— Lectures on —

DIFFERENTIAL GEOMETRY AND TOPOLOGY

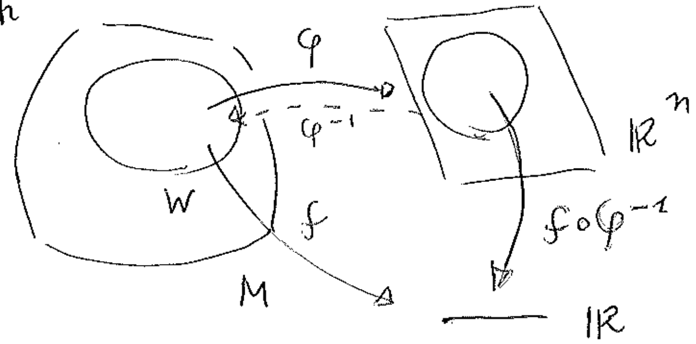
Marco Spera

Lecture XI

* Smooth functions on manifolds

Let M be a differentiable manifold (smoothness assumed throughout) [we omit the explicit specification of an atlas]

Def. A function $f : M \to \mathbb{R}$ is said to be smooth if $\forall$ chart $\varphi : U \to \mathbb{R}^n$, the function $f \circ \varphi^{-1} : \underset{\varphi(U)}{\overset{W}{\shortparallel}} \to \mathbb{R}$ is smooth



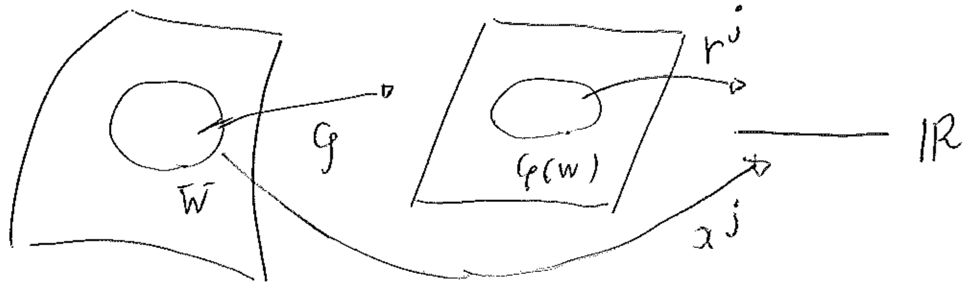
Notice that this concept is intrinsic, i.e. independent of the choice of a chart: in a non-empty intersection of the domains of two charts φ and ψ, one has
$$f \circ \varphi^{-1} = f \circ \psi^{-1} \circ \psi \circ \varphi^{-1} = (f \circ \psi^{-1}) \circ (\psi \circ \varphi^{-1})$$
which is smooth if and only if $f \circ \psi^{-1}$ is smooth (being $\psi \circ \varphi^{-1}$ smooth)

Let us now define local coordinates (or local coordinate functions). Let $r^i : \mathbb{R}^n \to \mathbb{R}$ the standard coordinate functions on $\mathbb{R}^n$: $r^i(a) = a^i$ $\quad (a = (a^i))$

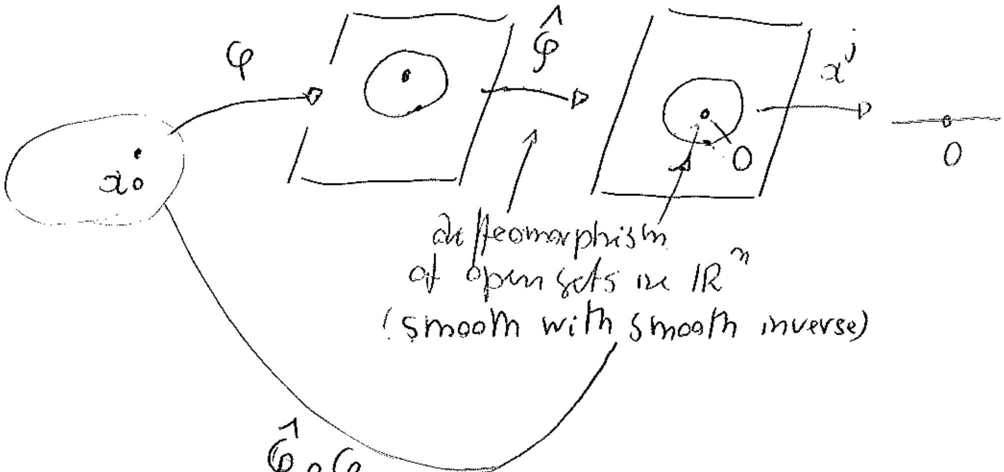
Then, given a chart $\varphi : W\ (\subset M) \to \mathbb{R}$,

set $x^j : W \to \mathbb{R}$

$$x^j := r_j \circ \varphi \qquad \leftarrow \text{local coordinate functions}$$



Given $x_0 \in M$, one can devise a local coordinate system centred at x_0 (i.e. $x^j(x_0) = 0 \quad \forall\, j = 1..n$)



$\hat{\varphi} \circ \varphi \equiv \psi$ is a new chart

This can be achieved in view of maximality of the atlas (one can add to a fixed atlas any chart compatible with all others).

★ Tangent vectors as derivations ... The $\mathbb{R}^n$ case

Def. A derivation on an algebra A is a map
$D: A \to A$ such that
1. D is linear (A is in particular a vector space)
2. $D(a \cdot b) = D(a) \cdot b + a\, D(b)$ (Leibniz rule)

Def. A derivation of the algebra $\mathcal{C}^\infty(\mathbb{R}^n)$ at $0 \in \mathbb{R}^n$ is a map $v: \mathcal{C}^\infty(\mathbb{R}^n) \to \mathbb{R}$ such that

1. v is linear
2. $v(f \cdot g) = v(f)\, g(0) + f(0)\, v(g)$ (Leibniz)

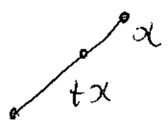
★ **Theorem.** $v = \displaystyle\sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}\Big|_0$ for a unique vector $(a^i) \in \mathbb{R}^n$

Namely, $T_0\mathbb{R}^n$ can be identified with the space of derivations of $C^\infty(\mathbb{R}^n)$ at 0 (true in general ...)

Proof. One needs the following

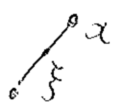
Lemma (Willmore). Let $f \in C^\infty(\mathbb{R}^n)$. Then
$$f(x) - f(0) = \sum_{j=1}^{n} x^j h_j(x), \quad h_j \in C^\infty(\mathbb{R}^n).$$

Indeed: $$f(x) - f(0) = \int_0^1 \frac{d}{dt} f(tx)\, dt \underset{\text{(chain rule)}}{=} \sum_{j=1}^{n} \int_0^1 x^j \frac{\partial f}{\partial x^j}(tx)\, dt$$



$$= \sum_{j=1}^{n} x^j \underbrace{\int_0^1 \frac{\partial f}{\partial x^j}(tx)\, dt}_{h_j(x)}$$

Notice that $h_j(x) = \dfrac{\partial f}{\partial x^j}(\xi) \cdot 1$ (Lagrange)



which proves the lemma

$$\Rightarrow h_j(0) = \frac{\partial f}{\partial x^j}(0)$$

Now, given a derivation at 0, say v, we find,

subsequently

Willmore

$$v(f) = v\Big(f(0) + \sum_j x^j h_j(x)\Big)$$

(linearity)

$$= \underbrace{v(f(0))}_{=\,0} + \sum_j v(x^j h_j(x))$$

viewed as a constant function

$v(1) = v(1 \cdot 1) =$
$= 1 \cdot v(1) + v(1) \cdot 1$
$= 2\, v(1)$
$\Rightarrow v(1) = 0$
also: $v(c) = 0$

$$= \sum_j \underbrace{v(x^j)}_{\equiv\, a^j} \underbrace{h_j(0)}_{=\,\frac{\partial f}{\partial x^j}(0)} + \underbrace{\sum_j 0 \cdot v(h_j)}_{=\,0}$$

$$= \sum_{j=1}^{n} a^j \frac{\partial f}{\partial x^j}(0)$$

Conversely, any tangent vector at 0 fulfils the two properties of a derivation at 0.

This completes the proof of the theorem. $\square$

★ The tangent space $T_{x_0}M$ at $x_0 \in M$.

⚠ a bit abstract

Def. $T_{x_0}M$ (tangent space to M at x_0)

consists of maps

$$v : \mathcal{C}^\infty(M, x_0, \mathbb{R}) \longrightarrow \mathbb{R}$$

smooth functions on a neighbourhood $\mathcal{U}$ of x_0, domain of a chart φ ⋯→ we have already introduced the notion of smooth function on a manifold (or on an open subset thereof)

such that $$v(f) = \sum_{i=1}^{n} a^i \frac{\partial}{\partial r^i} (f \circ \varphi^{-1})\Big|_{\varphi(x_0)}$$

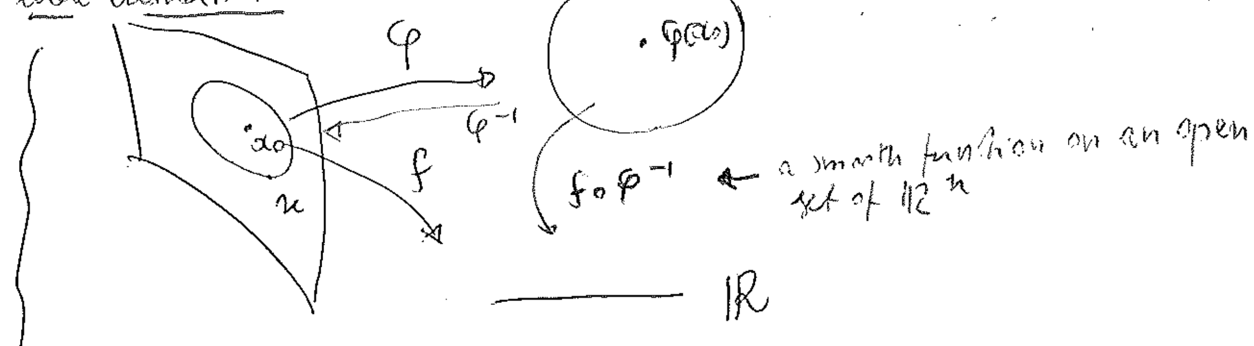
for some $(a^i) \in \mathbb{R}^n$. v is called a tangent vector at x_0

Namely: tangent vectors are derivations (at x_0) of the above algebra of functions.

One uses the suggestive notation $$v = \sum a^i \frac{\partial}{\partial x^i}\Big|_{x_0}$$

$$\frac{\partial f}{\partial x^j}(x_0) := \frac{\partial (f \circ \varphi^{-1})}{\partial r^j}(\varphi(x_0))$$

★ partial derivatives with respect to local coordinates.



They give rise to a basis of $T_{x_0}M$, $\left(\frac{\partial}{\partial x^i}\Big|_{x_0}\right)_{i=1\dots n}$

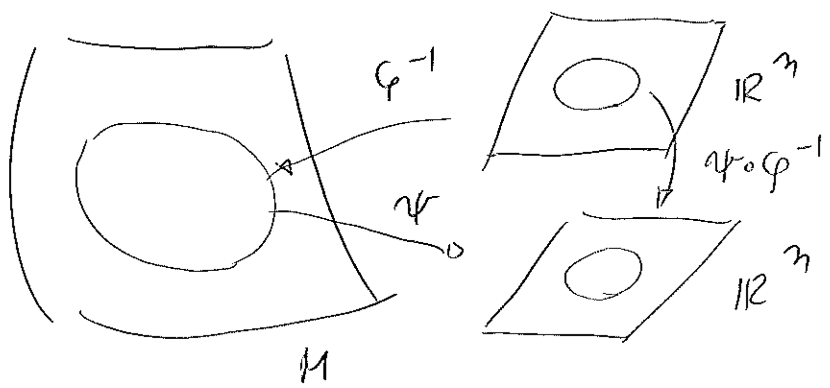
or $\left(\partial_j|_{x_0}\right)_{j=1\dots n}$

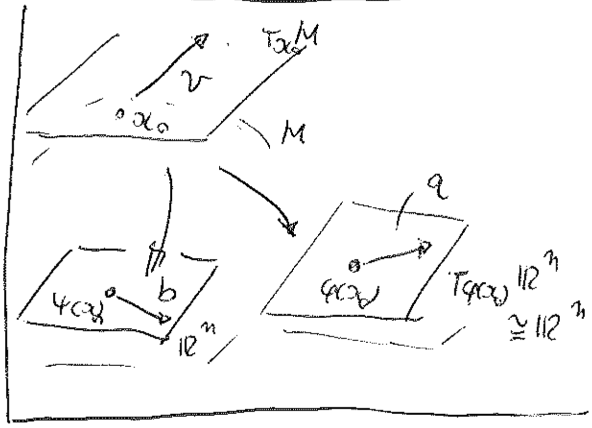
It is necessary, and instructive, to check chart-independence

$$v(f) = \sum_i a^i \frac{\partial}{\partial r^i}\left(f\circ\varphi^{-1}\right)\Big|_{\varphi(x_0)} =$$

$$= \sum_i a^i \frac{\partial}{\partial r^i}\Big(f\circ \psi^{-1}\circ \overset{\text{another chart}}{\overset{\downarrow}{\psi}}\circ\varphi^{-1}\Big)\Big|_{\varphi(x_0)}$$

$$= \sum_i a^i \sum_j \frac{\partial}{\partial r^j}\left(f\circ\psi^{-1}\right)\Big|_{\psi(x_0)} \cdot \overset{(J^T)}{\overset{\shortparallel}{J_{ji}}}\left(\psi\circ\varphi^{-1}\right)\Big|_{\varphi(x_0)}$$





$$= \sum_j \frac{\partial}{\partial r^j}\left(f\circ\psi^{-1}\right)\Big|_{\psi(x_0)} \underbrace{\left(\sum_i a^i J_{ji}\left(\psi\circ\varphi^{-1}\right)\right)}_{b^j}$$

$$= \sum_j b^j \frac{\partial}{\partial r^j}\left(f\circ\psi^{-1}\right)\Big|_{\psi(x_0)}$$

We shall use the stenographic notation already employed for $\mathbb{R}^n$, omitting explicit indication of charts $v(f) = \sum_i a^i \frac{\partial}{\partial x^i}$ (or simply $a^i \frac{\partial}{\partial x^i}$ (Einstein)). We still have for $y = y(x)$

$$\frac{\partial}{\partial x^j} = \frac{\partial y^i}{\partial x^j}\frac{\partial}{\partial y^i} \qquad \frac{\partial}{\partial x} = J^t \frac{\partial}{\partial y} \qquad \frac{\partial}{\partial y} = (J^{-t})\frac{\partial}{\partial x}$$

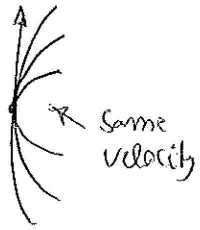
coordinate change

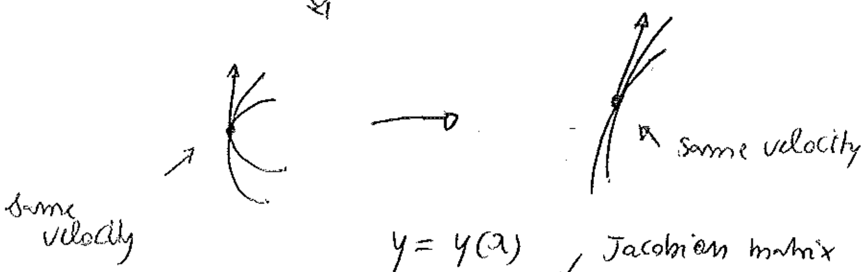
contravariance

★ that is, we still employ the $\mathbb{R}^n$-notation, with a generalized meaning

Another approach to the tangent space (Hint)

$\boxed{\text{In } \mathbb{R}^n}$, two curves passing through a point x said to be tangent at x if they have the same velocity vector. This defines an equivalence relation. This notion is clearly diffeomorphism-invariant





$y = y(x)$, Jacobian matrix

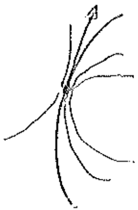
Schematically:

$$y(t) = y(x(t)) \qquad \dot{y}(t) = J \cdot \dot{x}(t)$$

So the tangent space at $p \in \mathbb{R}^n$ is the space of velocity vectors of curves passing through it (an equivalence class of curves)

$\boxed{\text{On a manifold } M}$, two curves are said to be tangent at a point x if they are such for a chart (whose domain contains x), i.e. if their images through the chart are tangent in $\mathbb{R}^n$. This notion is indeed chart-independent (by the observed diffeomorphism invariance).

A tangent vector at x is then an equivalence class of tangent curves at x (all curves containing whose images in a (and hence in all) chart have the same velocity at the image point



We have already appreciated the importance of this point of view.