $(u, v)^T \in S_0$ and $(u, v)^T \geq (\bar{u}, \bar{v})^T$. If $(u, v)^T \neq (\bar{u}, \bar{v})^T$, then $g(u, v) > g(\bar{u}, \bar{v})$. The obtained contradiction shows that axiom 3 also is fulfilled.
• Since $(\bar{u}, \bar{v})^T$ maximizes g on S_0, then $(\bar{u}, \bar{v})^T$ maximizes g on the smaller set T_0. Thus axiom 4 holds true.
• If we set $u' = \alpha_1 u + \beta_1$ and $v' = \alpha_1 v + \beta_1$, then

$$g'(u', v') = (u' - u'_0)(v' - v'_0) = \alpha_1 \alpha_2 (u - u_0)(v - v_0) = g(u, v).$$

Since the point $(\bar{u}, \bar{v})^T$ maximizes g on S_0, then the point $(\bar{u}', \bar{v}')^T$ maximizes g' on S'_0. So, axiom 5 also is fulfilled.

Proof of Theorem 1 (continuation).

- Assume that the set S is symmetric, $u_0 = v_0 = w_0$ and $(\bar{u}, \bar{v})^T$ maximizes g on S_0. Then $(\bar{v}, \bar{u})^T$ maximizes g on S_0 too. Since the maximum of g is reached at a unique point, we have that $(\bar{u}, \bar{v})^T = (\bar{v}, \bar{u})^T$, i.e. $\bar{u} = \bar{v}$. Thus axiom 6 holds true too.

- We have obtained that the point $(\bar{u}, \bar{v})^T$ satisfies the axioms $1 \div 6$. To complete the proof, we have to show that $(\bar{u}, \bar{v})^T$ is the unique point that satisfies these axioms. Indeed, let us consider the set $U := \{(u, v)^T : \ h(u, v) \leq h(\bar{u}, \bar{v})\}$. According to Lemma 2, we have that $S \subset U$. Let the set U' be obtained from the set U by the linear map: $L(u, v) = (u', v')^T$, where

$$u' := \frac{u - u_0}{\bar{u} - u_0}, \ \ v' := \frac{v - v_0}{\bar{v} - v_0}.$$

Proof of Theorem 1 (continuation).

One can directly checked that $L(u_0, v_0) = (0,0)^T$, $L(\bar{u}, \bar{v}) = (1,1)^T$ and $U' = \{(u', v') : u' + v' \leq 2\}$. Because U' is symmetric, axiom 6 implies that the solution for the triple $U', 0, 0)$ have to belong to the line $u' = v'$. According to axiom 3, the solution has to be the point $(1,1)^T$. Let us consider the linear map L^{-1}. Then U is the image of U' under the map L^{-1}. Then axiom 5 implies that $(\bar{u}, \bar{v})^T$ is the solution of the bargaining problem for (U, u_0, v_0). Because $(\bar{u}, \bar{v})^T \in S \subset U$, the axiom 4 implies that $(\bar{u}, \bar{v})^T$ is the solution of the bargaining problem for (S, u_0, v_0). This completes the proof of Theorem 1 whenever there exists a point $(u, v)^T \in S$ such that $u > u_0$ and $v > v_0$.

Proof of Theorem 1 (continuation).

Let us assume that there does not exist a point $(u, v)^T \in S$ such that $u > u_0$ and $v > v_0$. If there exist two points $(u_1, v_0)^T \in S$ and $(u_0, v_2)^T \in S$ with $u_1 > u_0$ and $v_2 > v_0$, then the convexity of S implies that $(u_3, v_3)^T := \frac{1}{2}(u_1, v_0)^T + \frac{1}{2}(u_0, v_2)^T \in S$. Clearly $u_3 > u_0$ and $v_3 > v_0$, which is impossible by our assumption. Hence, if a point $(u, v_0)^T \in S$ with $u > u_0$, then there does exist a point $(u_0, v)^T \in S$ with $v > v_0$. Then $(\bar{u}, \bar{v})$ is the point with maximal first component and $\bar{v} = v_0$. Similarly, if a point $(u_0, v)^T \in S$ with $v > v_0$, then there does exist a point $(u, v_0)^T \in S$ with $u > u_0$. Then $(\bar{u}, \bar{v})$ is the point with maximal second component and $\bar{u} = u_0$. It can be easily checked that these solutions satisfy all axioms $1 \div 6$. Moreover, axioms $1 \div 3$ imply that there is no other point satisfying these axioms.

n-person cooperative games

n-person noncooperative games

First, let us consider an n -person noncooperative game. Then cooperation between the players is forbidden by the rules of the game. Then there exist at least one Nash equilibrium. The idea of the proof is the same as in the case of a noncooperative bi-matrix game. In general there is no big difference between a noncooperative bi-matrix game and an n -person noncooperative game.

n-person cooperative games

Let us consider an n-cooperative bi-matrix game for which cooperation are permitted. In this case a new idea appears: coalition. In the n-person case, there are many possible coalitions. This means that is a coalition exists ant remains for some time, then the members of the coalition have to reach some sort of equilibrium or stability. Hence this idea for stability has to be analyzed in any meaningful theory.

n-person cooperative games

Definition

For an n-person game we denote by $N := \{1, 2, 3, \dots, n\}$ the set of all players. Any nonempty subset S of N is called a coalition.

Definition

By the characteristic function v of an n-person game we mean a real valued function v defined on the subsets of N that assigns to each $S \subset N$ the maximum value for S of the two-person game played between S and $N \setminus S$ (assuming that these two coalitions form). This means that $v(S)$ is the payoff of the members of the coalition S can obtain from the game, whatever the remaining players may do.

Properties of the characteristic function

1. $v(\emptyset) = 0$;
2. $v(S \cup T) \geq v(S) + v(T)$ for each $S \subset N \supset T$ with $S \cap T = \emptyset$.

The second property means that if S and T are disjoint coalitions, they can accomplish at least as much by joining forces as by remaining separate.

Remark.

The essence of n-person games is the formation of coalition. For that reason we shall study in details the characteristic function. This function tells us the capacities of the coalitions. This we shall study an n-person cooperative game with its characteristic function.

n-person cooperative games

Definition.

A game in characteristic function form is said to be constant sum if $v(S) + v(N \setminus S) = v(N)$ for all subsets sS of N.

Remark.

Let us assume that an n-person sum is played and the players have a total payoff $v(N)$ to divide. This can be divided in any way, but clearly no player will accept less than the minimum that he can attain for himself.

Definition.

An imputation for an n-person sum determined by its characteristic function is any vector $x = (x_1, x_2, \ldots, x_n)$, satisfying

1. $\sum_{i=1}^{n} x_i = v(N)$;
2. $x_i \geq v(\{i\}$ for all $i = 1, \ldots, n$.

We denote by $E(v)$ the set of all imputations of the game determined by the characteristic function v.
The main question of an n-person cooperative game is: Which of all the imputations will be obtained?
The answer is trivial when the set $E(v)$ contains only one element. In that case the unique imputations will be the obvious result. It does not matter what coalition form. This give rise to distinction between essential and inessential games. It is clear by super additivity (repeated n times) that $v(N) \geq \sum_{i=1}^{n} v(\{i\})$. If

$v(N) = \sum_{i=1}^{n} v(\{i\})$, then the following relations

$v(N) = \sum_{i=1}^{n} x_i \geq \sum_{i=1}^{n} v(\{i\}) = v(N)$ imply that

$E(v) = \{(v(1)\}, v(2)\}, \ldots, v(n)\})\}$, and the game is trivial.

Definition

A game v is essential if

$$v(N) > \sum_{i=1}^{n} v(\{i\}).$$

It is otherwise inessential (we study further only essential games).

Let x and y be two imputations. Suppose that the players are confronted by a choice between x and y. It is not enough that some players prefer the imputation y to x because it is impossible that all players prefer y to x (since the sum of the components of each imputations is $v(N)$). It is important that the players that prefer y to x are sufficiently strong to enforce the choice of y.

n-person cooperative games

Definition.

Let y and x be two imputations, and let S be an arbitrary coalition. It is said that y dominates x trough S (the notation is $y \underset{S}{\succ} x$) if

1. $y_i > x_i$ for each $i \in S$;
2. $\sum_{i \in S} y_i \leq v(S)$.

It is said that y dominates x (the notation is $y \succ x$) if there exists a coalition S with $y \underset{S}{\succ} x$.

Remark

The condition 1. says that the players from the coalition S prefer y to x, while condition 2 says that the players from S are capable of obtaining what y gives them.

One can check that the relation $y \underset{S}{\succ} x$ (for any given S is a partial order relation in $E(v)$, while $\succ$ is not irreflexive, symmetric and transitive (since the coalition can be different in different cases).

n-person cooperative games

Definition

It is said that two n-person cooperative games u and v are isomorphic if there exists a bijection $f : E(u) \to E(v)$ such that for any $x \in E(u) \ni E(v)$ and $S \subset N$ we have that

$$y \underset{S}{\succsim} x \text{ iff } f(y) \underset{S}{\succsim} f(x).$$

Definition

It is said that two n-person cooperative games u and v are S-equivalent if there exist $r > 0$ and n constants $\alpha_1, \alpha_2, \ldots, \alpha_n$, so that

$$v(S) = ru(S) + \sum_{i=1}^{n} \alpha_i$$

for each coalition S.

Theorem.

If u and v are S-equivalent, then they are isomorphic.

Proof.

Assume that u and v are S-equivalent, i.e. there exist $r > 0$ and n constants $\alpha_1, \alpha_2, \ldots, \alpha_n$, so that

$$v(S) = ru(S) + \sum_{i=1}^{n} \alpha_i$$

for each coalition S. Consider the following bijection $f : E(u) \to E(v)$ such that if $y = f(x)$ then $y_i = rx_i + \alpha_i$ for each $i = 1, 2, \ldots, n$. One can check that f is well defined. Moreover,

$$y \underset{S}{\succ} x \text{ iff } f(y) \underset{S}{\succ} f(x).$$

This completes the proof.

Definition.

It is said that a game u is in (0,1) normalization if

1. $u(i) = 0$ for each $i = 1, 2, \ldots, n$;
2. $u(N) = 1$.

Theorem.

If u is an essential game, then u is S-equivalent to exactly one game in (0,1) normalization.

Proof.

Let v be a game in (0,1) normalization which is S-equivalent to the original game u. Then $0 = v(\{i\}) = ru(\{i\}) + \alpha_i$ for each $i = 1, 2, \ldots, n$, and $1 = v(N) = ru(N) + \sum_{i=1}^{n} \alpha_i$. From here we obtain that

Proof (continuation.

$$r = \frac{1}{u(N) - \sum_{i=1}^{n} u(\{i\})} \text{ and } \alpha_i = -\frac{u(\{i\})}{u(N) - \sum_{i=1}^{n} u(\{i\})}$$

for each $i = 1, 2, \ldots, n$. This completes the proof.

Definition.

A game u is said to be symmetric if $u(S)$ depends only on the number of elements of S.

Definition.

A game u in (0,1) normalization is said to be simple if for each coalition S we have that $u(S) \in \{0, 1\}$. A game u is said to be simple if its (0,1) normalization is simple.

Definition

The set of all undominated imputations for a game u is called core and it is denoted by $C(u)$.

Theorem.

The core of a game u is the set of all vectors $x = (x_1, x_2, \ldots, x_n)$ satisfying

1. $\sum_{i \in S} x_i \geq u(S)$ for all coalitions S;
2. $\sum_{i=1}^{n} x_i = u(N)$.

Proof.

If we set $S := \{i\}$, $i = 1, 2, \ldots, n$, then condition 1. implies that $x_i \geq u(\{i\})$, $i = 1, 2, \ldots, n$. These inequalities and condition 2 imply that all vector x are imputations.

Let us assume that the vector x satisfies conditions 1 and 2, but there exists a vector y with $y \underset{S}{\succ} x$, i.e. $y_i > x_i$ for all $i \in S$ and $\sum_{i \in S} y_i \leq u(S)$. By adding the inequalities we obtain that $\sum_{i \in S} y_i > u(S)$. The obtained contradiction shows that there does not exist a vector y with $y \underset{S}{\succ} x$.

Proof (continuation).

Conversely, let us assume that the imputation x does not satisfy the conditions 1. and 2. Then there exist $\varepsilon > 0$ and a coalition S such that $\sum_{i \in S} x_i = u(S) - \varepsilon$. We set

$$\alpha := u(N) - u(S) - \sum_{i \in N \setminus S} u(\{i\}).$$

Clearly, $\alpha \geq 0$. Let s be the number of the elements of S. We set

$$y_i := \begin{cases} x_i + \dfrac{\varepsilon}{s}, & \text{if } i \in S; \\ u(\{i\}) + \dfrac{\alpha}{n-s}, & \text{if } i \notin S. \end{cases}$$

One can check that y is an imputation such that $y \succ_S x$. Hence $x \notin C(u)$. This completes the proof.

Remark.

The core can contain more than one point. This means that more than one outcome is stable. But the core can be an empty set:

Theorem.

If u is an essential constant-sum game, then $C(u) = \emptyset$.

Proof.

Let us assume that $x \in C(u)$. Then

$$u(N) - x_i = \sum_{j \in N \setminus \{i\}} x_j \geq u(N \setminus \{i\}) = u(N) - u(\{i\}),$$

Proof (continuation).

and hence $x_i \leq u(\{i\})$. Then

$$u(N) = \sum_{i \in N} x_i \leq \sum_{i \in N} u(\{i\}) < u(N)$$

because the game is essential. The obtained contradiction shows that $C(u) = \emptyset$ and completes the proof.

Example.

Let u be a simple game in (0,1) normalization. It is said that the player i is a veto player if $u(N \setminus \{i\}) = 0$.
Let us assume that u has no veto players. Then $u(N \setminus \{i\}) = 1$ for each i.

Example (continuation).

Let x be an imputation belonging to the core $C(u)$. Then the following inequalities hold true:

$$\sum_{i \in N} x_i = u(N) = 1 \text{ and } \sum_{i \in N \setminus \{j\}} x_i \geq u(N \setminus \{j\}) = 1,$$

which is impossible, because $x_i \geq u(\{i\}) = 0$ for each $i = 1, 2, \ldots, n$. Hence $C(u) = \emptyset$.

Assume now that the game has one or more veto players and denote by S the set of all veto players. Let x be an imputation such that:

$$\sum_{i \in S} x_i = 1, x_i \geq 0 \text{ for all } i \in S, \text{ and } x_i = 0 \text{ for all } i \notin S.$$

Example (continuation).

Let us assume that $y \succ_T x$. Then $y_i > x_i$ for each $i \in T$ and $\sum_{i \in T} y_i \leq u(T)$. If $u(T) = 0$, then we have that

$$0 = u(T) \geq \sum_{i \in T} y_i > \sum_{i \in T} x_i \geq 0,$$

which is impossible. Hence T is a winning coalition, i.e. $u(T) = 1$. Then we must have $S \subset T$, so

$$1 = \sum_{i \in N} y_i \geq \sum_{i \in T} y_i > \sum_{i \in T} x_i \geq \sum_{i \in S} x_i = 1,$$

which is impossible. Hence, $x \in C(v)$.

Definition of "solution" in the sense of fon Neuman and Morgenstern

A set V is said to be stable subset of the set $E(u)$ of all imputations iff

1. internal stability: if $x \in V \ni y$, then $y \not\succ x$;
2. external stability: if $x \notin V$, then there exists $y \in V$ so that $y \succ x$.

Example.

Let us consider the game determined by the following characteristic function:

$$u(S) = \begin{cases} -2, & \text{if } S \text{ has one member;} \\ 2, & \text{if } S \text{ has two members;} \\ 0, & \text{if } S \text{ has three members.} \end{cases}$$

Example (continuation).

One can check that this game is S-equivalent to a (0,1) normalization (with $r = 1/6$, $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$):

$$v(S) = \begin{cases} 0, & \text{if } S \text{ has one member;} \\ 1, & \text{if } S \text{ has two members;} \\ 1, & \text{if } S \text{ has three members.} \end{cases}$$

We set

$$V := \left\{ \left(\frac{1}{2}, \frac{1}{2}, 0\right), \left(\frac{1}{2}, 0, \frac{1}{2}\right), \left(0, \frac{1}{2}, \frac{1}{2},\right) \right\}$$

and show that this set is a "solution" in the sense of of fon Neuman and Morgenstern: One can check the no one of these imputations dominates any one of the others, i.e. 1. holds true.

Example (continuation).

Let $x = (x_1, x_2, x_3)$ be any imputation of the game, i.e. $x_1 + x_2 + x_3 = 1$ with $x_1 \geq 0$, $x_2 \geq 0$ and $x_3 \geq 0$. Then at most two components of x can be large as $1/2$. If this actually happens, then both of those components must be equal to $1/2$, while the remaining component is zero. Hence in this case x belongs to V. Thus if $x \notin V$, the at most one of the components can be as large as $1/2$. Clearly, then exists an element of V that dominates x with respect to the coalition determined by the indices of the remainder two components of x.

However, V is not the only stable set: if c is an arbitrary number from the interval $[0, 1/2)$, then the set

Example (continuation).

$$V_{3,c} := \{x_1, 1 - c - x_1, c) : \ 0 \leq x_1 \leq 1 - c\}$$

is also a "solution" in the sense of of fon Neuman and Morgenstern: clearly, the set $V_{3,c}$ is internally stable. Let $y = (y_1, y_2, y_3)$ be any imputation of the game that does not belong to $V_{3,c}$, i.e. $y_1 + y_2 + y_3 = 1$ with $y_1 \geq 0$, $y_2 \geq 0$ and $c \neq y_3 \geq 0$. If $y_3 = c + \varepsilon$ for some $\varepsilon > 0$, then the imputation $x = (x_1, x_2, x_3)$ with $x_1 = y_1 + \varepsilon/2$, $x_2 = y_2 + \varepsilon/2$ and $x_3 = c$ dominates y with respect to the set $\{1, 2\}$. If $y_3 < c$, the at last one of the remainder components is less than or equal to $1/2$ (or else their sum will be greater than 1). Say that $y_1 \leq 1/2$. Then the imputation $x = (x_1, x_2, x_3)$ with $x_1 = 1 - c$, $x_2 = 0/2$ and $x_3 = c$ dominates y with respect to the set $\{1, 3\}$.

Remark.

The weakness of the concept for a "solution" in the sense of of fon Neuman and Morgenstern is that there is no theorems for existence of this solution in the general case. There are obtained only partial results.

Theorem.

Let u be a simple game in (0,1) normalization, and let S be the minimal winning coalition, i.e. a coalition such that $u(S) = 1$, but $u(T) = 0$ for each subset T of S. Let V_S be the set of all imputations for which $x_i = 0$ for each $i \notin S$. Then V_S is a "solution" in the sense of of fon Neuman and Morgenstern.

Proof.

Let x and y be two arbitrary elements of V_S such that $y \underset{T}{\succ} x$. Clearly, $T \subseteq S$. If $T \subset S$, then

$$0 = u(T) \geq \sum_{i \in T} y_i > \sum_{i \in T} x_i \geq 0,$$

which is impossible. If $T = S$, then

$$1 = u(T) \geq \sum_{i \in T} y_i > \sum_{i \in T} x_i = 1,$$

which is also impossible. Hence, V_S is internally stable.
Let y be an arbitrary imputation that does not belong to V_S. Then $\sum\limits_{i \notin S} y_i = \varepsilon > 0$. Let s be the number of elements of S. We set $x_i = y_i + \varepsilon/s$ for each $i \in S$, and $x_i = 0$ for each $i \notin S$. Then $x \underset{S}{\succ} y$. Hence, V_S is externally stable. This completes the proof.

The Shapley value (1953)

Definition.

It is said that the coalition T of the game u is a carrier iff $u(S) = u(S \cap T)$ for any coalition S.

The meaning of this definition is that if a player does not belong to a carrier, then this player is dummy, i.e. he can to contribute nothing to any coalition.

Definition.

Let u be an n-person game and π be a permutation of the elements of the set N. Then by πu we mean the game v such that $v(S) := u(\{\pi^{-1}(i_1), \pi^{-1}(i_2), \dots, \pi^{-1}(i_s)\})$ for any coalition $S = \{i_1, i_2, \dots, i_s\}$.

Remark.

Clearly, $\pi u(S) = u(\pi^{-1}(S))$. Then we have that

1. $\pi u(\emptyset) = u(\pi^{-1}(\emptyset)) = u(\emptyset) = 0$.
2. If $S \cap T = \emptyset$, then $\pi u(S \cup T) = u(\pi^{-1}(S \cup T)) =$
$u(\pi^{-1}(S) \cup \pi^{-1}(T)) \geq u(\pi^{-1}(S)) + u(\pi^{-1}(T)) = \pi u(S) + \pi u(T)$.

Hence, πu is a characteristic function.

The Shapley value

In fact, the game πu is the game v with the role of the players interchanged by the permutation π.

Axioms (Shapley)

By the value of a game u we mean a vector $\varphi[u]$ satisfying the following axioms:

$S1$. If T is any accrier, then $\sum_{i \in S} \varphi[u]_i = u(T)$;

$S2$. $\varphi[\pi u]_{\pi(i)} = \varphi[u]_i$ For any permutation π and for any $i \in N$;

$S3$. If u and v are any games, then $\varphi[u+v] = \varphi[u] + \varphi[v]$.

It is remarkable, that these three axioms determine unique vector $\varphi[u]$ for each n-person game u.

The Shapley value

Theorem.

There exists a unique function φ defined on all games and satisfying Axioms $1 \div 3$.

The proof is based on the following lemmas:

Lemma 1.

For any coalition S the game w_S is defined as

$$w_S(T) = \begin{cases} 0, \text{if } S \not\subset T; \\ 1, \text{if } S \subset T; \end{cases}$$

Then, if $s := |S|$, then

$$\varphi_i[w_S] = \begin{cases} \dfrac{1}{s}, & \text{if } i \in S; \\ \\ 0, & \text{if } i \notin S. \end{cases}$$

Remark 1.

Clearly, we have that

1. $w_S(\emptyset) = 0$ because $S \not\subset \emptyset$.

2. Let $T_1 \cap T_2 = \emptyset$. If $T_1 \cup T_2 \supseteq S$, then $w_S(T_1 \cup T_2) = 1$ and there are possible the following cases:

a) $T_1 \supseteq S$ and $T_2 \not\supseteq S$. Then $w_S(T_1) = 1$ and $w_S(T_2) = 0$;
b) $T_1 \not\supseteq S$ and $T_2 \supseteq S$. Then $w_S(T_1) = 0$ and $w_S(T_2) = 1$;
c) $T_1 \not\supseteq S$ and $T_2 \not\supseteq S$. Then $w_S(T_1) = 0$ and $w_S(T_2) = 0$;

Hence $w_S(T_1 \cup T_2) \geq w_S(T_1) + w_S(T_2)$ in each of these three cases.

Hence, w_S is a characteristic function.

The Shapley value

Remark 2.

We show that if $\bar{S} \supseteq S$, then $\bar{S}$ is a carrier of the game w_S. Indeed, if a coalition $T \supseteq S$, then we have that $(\bar{S} \cap T) \supseteq S$, and hence $w_S(\bar{S} \cap T) = 1 = w_S(T)$.
If a coalition $T \not\supseteq S$, then we have that $(\bar{S} \cap T) \not\supseteq S$, and hence $w_S(\bar{S} \cap T) = 0 = w_S(T)$.
Hence, $\bar{S}$ is a carrier for the game w_S.

Remark 3.

Let π is any any permutation which carries S into itself, i.e. $\pi(S) := \{\pi(s) : \ s \in S\} = S$. If $T \supseteq S$, then $\pi^{-1}(T) \supseteq \pi^{-1}(S) = S$, and hence $\pi w_S(T) = w_S(\pi^{-1}(T)) = 1 = w_S(T)$. If $T \not\supseteq S$, then $\pi^{-1}(T) \not\supseteq \pi^{-1}(S) = S$, and hence $\pi w_S(T) = w_S(\pi^{-1}(T)) = 0 = w_S(T)$. So, $\pi w_S = w_S$.

The Shapley value

Proof of Lemma 1.

It is clear the S is a carrier for w_S. Also, any superset T of S is also carrier for w_S. By Axiom 2, then it follows that $\sum_{i \in T} \varphi_i[w_S] = 1$ for any coalition $T \supseteq S$. And this implies that $\varphi_i[w_S] = 0$ for any $i \notin S$.

If π is any any permutation which carries S into itself, it is clear that $\pi w_S = w_S$. Then Axiom 2 implies that $\varphi_{\pi(i)}[w_S] = \varphi_{\pi(i)}[\pi w_S] = \varphi_i[w_S]$. In particular, if $\pi(i) = j$, then we have that $\varphi_j[w_S] = \varphi_{\pi(i)}[w_S] = \varphi_{\pi(i)}[\pi w_S] = \varphi_i[w_S]$. This implies that $\varphi_j[w_S] = \varphi_i[w_S]$ for each two indices i and j from S. As there are s of this indices and their sum is equal to 1, it follows that $\varphi_i[w_S] = \dfrac{1}{s}$ and this completes the proof of Lemma 1.

The Shapley value

Corollary.

If $c > 0$, we set $(cw_S)(T) := c.w_S(T)$ for each coalition T. Clearly, (cw_S) is a characteristic function. Then $\varphi_i[cw_S] = c\varphi_i[w_S]$.

Proof.

As in the proof of Lemma 1, we can obtain that $\varphi_i[(cw_S)] = 0$ for any $i \notin S$, $\sum_{i \in S} \varphi_i[(cw_S)] = c$ and $\varphi_j[(cw_S)] = \varphi_i[(cw_S)]$ for each two indices i and j from S. From here, it follows that $\varphi_i[(cw_S)] = \dfrac{c}{s}$ and this completes the proof of the Corollary.

The Shapley value

Lemma 2.

If u is an arbitrary game, then

$$u(U) = \sum_{S \subseteq N} c_S \; w_S(U)$$

for each coalition U, where

$$c_S = \sum_{T \subseteq S} (-1)^{s-t} u(T),$$

where $s := |S|$ and $t := |T|$, i.e. by the number of the elements of the sets S and T are denoted by s and t.

Proof of Lemma 2.

Let U be an arbitrary coalition with $u = |U|$. Then the following equalities hold true:

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Proof of Lemma 2 (continuation).

$$\sum_{S \subseteq N} c_S \; w_S(U) = \sum_{S \subseteq U} c_S \; =$$

$$= \sum_{S \subseteq U} \left(\sum_{T \subseteq S} (-1)^{s-t} u(T) \right) = \sum_{T \subseteq U} \left(\sum_{T \subseteq S \subseteq U} (-1)^{s-t} \right) u(T) =$$

$$= u(U) + \sum_{T \subset U} \left(\sum_{T \subseteq S \subseteq U} (-1)^{s-t} \right) u(T) =$$

$$= \sum_{T \subseteq U} \left(\sum_{s=t}^{u} \binom{u-t}{s-t} (-1)^{s-t} \right) u(T)$$

because for every value of s between t and u, there will be

Proof of Lemma 2 (continuation).

$\left(\begin{array}{c} u-t \\ s-t \end{array} \right)$ sets S with s elements such that $T \subset S \subseteq U$. Since

$$\sum_{s=t}^{u} \left(\begin{array}{c} u-t \\ s-t \end{array} \right) (-1)^{s-t} = \sum_{j=0}^{u-t} \left(\begin{array}{c} u-t \\ j \end{array} \right) (-1)^{j} = (1-1)^{u-t} = 0.$$

(remind that $\sum_{j=0}^{k} \left(\begin{array}{c} k \\ j \end{array} \right) a^{k-j}(-b)^{j} = (a-b)^{k}$). So, we have obtained that

$$\sum_{S \subseteq N} c_S \; w_S(U) = \sum_{S \subseteq U} c_S \; = u(U)$$

which completes the proof.

The Shapley value

Remark.

Let u and v be characteristic functions such that $u - v$ is also a characteristic function. Since $u = v + (u - v)$, Axiom 3 implies that $\varphi[u] = \varphi[v] + \varphi[u - v]$, i.e. $\varphi[u - v] = \varphi[u] - \varphi[v]$. Since

$$u = \sum_{S \subseteq N} c_S \; w_S,$$

we obtain that for each index i

$$\varphi_i[u] = \sum_{S \subseteq N} c_S \; \varphi_i[w_S] = \sum_{i \in S \subseteq N} c_S \frac{1}{s}.$$

Taking into account the definition of c_S, we obtain that

The Shapley value

$$\varphi_i[u] = \sum_{i \in S \subseteq N} \sum_{T \subseteq S} \frac{1}{s}(-1)^{s-t} u(T) =$$

$$= \varphi_i[u] = \sum_{T \subseteq N} \left(\sum_{\{i\} \cup T \subseteq S \subseteq N} \frac{1}{s}(-1)^{s-t} \right) u(T).$$

We set

$$\gamma_i(T) := \sum_{\{i\} \cup T \subseteq S \subseteq N} \frac{1}{s}(-1)^{s-t}.$$

It is easy to see that if $i \notin T$ and $T' := T \cup \{i\}$, then $\gamma_i(T) = -\gamma_i(T')$ (because the terms in the sum of $\gamma_i(T)$ will be the same, except that $t' = t + 1$ and hence, there is a change of the sign throughout. Therefore,

The Shapley value

$$\varphi_i[u] = \sum_{i \in T \subseteq N} \gamma_i(T)(u(T) - u(T \setminus \{i\}).$$

Now, if $i \in T$, then there exist $\begin{pmatrix} n-t \\ s-t \end{pmatrix}$ coalitions S with s elements such that $S \supseteq T$. Hence,

$$\gamma_i(T) := \sum_{s=t}^{n} \begin{pmatrix} n-t \\ s-t \end{pmatrix} \frac{1}{s}(-1)^{s-t} =$$

$$= \sum_{s=t}^{n} \begin{pmatrix} n-t \\ s-t \end{pmatrix} (-1)^{s-t} \int_0^1 x^{s-1} ds$$

$$= \int_0^1 x^{t-1} \sum_{s=t}^{n} \begin{pmatrix} n-t \\ s-t \end{pmatrix} (-1)^{s-t} x^{s-t} ds$$

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$$= \int_0^1 x^{t-1}(1-x)^{n-t}dx =: I(t) = \frac{(t-1)!(n-t)!}{n!}. \qquad (1)$$

To check this equality, we shall use induction on t. For $t = 1$ we have that $I(1) =$

$$= \int_0^1 (1-x)^{n-1}dx = -\int_0^1 (1-x)^{n-1}d(1-x) = -\left.\frac{(1-x)^n}{n}\right|_0^1 = \frac{1}{n},$$

and (1) holds true. Let us assume that (1) is fulfilled for some $n > t \geq 1$. The we shall prove (1) for $t+1$. Indeed, we have that

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$$I(t+1)=\int_0^1 x^t(1-x)^{n-t-1}dx=-\int_0^1 x^t(1-x)^{n-t-1}d(1-x)=$$

$$=-\int_0^1 x^t d\frac{(1-x)^{n-t}}{n-t}=-\,x^t\frac{(1-x)^{n-t}}{n-t}\Big|_0^1$$

$$+\frac{t}{n-t}\int_0^1 x^{t-1}(1-x)^{n-t}dx=\frac{t}{n-t}\frac{(t-1)!(n-t)!}{n!}=$$

$$=\frac{t!(n-t-1)!}{n!},$$

and, hence, (1) is true for $t+1$ too. This shows that (1) is fulfilled for each positive integer n and for each positive integer $t \leq n$.

The Shapley value

Thus we have obtain that

$$\varphi[u]_i = \sum_{i \in T \subseteq N} \frac{(t-1)!(n-t)!}{n!}(u(T) - u(T \setminus \{i\})).$$

This formula gives the Shapley value explicitly. One can check that Axioms 1 $\div$ 3 hold true. Indeed,

The Shapley value

$$\varphi[u_1 + u_2]_i = \sum_{i \in T \subseteq N} \frac{(t-1)!(n-t)!}{n!}$$

$$(u_1(T) + u_2(T) - u_1(T \setminus \{i\}) - u_2(T \setminus \{i\})) =$$

$$= \sum_{i \in T \subseteq N} \frac{(t-1)!(n-t)!}{n!}(u_1(T) - u_1(T \setminus \{i\})) +$$

$$+ \sum_{i \in T \subseteq N} \frac{(t-1)!(n-t)!}{n!}(u_2(T) - u_2(T \setminus \{i\})) = \varphi_i[u_1] + \varphi_i[u_2],$$

i.e. Axiom 1 holds true.

The Shapley value

Also, denoting $T' := \pi^{-1}T$, we have that

$$\varphi[\pi\, u]_{\pi(i)} = \sum_{\pi(i)\in T\subseteq N} \frac{(t-1)!(n-t)!}{n!}(\pi\, u(T) - \pi\, u(T \setminus \{\pi(i)\})) =$$

$$= \sum_{\pi(i)\in T\subseteq N} \frac{(t-1)!(n-t)!}{n!}(u(\pi^{-1}(T)) - \ u(\pi^{-1}(T \setminus \ \{\pi(i)\})) =$$

$$= \sum_{i\in T'\subseteq N} \frac{(t-1)!(n-t)!}{n!}(u(T') - \ u(T') \setminus \{i\})) = \varphi[u]_i,$$

i.e. Axiom 2 holds true too.

The Shapley value

Let T be a carrier of the game u. Then for each imputation U

$$u(U) = u(U \cap T) = \sum_{S \subseteq N} c_S \; w_S(U \cap T) =$$

$$= \sum_{S \subseteq T} c_S \; w_S(U \cap T) = \sum_{S \subseteq T} c_S \; w_S(U).$$

Thus $u = \sum_{S \subseteq T} c_S \; w_S$ and, hence, using that T is a carrier for any w_S with $S \subseteq T$, we obtain that

$$\sum_{i \in T} \varphi_i[u] = \sum_{i \in T} \sum_{S \subseteq T} c_S \; \varphi_i[w_S] =$$

$$= \sum_{S \subseteq T} c_S \; \sum_{i \in T} \varphi_i[w_S] = \sum_{S \subseteq T} c_S \; w_S(T) = u(T),$$

and, so, Axiom 3 also holds true.

The Shapley value

Since $\varphi_i[u] = \sum_{i \in T \subseteq N} \frac{(t-1)!(n-t)!}{n!}(u(T) - u(T \setminus \{i\}) \geq$

$$\geq \sum_{i \in T \subseteq N} \frac{(t-1)!(n-t)!}{n!} u(\{i\}) =$$

$$= \sum_{t=1}^{n} \left(\begin{array}{c} n-1 \\ t-1 \end{array} \right) \frac{(t-1)!(n-t)!}{n!} u(\{i\}) =$$

$$= \sum_{t=1}^{n} \frac{(n-1)!}{(t-1)!(n-t)!} \frac{(t-1)!(n-t)!}{n!} u(\{i\}) =$$

$$= \sum_{t=1}^{n} \frac{1}{n} u(\{i\}) = u(\{i\}),$$

and because N is a carrier for each game u, we have that

$$\sum_{i=1}^{n} \varphi_i[u] = u(N),$$

and hence the Shapley vector is an imputation for each game u.

Example.

Consider a corporation with 4 stockholders, having respectively 10, 20, 30 and 40 shares of stock. It is assumed that each decision is taken by approval of stockholders holding a simple majority of the shares. This can be treated by as a simple 4-person game in which the winning coalitions are the following: $\{2,4\}$, $\{3,4\}$, $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, $\{2,3,4\}$, and $\{1,2,3,4\}$.

Example (continuation).

To find φ_1, we note that the only winning coalition T for which $T \setminus \{1\}$ is not winning, is $\{1,2,3\}$. Hence

$$\varphi_1 = \frac{2!1!}{4!} = \frac{1}{12}.$$

Analogously,

$$\varphi_2 = \frac{1!2!}{4!}(u(\{2,4\}) - u(\{4\})) + \frac{2!1!}{4!}(u(\{1,2,3\}) - u(\{1,3\}))$$

$$+\frac{2!1!}{4!}(u(\{1,2,4\}) - u(\{1,4\})) = \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{1}{4}.$$

The Shapley value

Example (continuation).

$$\varphi_3 = \frac{1!2!}{4!}(u(\{3,4\}) - u(\{4\})) + \frac{2!1!}{4!}(u(\{1,2,3\}) - u(\{1,2\}))$$

$$+\frac{2!1!}{4!}(u(\{1,3,4\}) - u(\{1,4\})) = \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{1}{4}.$$

$$\varphi_4 = \frac{1!2!}{4!}(u(\{2,4\}) - u(\{2\})) + \frac{2!1!}{4!}(u(\{1,2,4\}) - u(\{1,2\}))$$

$$+\frac{2!1!}{4!}(u(\{3,4\}) - u(\{3\})) + \frac{2!1!}{4!}(u(\{1,3,4\}) - u(\{1,3\}))$$

$$+\frac{2!1!}{4!}(u(\{2,3,4\}) - u(\{2,3\})) == \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{5}{12}.$$

The Shapley value

Example (continuation).

Hence, the Shapley value is the vector $\left(\frac{1}{12}, \frac{1}{4}, \frac{1}{4}, \frac{5}{12}\right)$. This contrast to the "vote vector" $\left(\frac{1}{10}, \frac{1}{5}, \frac{1}{3}10, \frac{2}{5}\right)$. Note that player 3 have more shares then player 2, but they have the same value. This is not surprising because player 3 has not greater opportunity than player 2 to form winning coalitions.