TOPOLOGIA E GEOMETRIA DIFFERENZIALE
a.a. 2009/10 Prof. M. Spera

Lezione V

Lettura

"Pull-back" di forme differenziali

"il trave nell'altro"

Sia data $f : \mathbb{R}^n \to \mathbb{R}^m$ liscia

data $w \in \Lambda^k((\mathbb{R}^m)^*)$, costruiamo

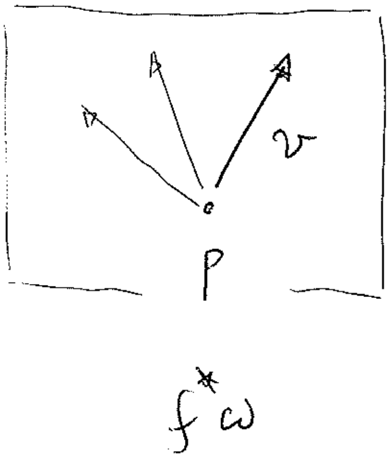
$f^*\omega \in \Lambda^k((\mathbb{R}^n)^*)$ in questo modo

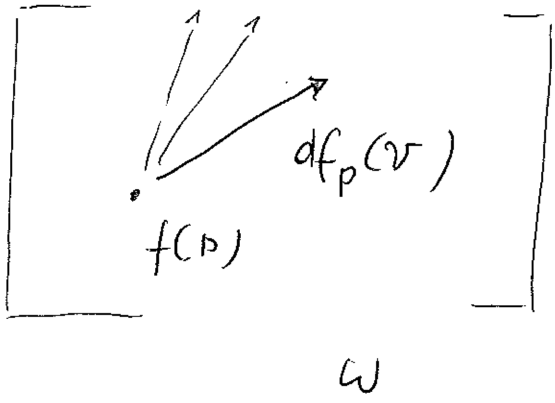
pull-back di ω

diff. di f ≡ spingere avanti push-forward of a vector

$$(f^*\omega)(p)\,(v_1 \ldots v_k) := \omega(f(p))\big(df_p(v_1) \ldots df_p(v_k)\big)$$

$v_i \in T_p\mathbb{R}^n$, $df_p(v_i) \in T_{f(p)}\mathbb{R}^m$





Se $g \in \Lambda^0((\mathbb{R}^m)^*)$ (cioè, è una funzione liscia $g : \mathbb{R}^m \to \mathbb{R}$)

Si pone $f^*g := f \circ g$

Interpretiamo la formula

$f: \mathbb{R}^n \to \mathbb{R}^m$

[si può lavorare in $\mathcal{U} \subset \mathbb{R}^n$ aperto]

$$\begin{cases} y_1 = f_1(x_1 \ldots x_n) \\ \vdots \\ y_m = f_m(x_1 \ldots x_n) \end{cases}$$

$$(f^* dy_i)(v) = dy_i(df(v)) = d(\underbrace{y_i \circ f}_{f_i})(v)$$

notare $\rightsquigarrow$ $d(f^* y_i)(v)$

$$= df_i(v)$$

funzione di x

$$\omega = \sum a_I(y)\, dy_I$$

dunque operativamente

$$\Rightarrow f^*\omega = \sum a_I(f(x))\, df_I$$

Ex: $$\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

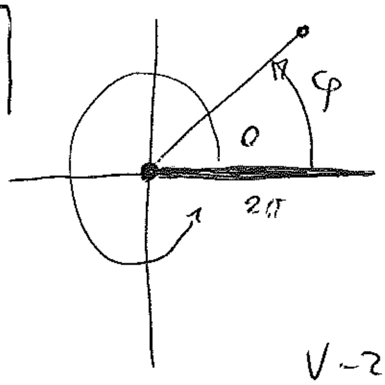
$U = \{r > 0,\ 0 < \varphi < \pi\}$

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \leftarrow f = f(r,\varphi)$$

Importantissima forma "angolo" o angolare angular form

$$f^*\omega = \ldots = d\varphi$$

$\boxed{\text{v. Note di geometria lezione V}}$



$\varphi = \arctan \frac{y}{x} \quad x \neq 0$

Concretamente , se
concretely if

$$\omega = \sum_I a_I(y)\, dy_I$$

$$f: \mathbb{R}^n \to \mathbb{R}^m$$
$$x \mapsto y = f(x)$$

$$f^*\omega = \sum_I \underbrace{a_I(y(x))}_{a'_I(x)}\, d\,y_I(x)$$

$$\text{"}\ dy = \underbrace{\frac{\partial y}{\partial x}}_{J} dx\ \text{"}$$

matrice jacobiana di $f \equiv df$

es:

$y = x^3$

$dy = d\,x^3 = 3x^2 dx$

$(x+iy)^2 = x^2 - y^2 + 2xy \cdot i$

$$\begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$$

$$\begin{cases} du = 2x\,dx - 2y\,dy \\ dv = 2y\,dx + 2x\,dy \end{cases}$$

$$\begin{pmatrix} du \\ dv \end{pmatrix} = \begin{array}{cc} u_x & u_y \\ \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix} \\ v_x & v_y \end{array} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

$$du \wedge dv = (2x\,dx - 2y\,dy) \wedge (2y\,dx + 2x\,dy)$$

$$= 4\,(x\,dx - y\,dy) \wedge (y\,dx + x\,dy)$$

$$= 4\,(-y^2\,dy \wedge dx + x^2\,dx \wedge dy)$$

$$= 4\,(y^2\,dx \wedge dy + x^2\,dx \wedge dy)$$

$$= \underbrace{4\,(x^2+y^2)}_{= \frac{\partial(u,v)}{\partial(x,y)}}\,dx \wedge dy$$

In generale

$$du = u_x\,dx + u_y\,dy$$
$$dv = v_x\,dx + v_y\,dy$$

$$du \wedge dv = (\underbrace{u_x v_y - v_x u_y}_{\frac{\partial(u,v)}{\partial(x,y)}})\,dx \wedge dy$$

$$dy_{i_1} \wedge \dots \wedge dy_{i_k} = \frac{\partial y_{i_1}}{\partial x_{l_1}} dx_{l_1} \wedge \dots \frac{\partial y_{i_k}}{\partial x_{l_k}} dy_{l_k}$$

$\sum_{l_1=1}^{m}$ omessa (Einstein)
omitted

Es:
$f: \mathbb{R}^3 \to \mathbb{R}^2$

$$= \frac{\partial(y_{i_1} \ldots y_{i_k})}{\partial(x_{l_1} \ldots x_{l_k})} dx_{l_1} \wedge \ldots \wedge dx_{l_k}$$

Jacobiani "parziali"

$$\underset{\text{2-forma}}{dy_1 \wedge dy_2} = \frac{\partial y_1}{\partial x_{l_1}} \frac{\partial y_2}{\partial x_{l_2}} dx_{l_1} \wedge dx_{l_2}$$

$$= \underbrace{\left(\frac{\partial y_1}{\partial x_1} \frac{\partial y_2}{\partial x_2} - \frac{\partial y_1}{\partial x_2} \frac{\partial y_2}{\partial x_1} \right)} dx_1 \wedge dx_2 + \text{termini analoghi}$$

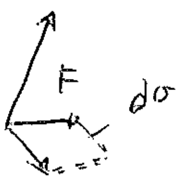
$$\frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} dx_1 \wedge dx_2 + \ldots$$

$$\begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix}$$

Es: $\varphi: \underset{(u,v)}{\mathbb{R}^2} \to \underset{(x,y,z)}{\mathbb{R}^3}$ $\quad \varphi \equiv \underline{r}$

2-forma di flusso $(\in \Lambda^2(\mathbb{R}^3))$
$$F = F_1\, dy \wedge dz + F_2\, dz \wedge dx + F_3\, dx \wedge dy$$

$$\underset{\Lambda^2(\mathbb{R}^2)}{\varphi^* F \atop \cap} = \Big[F_1 (\underbrace{x(u,v), y(u,v), z(u,v)}_{\varphi(u,v)}) \frac{\partial(y,z)}{\partial(u,v)} + F_2 \cdot \frac{\partial(z,x)}{\partial(u,v)} + F_3 \frac{\partial(x,y)}{\partial(u,v)} \Big] du \wedge dv$$



$\underline{E} \cdot \underline{n} \cdot d\sigma$ $\quad \overset{d\underline{\sigma}}{\sim}$ (vettore area)

$$= \langle \underline{E}, \underbrace{\underline{r}_u \times \underline{r}_v} \rangle du\, dv$$
$$= \det(\underline{E}, \underline{r}_u, \underline{r}_v)$$

Proprietà:

a) $f^*(w + \varphi) = f^*\omega + f^*\varphi$

$\uparrow$ k-forme

b) $f^*(g \cdot \omega) = \underbrace{f^*(g)}_{= \, g \circ f} f^*\omega$

$\uparrow$ Λ^0

c) $f^*(\varphi_1 \wedge .. \varphi_k) = f^*\varphi_1 \wedge \cdots f^*\varphi_k$

1-forme [varrà in generale]

Dim. a) e b) sono semplici

- c): $f^*(\varphi_1 \wedge .. \varphi_k)(v_1 \dots v_k) =$

(tutto calcolato in p ...)

$= (\varphi_1 \wedge .. \wedge \varphi_k)(df(v_1) .. df(v_k))$

$= \det(\varphi_i(df(v_j)))$

$= \det(\underbrace{f^*\varphi_i}(v_j))$

$= (f^*\varphi_1 \wedge .. f^*\varphi_k)(v_1 \dots v_k)$

Altre proprietà "funtoriali"
functorial

(a) $f^*(\omega \wedge \varphi) = f^*\omega \wedge f^*\varphi$ in generale

(b) $(f \circ g)^*\omega = g^*(f^*\omega)$

$\mathbb{R}^p \xrightarrow{g} \mathbb{R}^n \xrightarrow{f} \mathbb{R}^m$

Dim.

(a) $\omega = \sum a_I \, dy_I$
$\varphi = \sum b_J \, dy_J$

$f^*(\omega \wedge \varphi) = f^*\Big(\sum_{IJ} a_I b_J \, \underline{dy_I \wedge dy_J}\Big)$ ⚠ la proprietà è vera per prodotti di ℓ-forme

$= \sum a_I(f_1 \ldots f_m)\, b_J(f_1 \ldots f_m)\, df_I \wedge df_J$

$= \underbrace{f^*\omega}_{= \sum a_I(f)\, df_I} \wedge f^*\varphi$

(b) $(f \circ g)^*\omega = \sum a_I\big((f\circ g)_1 \ldots (f \circ g)_m\big)\, d(f\circ g)_I$ $\quad g^*f$

$= \sum a_I\big(f_1(g_1 \ldots g_n) \ldots f_m(g_1 \ldots g_n)\big) \times$
$\times \; df_I(dg_1 \ldots dg_n)$
$= g^*(f^*\omega)$

Differenziale esterno

exterior differential

Procediamo dapprima in modo formale $d: \Lambda^k \to \Lambda^{k+1}$

$$\omega = \sum a_I \, dx_I$$

$$\boxed{d\omega := \sum da_I \wedge dx_I}$$

in particolare $d(dx_I) = d(1 \cdot dx_I) = \underset{\overset{\parallel}{0}}{d1} \wedge dx_I = 0$

Es: $d\,[\,\underset{1 \cdot dx \wedge dz}{dx \wedge dz} + \sin z \, dx \wedge dy\,]$

$$= d1 \wedge dx \wedge dz\,] + d(\sin z) \wedge dx \wedge dy$$

$$= 0 + \cos z \, dz \wedge dx \wedge dy$$

$$= \cos z \, dx \wedge dy \wedge dz$$

Proprietà:

a) $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$ (vale per le k forme, si postula in generale)

b) $d(\omega \wedge \varphi) = d\overset{k}{\omega} \wedge \varphi + (-1)^k \omega \wedge d\varphi$

$\deg \omega = k$ ⚠ d è un'antiderivazione

c) $d(d\omega) = d^2\omega = 0$ ← importantissima

d) $d(f^*\omega) = f^* d\omega$ ← d commuta con il pull-back

(b) $\omega = \sum a_I \, dx_I$

$\varphi = \sum b_J \, dx_J$

$$d(\omega \wedge \varphi) = d\left[\sum_{I,J} a_I b_J \, dx_I \wedge dx_J\right]$$

$$= \sum_{I,J} d(a_I b_J) \, dx_I \wedge dx_J$$

$$= \sum_{I,J} (da_I \, b_J + a_I \, db_J) \, dx_I \wedge dx_J$$

$$= \sum_{I,J} b_J \, da_I \wedge dx_I \wedge dx_J + \quad \leftarrow \quad d\omega \wedge \varphi$$

$$+ \sum_{I,J} a_I \, db_J \wedge dx_I \wedge dx_J \quad \leftarrow \omega \wedge (-1)^k d\varphi$$

k scambi

$$= \sum_{I,J} a_I \, dx_I \wedge db_J \wedge dx_J$$

$$= \omega \wedge (-1)^k \, d\varphi$$

$$db_J \wedge dx_I = (-1)^{k \cdot 1} dx_I \wedge db_J = (-1)^k dx_I \wedge db_J$$

(c) Sia f una funzione

$$d(df) = d\left[\sum \frac{\partial f}{\partial x_i} dx_i\right] =$$

$$= \sum_{i,j} \underbrace{\frac{\partial^2 f}{\partial x_i \partial x_j}}_{= \frac{\partial^2 f}{\partial x_j \partial x_i} \text{ (Schwartz)}} dx_i \wedge dx_j = \ldots = 0$$

$dx_i \wedge dx_j = - dx_j \wedge dx_i$

e $dx_i \wedge dx_i = 0$

Sia $\omega = \sum a_I \, dx_I$

per linearità, basta limitarsi al caso

$\omega = a_I \, dx_I$ (monomio)

$$d\omega = da_I \wedge dx_I$$

$$d^2\omega = d(da_I \wedge dx_I) =$$

$$= \underbrace{d^2 a_I \wedge dx_I}_{= 0} - \underbrace{da_I \wedge d(dx_I)}_{= 0}$$

$d(dx_I) = d1 \wedge dx_I = 0$

$\therefore = 0$

(d) verifichiamo $f^*(dg) = d(f^*g)$

$$f^*(dg) = f^*\left(\sum \frac{\partial g_i}{\partial y_i} dy_i\right) = \sum \frac{\partial g_i}{\partial y_i} \frac{\partial f_i}{\partial x_j} dx_j$$

$$= \sum \frac{\partial (g \circ f)}{\partial x_i} dx_j = d(g \circ f) = d(f^* g)$$

Sia ora $\varphi = \sum a_I \, dx_I$ ⚠

$$d(f^*\varphi) = d\left(\sum_I f^*(a_I)\, f^*(dx_I)\right)$$

$$= \sum d(f^*(a_I)) \wedge f^* dx_I$$

$$= \sum f^*(da_I) \wedge f^*(dx_I)$$

$$= \sum f^*(da_I \wedge dx_I)$$

$$= f^* d\varphi$$