Types of Fourier transforms

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Types of Fourier Transforms

- Continuous Time Fourier Transform (CTFT), or Fourier Transforms
- Continuous Time Fourier Series (CTFS), or Fourier Series
- Discrete Time Fourier Transform (DTFT)
- Discrete Time Fourier Series (DTFS)

Reference textbook: *Wavelets and Subband Coding*, M. Vetterli, J. Kovacevic, Prentice Hall Signal Processing Series.

Continuous Time FT

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt \quad \text{analysis formula} \tag{1}$$
$$f(t) = \int_{-\infty}^{+\infty} F(\omega)e^{j\omega t}d\omega \quad \text{synthesis formula} \tag{2}$$

Both $f(t)$ and $F(\omega)$ are *continuous* functions.

- Linearity

$$\alpha f(t) + \beta g(t) \leftrightarrow \alpha F(\omega) + \beta G(\omega) \tag{3}$$

- Symmetry

$$f(t) \leftrightarrow F(\omega) \Rightarrow F(t) \leftrightarrow 2\pi f(-\omega) \tag{4}$$

Continuous Time FT

- Shifting

$$f(t-t_0) \leftrightarrow e^{-j\omega t_0}F(\omega) \quad (5)$$

$$e^{j\omega t_0}f(t) \leftrightarrow F(\omega-\omega_0) \quad (6)$$

- Scaling

$$f(at) \leftrightarrow \frac{1}{|a|}F\left(\frac{\omega}{a}\right) \quad (7)$$

- Differentiation/Integration

$$\frac{\partial^n f(t)}{\partial t^n} \leftrightarrow (j\omega^n)F(\omega) \quad (8)$$

$$\int_{-\infty}^{t} f(\tau)d\tau \leftrightarrow \frac{F(j\omega)}{j\omega} \quad (9)$$

Continuous Time FT

- Convolution:

$$h(t) = f(t) \star g(t) = \int_{-\infty}^{+\infty} f(\tau) g(t-\tau) d\tau \tag{10}$$

Convolution theorem:

$$f(t) \star g(t) \leftrightarrow F(\omega) G(\omega) \tag{11}$$

Modulation theorem:

$$f(t) g(t) \leftrightarrow \frac{1}{2\pi} F(\omega) \star G(\omega) \tag{12}$$

- Parseval's formula (energy conservation):

$$\int_{-\infty}^{+\infty} f^{\star}(t) g(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F^{\star}(\omega) G(\omega) \tag{13}$$

$$f(t) = g(t) \leftrightarrow \int_{-\infty}^{+\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |F(\omega)|^2 d\omega \tag{14}$$

Continuous Time Fourier Series

A *periodic* function $f(t)$ with period T:

$$f(t) = f(t+T) \tag{15}$$

can be expressed as the linear combination of complex exponentials with frequency $n\omega_0$ where $\omega_0 = 2\pi/T$. Accordingly:

$$F[k] = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jk\omega_0 t}dt \qquad \text{analysis} \tag{16}$$

$$f(t) = \sum_{k=-\infty}^{+\infty} F[k]e^{jk\omega_0 t} \qquad \text{synthesis} \tag{17}$$

Parseval's relation:

$$\int_{-T/2}^{T/2} f^\star(t)g(t)dt = T\sum_{k=-\infty}^{+\infty} F^\star[k]G[k] \tag{18}$$

$$f(t) = g(t) \leftrightarrow \|f(t)\|^2 = \int_{-T/2}^{T/2} |f(t)|^2 dt = T\sum_{k=-\infty}^{+\infty} |F[k]|^2 = \|F[k]\|^2 \tag{19}$$

Discrete Time Fourier Transform

A *discrete* signal, i.e. a sequence $f[n]$, can be described by its DTFT:

$$F(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} f[n]e^{-j\omega n} \quad \text{analysis} \tag{20}$$

$$f[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{j\omega})e^{j\omega n}d\omega \quad \text{synthesis} \tag{21}$$

$F(\omega)$ is continuos and 2π periodic.

Convolution: given two sequences $f[n]$ and $g[n]$ and their DFTF $F(e^{j\omega})$ and $G(e^{j\omega})$ then:

$$f[n] \star g[n] = \sum_{k=-\infty}^{+\infty} f[n-k]g[k] = \sum_{k=-\infty}^{+\infty} f[k]g[n-k] \leftrightarrow F(e^{j\omega})G(e^{j\omega}) \tag{22}$$

Discrete Time Fourier Transform

Parseval's equality:

$$\sum_{n=-\infty}^{+\infty} f^{\star}[n]g[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} F^{\star}(e^{j\omega})G(e^{j\omega})d\omega \tag{23}$$

and in particular when $f[n] = g[n]$:

$$\sum_{n=-\infty}^{+\infty} |f[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |F(e^{j\omega})|^2 d\omega \tag{24}$$

CTFS	$f(t)$ continuous and periodic	$F(j\omega)$ discrete
DTFT	$f[n]$ discrete	$F(e^{j\omega})$ continuous and periodic

$\Rightarrow$ CTFS and DTFT are *duals* of each other

CTFS: continuous time fourier series; DTFS: discrete time Fourier transforms

There is a mathematical relation between the CTFS of a signal $f(t)$ and the DTFT of a sequence $\{f[n]\}$ (sampling section).

Discrete Time Fourier series

If a discrete time sequence is *periodic* with period N: $f[n] = f[n+Nl], \forall l \in \mathcal{Z}$, then the Discrete Time Fourier Series (DTFS) is given by:

$$F[k] = \sum_{n=0}^{N-1} f[n] e^{\frac{-j2nk\pi}{N}} \quad \text{analysis} \tag{25}$$

$$f[n] = \frac{1}{N} \sum_{k=0}^{N-1} F[k] e^{\frac{j2nk\pi}{N}} \quad \text{synthesis} \tag{26}$$

Both $f[n]$ and $F[k]$ are discrete and periodic.

Similar relations hold for the convolution and Parseval's formulas.

The Discrete Fourier Transform (DFT) is derived by relaxing the periodicity constraint and considering only one period. It can be thought either as the transform of one period of a periodic signal or as the sampling of a DTFT of a continuous signal.

Summary

Transform	Time	Freq.	Analysis/Synthesis	Duality
Fourier Transform (CTFT)	C	C	$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt$ $f(t) = 1/2\pi \int_{\omega} F(\omega) e^{j\omega t} d\omega$	self-dual
Fourier Series (CTFS)	C,P	D	$F[k] = 1/T \int_{-T/2}^{T/2} f(t) e^{-j2\pi kt} dt$ $f(t) = \sum_k F[k] e^{j2\pi kt}$	dual with DTFT
Discrete Time Fourier Transform (DTFT)	D	C,P	$F(e^{j\omega}) = \sum_n f[n] e^{-j2\pi\omega n/\omega_s}$ $f[n] = 1/\omega_s \int_{-\omega_s/2}^{\omega_s/2} F(e^{j\omega}) e^{j2\pi\omega n/\omega_s} d\omega$	dual with CTFS
Discrete Time Fourier Series (DTFS)	D,P	D,P	$F[k] = \sum_{n=0}^{N-1} f[n] e^{-j2\pi nk/N}$ $f[n] = 1/N \sum_{n=0}^{N-1} F[k] e^{j2\pi nk/N}$	self-dual