— Lectures on —

DIFFERENTIAL GEOMETRY AND TOPOLOGY

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Lecture VIII

✶ Exterior differential

$\Lambda^k = \Lambda^k(\mathbb{R}^n)$ (or $\Lambda^k(\mathcal{U})$ $\mathcal{U} \subset \mathbb{R}^n$ open)

k-forms on $\mathcal{U}$

Let us define $d : \Lambda^k \longrightarrow \Lambda^{k+1}$

(exterior differential) via the position — if $\omega = a_I dx^I$

$$\underset{\Lambda^0}{\overset{}{d\,(\underset{\cap}{a_I}\, dx^I)}} := da_I \wedge dx^I$$

in particular $d\,(dx^I) = d\,(1 \cdot dx^I) = \overset{0}{\overset{\|}{d1}} \wedge dx^I = 0$

example: $\omega = dx \wedge dz + \sin z \, dx \wedge dy$

$$\begin{aligned} d\omega &= d\,(dx \wedge dz) + d(\sin z) \wedge dx \wedge dy \\ &= 0 + \cos z \; dz \wedge dx \wedge dy = \cos z \cdot dx \wedge dy \wedge dz \end{aligned}$$

Properties of d

(a) $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$

(valid if $\omega_1, \omega_2 \in \Lambda^k$, postulated in general)

(b) $d(\omega \wedge \varphi) = d\omega \wedge \varphi + (-1)^k \omega \wedge d\varphi$

$\omega \in \Lambda^k$, $\varphi \in \Lambda$ (Grassmann)

$k = \deg \omega$ — notice this

(antiderivation property)

generalizes Leibniz rule

$d(fg) = df \cdot g + f\, dg$

Let us prove it: $\omega = a_I\, dx^I$ k- form

it is enough to set $\varphi = b_J\, dx^J$ l- form

and use linearity.

Then

$\downarrow$ sum over I, J

$$d(\omega \wedge \varphi) = d(a_I b_J\, dx^I \wedge dx^J) =$$

$$= d(a_I b_J) \wedge dx^I \wedge dx^J \underset{\text{Leibniz}}{=} (\underbrace{da_I\, b_J}_{= b_J\, da_I} + a_I\, db_J)\, dx^I \wedge dx^J$$

can be brought here

$$= b_J \underbrace{da_I \wedge dx^I}_{d\omega} \wedge dx^J + a_I\, db_J \wedge dx^I \wedge dx^J$$

can be brought here by means of k switches yielding a $(-1)^k$ factor

$$= d\omega \wedge \varphi + (-1)^k \omega \wedge \underbrace{db_J \wedge dx^J}_{d\varphi}$$

$$= d\omega \wedge \varphi + (-1)^k \omega \wedge d\varphi$$

Also notice, as a consistency check

$$db_J \wedge dx^I = (-1)^{1 \cdot k} dx^I \wedge db_J = (-1)^k dx^I \wedge db_J$$

and $\underset{\Lambda^0}{f} \wedge \underset{\Lambda^k}{\omega} = (-1)^{0 \cdot k} \omega \wedge f = \omega \wedge f$

and actually $f \wedge \omega = \omega \wedge f = f\omega$

(c) ⚠ Most important

$$d(\underset{\Lambda}{d\omega}) = 0$$

namely $\boxed{d^2 = 0}$

Proof. If $f \in \Lambda^0$, then

$$d(df) = d\left(\frac{\partial f}{\partial x^i} dx^i\right) = d\left(\frac{\partial f}{\partial x^i}\right) \wedge dx^i =$$

$$= \frac{\partial^2 f}{\partial x^j \partial x^i}\, dx^j \wedge dx^i$$. But, in view of the Schwarz lemma

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}$$, so, by virtue of skew-symmetry,

$$d(df) = d^2 f = 0$$

Now take $\omega = \sum_I a_I \, dx^I \in \Lambda^k$

we use summation symbol

In order to check that $d^2\omega = 0$, it is enough to verify it for a monomial $a_I \, dx^I$

No summation

Then

$$d\Big(\underbrace{d(a_I \, dx^I)}_{\text{monomial}}\Big) = d\left(da_I \wedge dx^I\right) = \underbrace{d^2 a_I}_{=0} \wedge dx^I$$

$$- \; da_I \wedge \underbrace{d(dx^I)}_{=0} = 0 \qquad \square$$

notice this

(d) functoriality: $d(f^*\omega) = f^* d\omega$

⚠ pull-back commutes with d

Pf. First check that $f^* dg = d(f^* g)$

$$f^* dg = f^*\left(\frac{\partial g}{\partial y^i} dy^i\right) = \frac{\partial g}{\partial y^i}\frac{\partial y^i}{\partial x^j} dx^j$$

Einstein

$f: \; y = y(x)$

$$\underset{\text{(chain rule)}}{=} \frac{\partial(g \circ f)}{\partial x^j} dx^j = d(\underbrace{g \circ f}_{f^* g}) = d(f^* g)$$

$f: \; y = y(x)$

Now let $\quad \varphi = a_I \, dy^I$

Einstein again

Then

$$d(f^*\varphi) = d\left(f^*(a_I)\, f^*(dy^I) \right)$$

$$= df^*(a_I) \wedge f^*(dy^I) + \underbrace{f^*(a_I) \wedge d(f^*dy^I)}_{\text{but this is zero:}}$$

$= $ (by the previous property)

$$f^*(da_I) \wedge f^*(dx^I)$$

$$d f^*(dy^{i_1} \wedge \dots dy^{i_k}) = d(f^*dy^{i_1} \wedge \dots f^*dy^{i_k}) = d(df^*y^{i_1} \wedge \dots df^*y^{i_k}) \dots = 0$$

since $d^2 = 0$ $\quad dy^{i_1}(x)$

$$= f^*(da_I \wedge dx^I) = f^*(d\varphi)$$

Important examples. In $\mathbb{R}^3$ $\quad d^2 = 0$ $\quad$ encapsulates the properties

caveat!

$$\text{curl grad } \varphi = 0$$
$$\text{div curl } \underline{F} = 0$$

$$\varphi \mapsto d\varphi = \frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy + \frac{\partial \varphi}{\partial z} dz$$

z-component of curl $\underline{A}$

$$\underline{A} \leftrightarrow A_1 dx + A_2 dy + A_3 dz \xmapsto{d} \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) dx \wedge dy + \text{similar terms}$$

$$\underline{F} \leftrightarrow F_1 \, dy \wedge dz + F_2 \, dz \wedge dx + F_3 \, dx \wedge dy$$

$$\xmapsto{d} \frac{\partial F_1}{\partial x} dx \wedge dy \wedge dz + \dots \text{similar terms}$$

$$= \dots = \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx \wedge dy \wedge dz$$

$$\Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \xrightarrow{d} \Lambda^3 \xrightarrow{d} \dots$$

Z^k : closed k-forms: $d\omega = 0$

B^k : exact k-forms : $\omega = da$, $a \in \Lambda^{k-1}$

☆ de Rham complex $\quad B^k \le Z^k$ in view of $d^2 = 0$

$H^k = Z^k / B^k \quad$ k-de Rham cohomology group

The Poincaré lemma tells us that

$$H^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k=0 \\ \{0\} & k=1,\dots n \end{cases}$$ ← easy to prove!

that is, any closed k-form, for $k \geq 1$ is exact.

★ ★ ★ de Rham cohomology groups store topological information about a manifold M

(see second part of the course and M. S. notes "Topologia e geometria differenziale" (Topogeo))