Wavelets and multiresolution representations

Time meets frequency…

Time-Frequency resolution

- Depends on the time-frequency spread of the wavelet atoms

Assuming that ψ is centred in t=0

Signal domain

$$f_s(t) = \frac{1}{\sqrt{s}} f(t) \rightarrow \|f_s\|^2 = \|f\|^2$$

$$\sigma_t^2 = \int_{-\infty}^{+\infty} t^2 |\psi(t)|^2 \, dt$$

$$\int_{-\infty}^{+\infty} (t-u)^2 |\psi_{u,s}(t)|^2 \, dt = s^2 \sigma_t^2$$

Fourier domain

$$\eta = \frac{1}{2\pi} \int_0^{+\infty} \omega^2 |\hat{\psi}(\omega)|^2 \, d\omega$$

$$\hat{\psi}_{u,s}(\omega) = \sqrt{s}\psi(s\omega)e^{-i\omega u} \rightarrow \text{ center frequency } \eta/\text{s}$$

Energy spread around η/s

$$\frac{\sigma_\omega^2}{s^2} = \frac{1}{2\pi} \int_0^{+\infty} \left(\omega - \frac{\eta}{s}\right)^2 \left|\hat{\psi}_{u,s}(\omega)\right|^2 d\omega$$

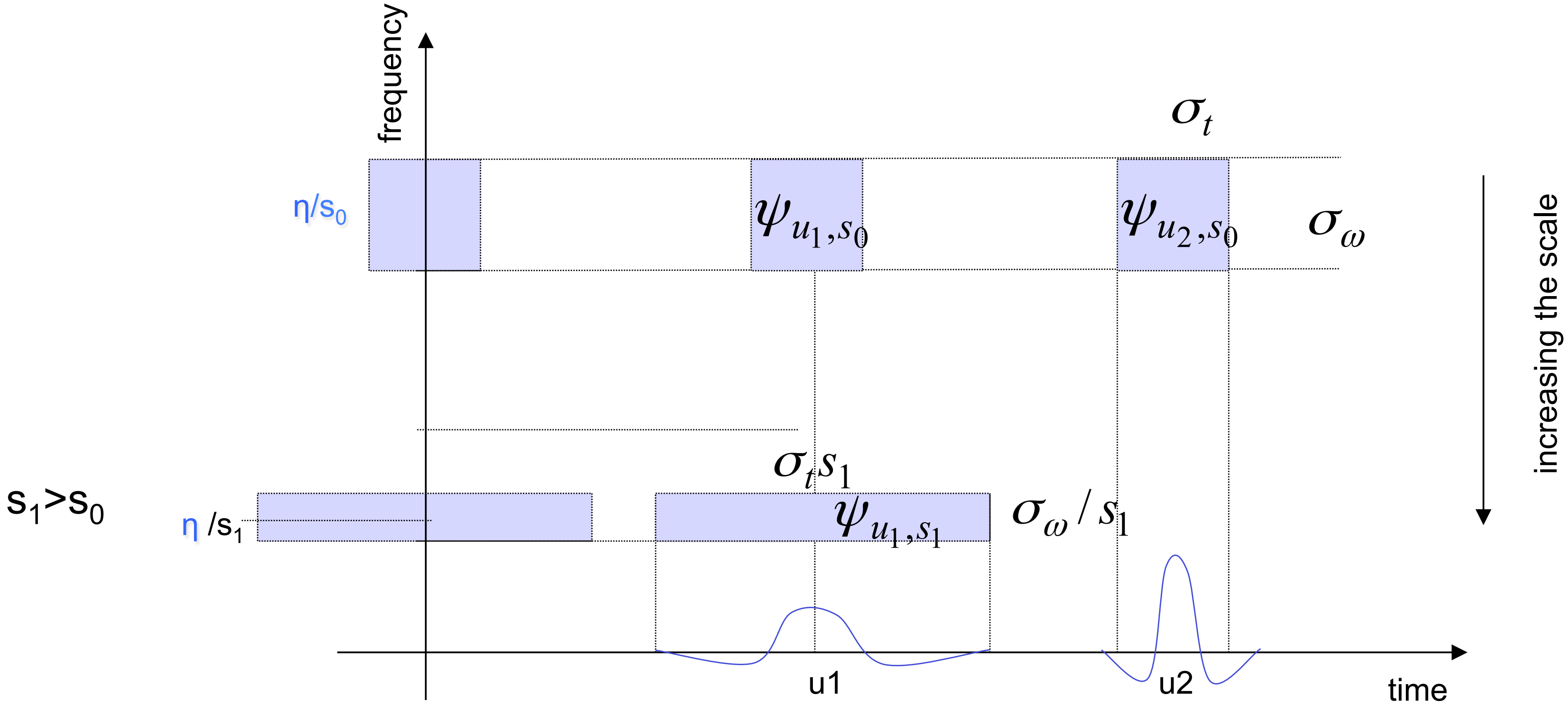
Time/frequency resolution

$$\sigma_{s,t}^{2} = s^{2}\sigma_{t}^{2}$$

$$\sigma_{s,\omega}^{2} = \frac{\sigma_{\omega}^{2}}{s^{2}}$$

- The energy spread of a wavelet time-frequency atom corresponds to an Heisemberg box centred at $(u,\eta/s)$ of size $s\sigma_t$ along the time and σ_ω/s along the frequency.
- The area of the rectangle **remains equal to $\sigma_t\ \sigma_\omega$ at all scales**, while the resolution in time and frequency depends on s.
- A wavelet defines a local time-frequency energy density $P_W f$ which measures the energy in the Heisemberg box of each wavelet centred at $(u, \eta /s)$. This energy density is called scalogram

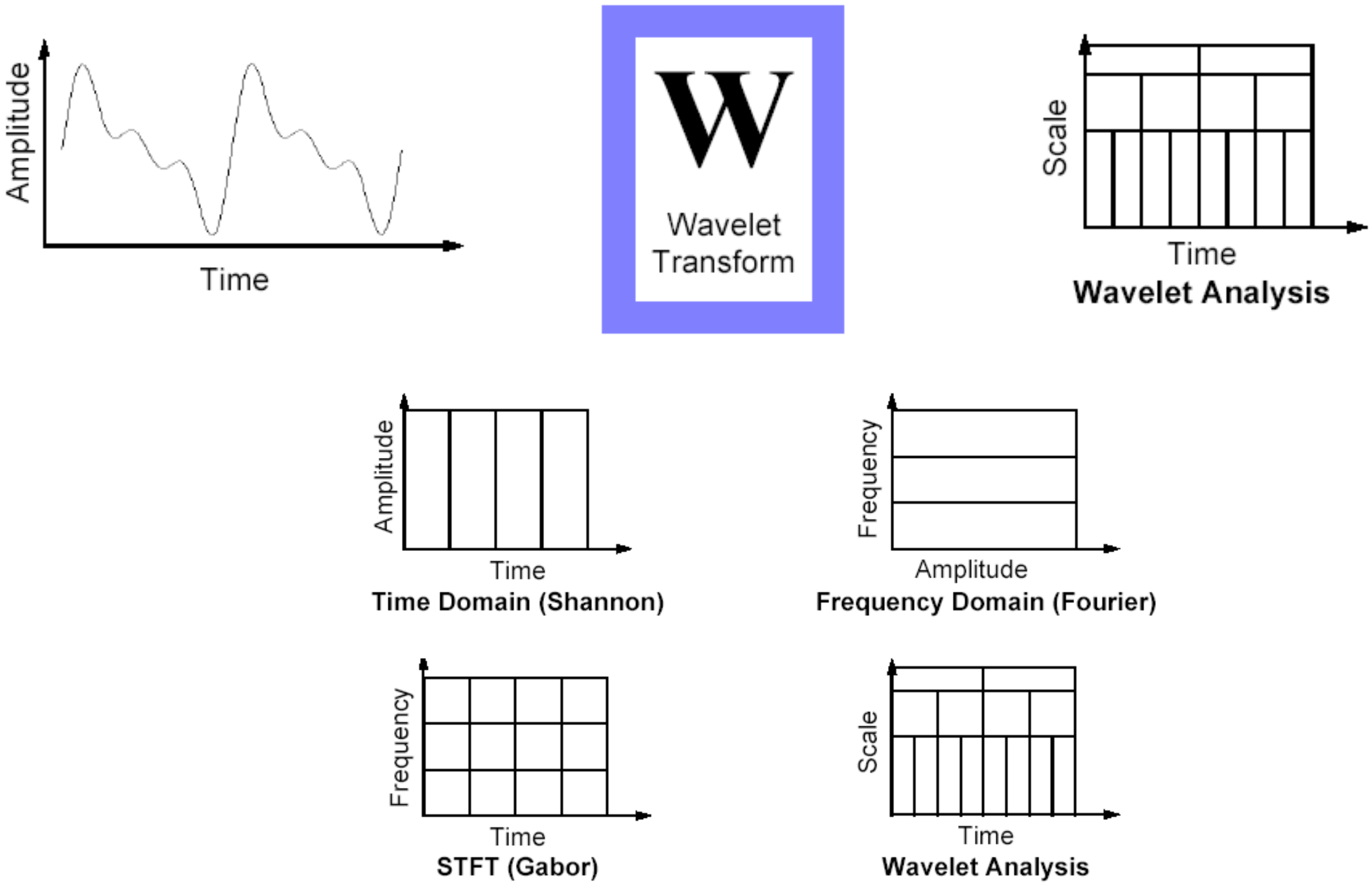
Time/frequency localization



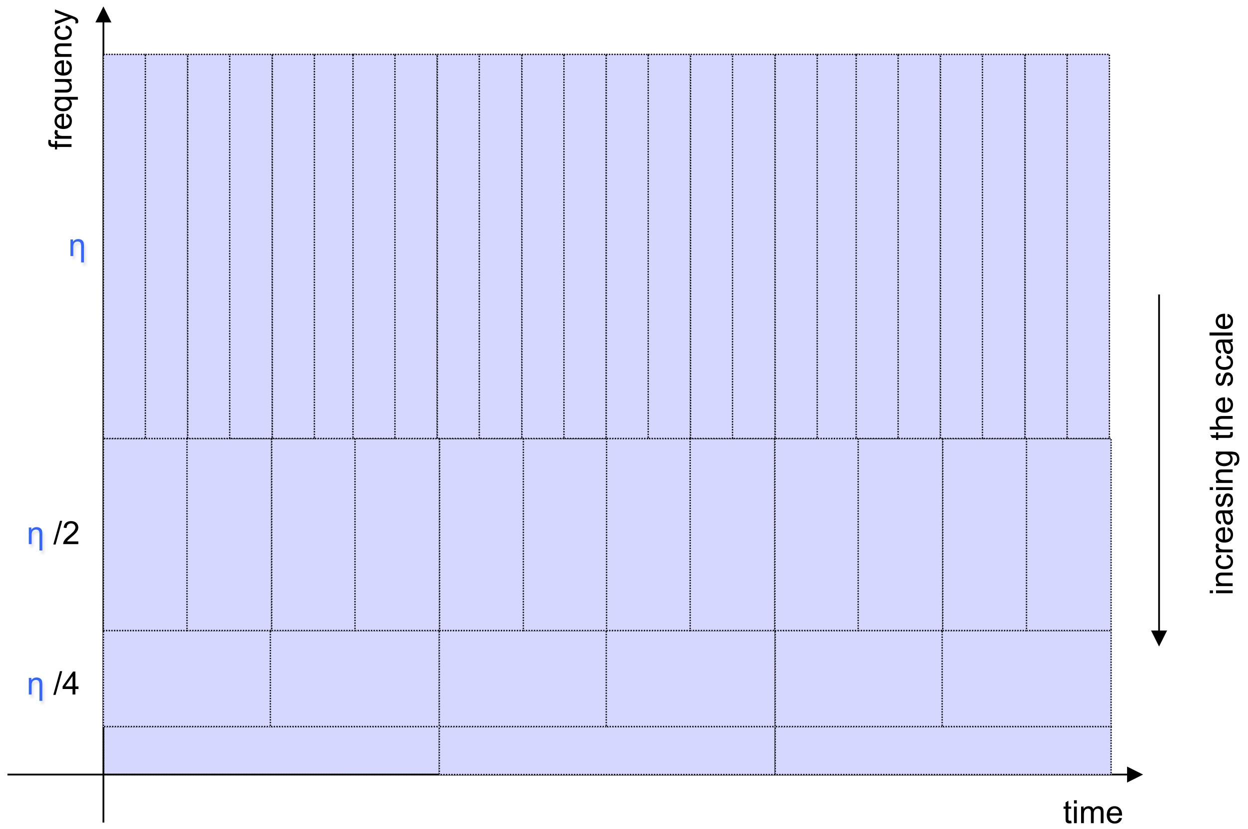
Increasing the scale (s gets larger) pushes the box towards low frequencies → frequency resolution increases, spatial resolution decreases

Time spread is proportional to scale
Frequency spread is proportional to 1/scale

Wavelet domain



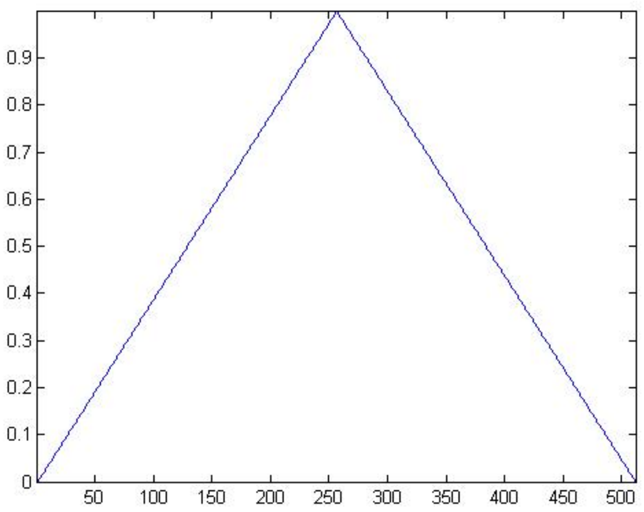
Dyadic Wavelets

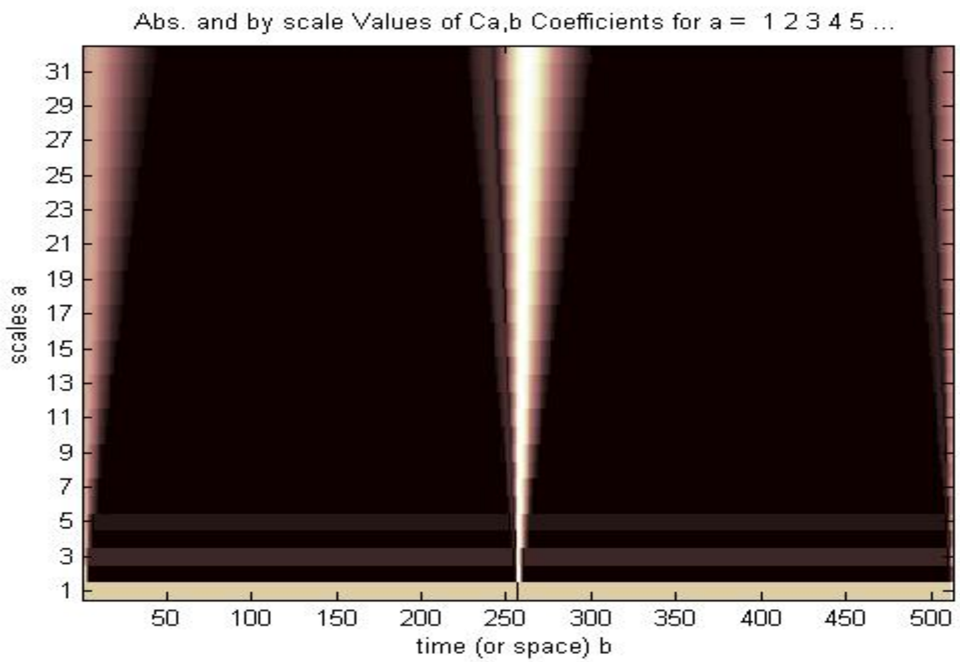


Scalogram

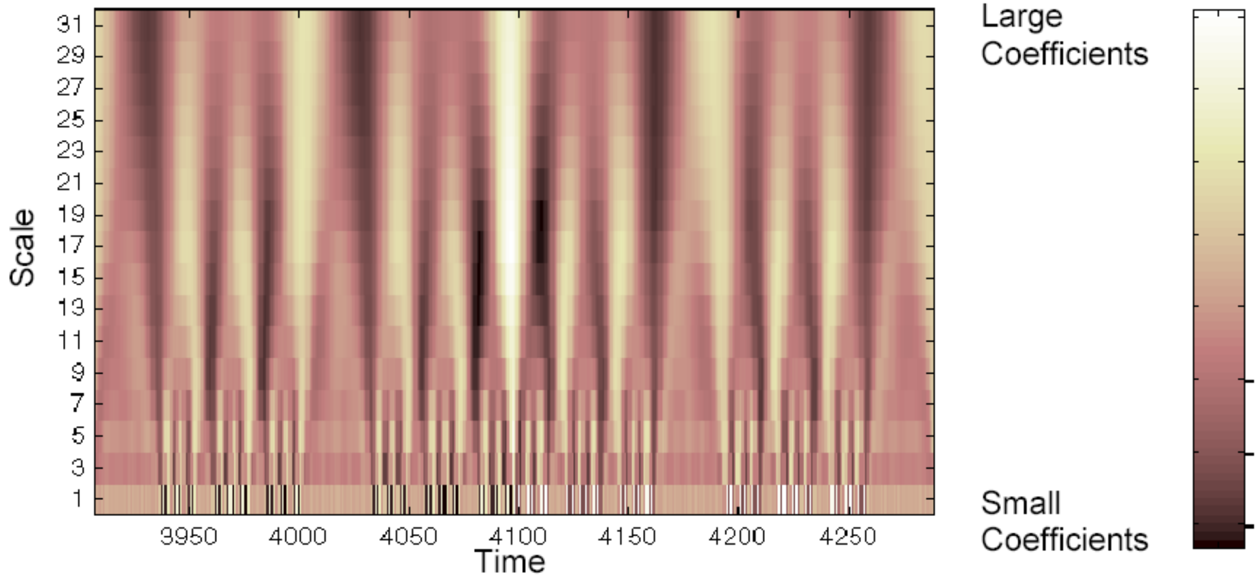
- The scalogram represents the local time/frequency energy density
  - Energy density in the Heisenberg box of each wavelet $\psi_{u,s}$

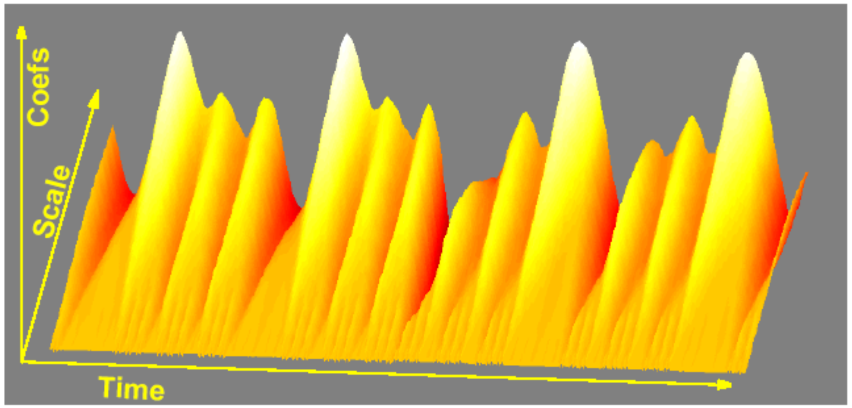
$$P_W f(u,\xi) = \left|Wf(u,s)\right|^2 = \left|Wf\left(u,\frac{\eta}{\xi}\right)\right|^2$$



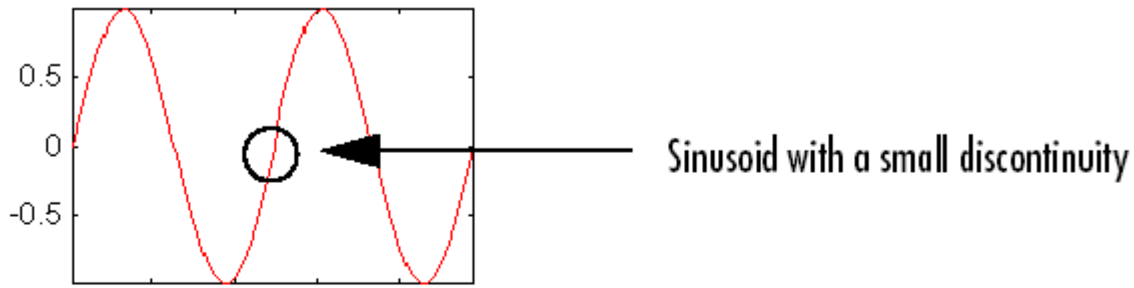


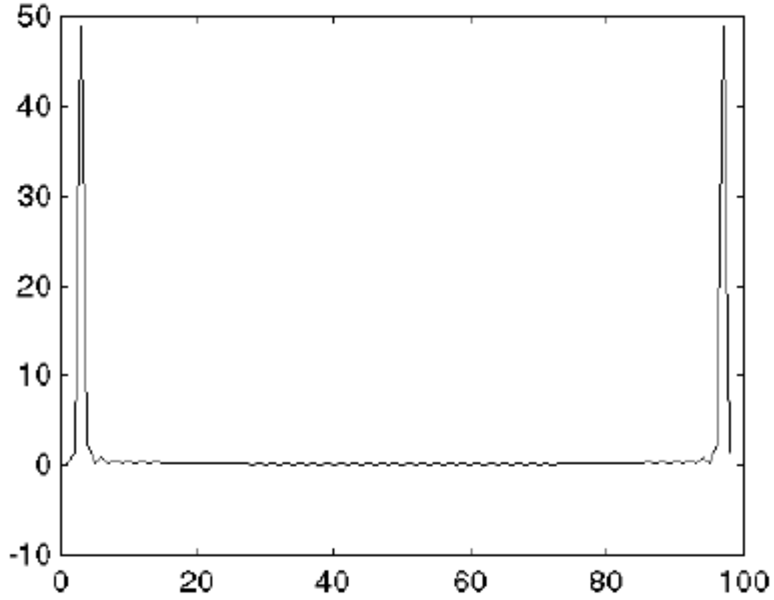
3D representation



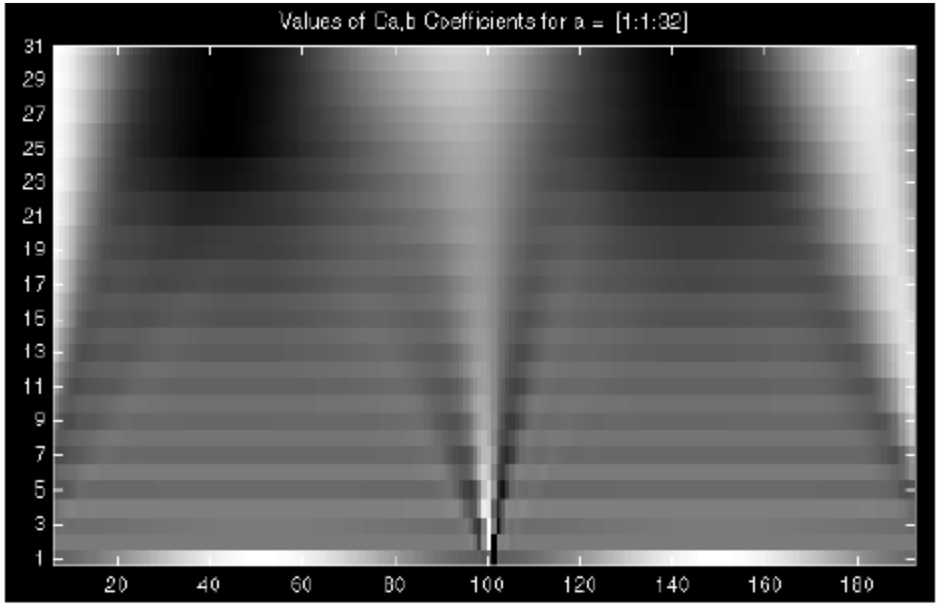


Local discontinuities





Fourier Coefficients



Wavelet Coefficients

Real Wavelets

- Detect sharp signal transitions

$$Wf(u,s) = \int_{-\infty}^{+\infty} f(t) \frac{1}{\sqrt{s}} \psi\left(\frac{t-u}{s}\right) dt$$

- Measures the variations of *f* in the neighborhood of *u* whose size is proportional to *s*
- A real WT is complete and maintains energy conservation as long as it satisfies a weak *admissibility condition* (Theorem 4.3, next slide)
- The **decay of the coefficients as s goes to zero** characterizes the **regularity** of f in the neighborhood of u

Real wavelets: Admissibility condition

- Theorem 4.3 (Calderon, Grossman, Morlet)

  Let ψ in $L^2(R)$ be a real function such that

$$C_\psi = \int_0^{+\infty} \frac{|\hat{\psi}(\omega)|^2}{\omega} d\omega < +\infty$$

Admissibility condition

  Any f in $L^2(R)$ satisfies

$$f(t) = \frac{1}{C_\psi} \int_0^{+\infty} \int_{-\infty}^{+\infty} Wf(u,s) \frac{1}{\sqrt{s}} \psi\left(\frac{t-u}{s}\right) du \frac{ds}{s^2}$$

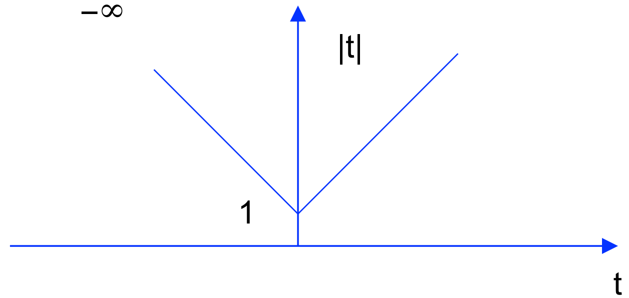
  and

$$\int_{-\infty}^{+\infty} |f(t)|^2 dt = \frac{1}{C_\psi} \int_0^{+\infty} \int_{-\infty}^{+\infty} |Wf(u,s)|^2 du \frac{1}{s^2} ds$$

Admissibility condition

- Consequences
  - The integral is finite if the wavelet has zero average $\hat{\psi}(0) = 0$
    - This condition is nearly sufficient →
  - If $\hat{\psi}(0) = 0$ and $\hat{\psi}(\omega)$ is continuously differentiable, than the admissibility condition is satisfied
    - This happens if it has a sufficient time decay

$$\int_{-\infty}^{+\infty} (1+|t|)\,|\psi(t)|\,dt < +\infty$$



→ The wavelet function must decay **sufficiently fast** in both time and frequency

Scaling function (1)

- When $Wf(u,s)$ is known only for $s<s_0$, to recover f we need a complement of information corresponding to $Wf(u,s)$ for $s>s_0$.
- This is obtained by introducing a *scaling function* φ that is an aggregation of wavelets at *scales larger than 1*.
- The modulus of the Fourier transform of φ is defined as follows and the complex phase can be arbitrarily chosen

$$\left|\hat{\phi}(\omega)\right|^2 = \int_1^{+\infty} \left|\psi(s\omega)\right|^2 \frac{ds}{s} = \int_\omega^{+\infty} \left|\psi(\xi)\right|^2 \frac{d\xi}{\xi}$$

- Remembering that

$$C_\psi = \int_0^{+\infty} \frac{|\hat{\psi}(\omega)|^2}{\omega} d\omega < +\infty$$

It appears that

$$C_\psi = \lim_{\omega \to 0} \left|\hat{\phi}(\omega)\right|^2$$

Scaling function (2)

- The scaling funcion can thus be seen as a *low-pass filter* with *unit gain* ($\|\phi\|^2 = 1$)
- Let us denote

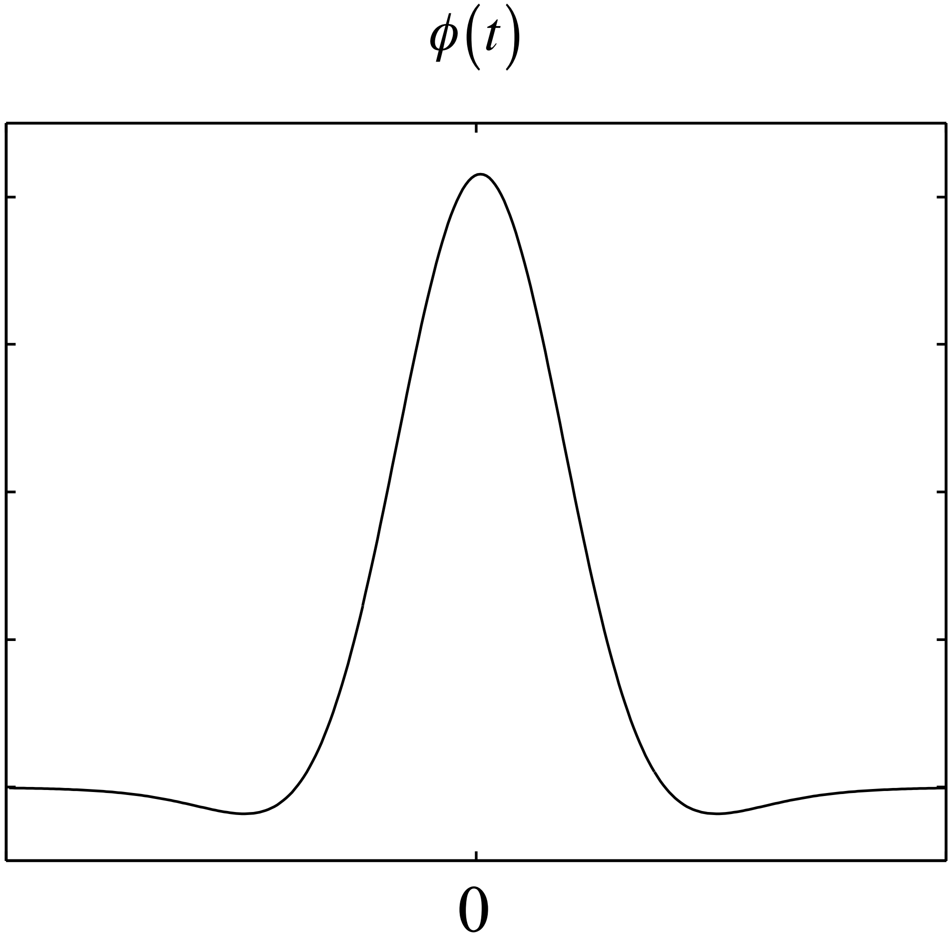
$$\phi_s(t) = \frac{1}{\sqrt{s}}\phi\left(\frac{t}{s}\right) \quad and \quad \bar{\phi}_s(t) = \phi_s^*(-t)$$

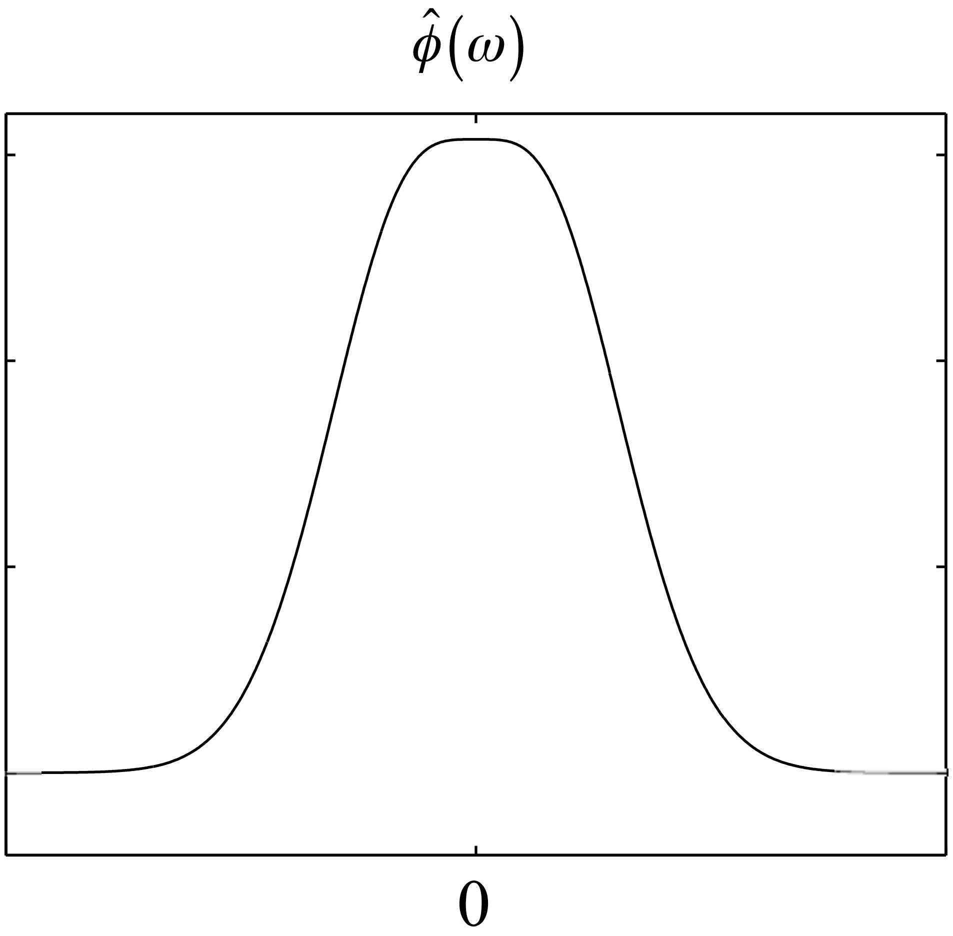
- The *low frequency approximation of f at scale s* is

$$Lf(u,s) = \left\langle f(t), \frac{1}{\sqrt{s}}\phi\left(\frac{t-u}{s}\right)\right\rangle = f * \bar{\phi}_s(u)$$

$$\bar{\phi}_s(t) = \phi\left(-\frac{t}{s}\right)$$

Mexican hat scaling function





Wavelet families

$$f(\vec{x}) \leftrightarrow Wf(u,s;\vec{x}) = c_{u,s}(\vec{x})$$

- In general, there is a *redundancy* in the representation
- The *amount* of redundancy depends on the *grids* over which the *u and s* parameters are sampled

  u,s are real : Continuous WT (CWT, overcomplete representation)

  u in Z, $s=a^{j}$, *j* in Z : Wavelet Frames (DWF, DDWF, overcomplete)

  – *a=2* Dyadic wavelet frames

  $u=k2^{j}$, $s=2^{j}$, *k in I* : Discrete Wavelet Transform (DWT) (*critically sampled*)

- Note: removing completely the redundancy leads to complete basis (*critically sampled*)

Wavelet bases

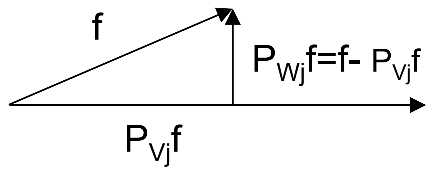
Mallat - Chapter VII

Wavelet *bases*

One can construct wavelets such that

$$\left\{ \psi_{j,n}(t) = \frac{1}{\sqrt{2^j}} \psi\left(\frac{t - 2^j n}{2^j} \right) \right\}_{j,n \in Z^2}$$

is an orthonormal basis for $L^2(R)$.



- Multiresolution approximations
  - The partial sum of wavelet coefficients giving $d_j(t)$ can be interpreted as the difference between two approximations of *f* at the scales 2^j and $2^{(j-1)}$
  - Multiresolution approximations compute the approximations of signals at various resolutions with orthogonal projections to different spaces $\{V_j\}_{j \text{ in } Z}$
  - The **approximation of *f* at scale 2^j** is specified by a discrete grid of samples that provides *local averages* of *f* on neighborhoods of size proportional to 2^j.
  - **A *multiresolution* consists of *embedded grids of approximations***

Orthogonal wavelet bases

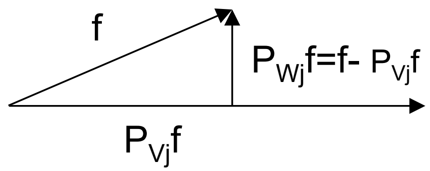
- The search for orthogonal wavelets begins with multiresolution approximations

  $f \in L^2(\Re) \rightarrow \sum_{n=-\infty}^{+\infty} \langle f, \psi_{j,n} \rangle \psi_{j,n}$ difference bewteen two approximations at resolutions 2^{-j+1} and 2^{-j}

- Resolution = 1/scale
  - The larger the scale, the smaller the resolution
- Multiresolution approximations compute the approximation of signals at various resolutions with orthogonal projections on different spaces $\{V_j\}_{j \in Z}$
  - These are characterized by a one particular discrete filter that governs the loss of information across resolutions

Multiresolution approximations

- The approximation of a function *f* at a resolution 2^j is specified by a discrete grid of samples that provides local averages of *f* over neighborhoods of size proportional to 2^j.
- Thus, a multiresolution approximation is composed of *embedded grids of approximation*.
- More formally:
  - *the approximation of a function at a resolution 2^j is defined as an* ***orthogonal projection*** *on a space $\mathbf{V}_j \subset \mathbf{L}^2(R)$.*
  - *The space $\mathbf{V}_j$ regroups* ***all possible*** *approximations at the resolution 2^j .*
  - *The orthogonal projection of f is the function $f_j \in \mathbf{V}_j$ that minimizes $||f - f_j||$.*



Multiresolution approximations

Definition 7.1 A sequence $\{V_j\}_{j \in Z}$ of closed subspaces of $L^2(R)$ is a multiresolution approximation if the following six conditions are satisfied

$$\forall (j,k) \in Z^2, f(t) \in V_j \Leftrightarrow f(t - 2^j k) \in V_j$$

V_j is invariant for translations proportional to the scale

$$\forall j \in Z, \qquad V_{j+1} \subset V_j$$

The *finer* approximation subspace encloses all the information concerning the coarser one

$$\forall j \in Z, f(t) \in V_j \Leftrightarrow f\left(\frac{t}{2}\right) \in V_{j+1}$$

Stretching the function by a factor 2 spans a coarser subspace

$$\lim_{j \to +\infty} V_j = \bigcap_{j=-\infty}^{+\infty} V_j = \{0\}$$

When the resolution goes to zero all the details are lost

$$\lim_{j \to +\infty} \| P_{\mathbf{V}_j} f \| = 0.$$

$$\lim_{j \to -\infty} V_j = Closure\left(\bigcup_{j=-\infty}^{+\infty} V_j\right) = L^2(R)$$

When the resolution goes to infinity the approximation converges to the signal

$$\lim_{j \to -\infty} \| f - P_{\mathbf{V}_j} f \| = 0.$$

There exists ϑ such that $\{\vartheta(t-n)\}_{n \in Z}$ is a Riesz basis of V_0

discretization theorem

j $\leftrightarrow$ scale
$2^{-j} \leftrightarrow$ resolution

Banach and Hilbert spaces

- A Hilbert space is an abstract vector space possessing the structure of an inner product that allows *length* and *angle* to be measured.
- Hilbert spaces are in addition required to be *complete*, a property that stipulates the existence of enough limits in the space to allow the techniques of calculus to be used.

Banach and Hilbert spaces

- Banach space

Signals are often considered as vectors. To define a distance, we work within a vector space $\mathbf{H}$ that admits a norm. A norm satisfies the following properties:

$$\forall f \in \mathbf{H}, \quad \|f\| \geqslant 0 \quad \text{and} \quad \|f\| = 0 \Leftrightarrow f = 0, \tag{A.3}$$

$$\forall \lambda \in \mathbb{C} \quad \|\lambda f\| = |\lambda| \, \|f\|, \tag{A.4}$$

$$\forall f, g \in \mathbf{H}, \quad \|f + g\| \leqslant \|f\| + \|g\|. \tag{A.5}$$

With such a norm, the convergence of $\{f_n\}_{n\in\mathbb{N}}$ to f in $\mathbf{H}$ means that

$$\lim_{n\to+\infty} f_n = f \Leftrightarrow \lim_{n\to+\infty} \|f_n - f\| = 0.$$

To guarantee that we remain in $\mathbf{H}$ when taking such limits, we impose a completeness property, using the notion of *Cauchy sequences*. A sequence $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence if for any $\varepsilon > 0$, if n and p are large enough, then $\|f_n - f_p\| < \varepsilon$. The space $\mathbf{H}$ is said to be *complete* if every Cauchy sequence in $\mathbf{H}$ converges to an element of $\mathbf{H}$.

Example 1

- For any integer p>0 we define over discrete sequences f[n]

$$\|f\|_p = \left(\sum_{n=\infty}^{+\infty} |f[n]| p \right)^{1/p}$$

- The space

$$l^p = \left\{ f : \|f\|_p < +\infty \right\}$$

is a Banach space with the norm $\|f\|_p$

Example 2

Example A.2 The space $\mathbf{L^p}(\mathbb{R})$ is composed of the measurable functions f on $\mathbb{R}$ for which

$$\|f\|_p = \left(\int_{-\infty}^{+\infty} |f(t)|^p \, dt\right)^{1/p} < +\infty.$$

This integral defines a norm and $\mathbf{L^p}(\mathbb{R})$ is a Banach space, provided one identifies functions that are equal almost everywhere.

Banach and Hilbert spaces

- Hilbert space

Whenever possible, we work in a space that has an inner product to define angles and orthogonality. A *Hilbert space* $\mathbf{H}$ is a Banach space with an inner product. The inner product of two vectors $\langle f, g\rangle$ is linear with respect to its first argument:

$$\forall \lambda_1, \lambda_2 \in \mathbb{C}, \quad \langle \lambda_1 f_1 + \lambda_2 f_2, g\rangle = \lambda_1 \langle f_1, g\rangle + \lambda_2 \langle f_2, g\rangle. \quad \textbf{(A.6)}$$

It has an Hermitian symmetry:

$$\langle f, g\rangle = \langle g, f\rangle^*.$$

Moreover,

$$\langle f, f\rangle \geqslant 0 \quad \text{and} \quad \langle f, f\rangle = 0 \Leftrightarrow f = 0.$$

One can verify that $\|f\| = \langle f, f\rangle^{1/2}$ is a norm. The positivity (A.3) implies the Cauchy-Schwarz inequality:

$$|\langle f, g\rangle| \leqslant \|f\|\,\|g\|, \quad \textbf{(A.7)}$$

which is an equality if and only if f and g are linearly dependent.

We write $\mathbf{V}^\perp$ the orthogonal complement of a subspace $\mathbf{V}$ of $\mathbf{H}$. All vectors of $\mathbf{V}$ are orthogonal to all vectors of $\mathbf{V}^\perp$ and $\mathbf{V} \oplus \mathbf{V}^\perp = \mathbf{H}$.

Example 3

Example A.3 An inner product between discrete signals $f[n]$ and $g[n]$ can be defined by

$$\langle f, g \rangle = \sum_{n=-\infty}^{+\infty} f[n]\, g^*[n].$$

It corresponds to an $\mathbf{l}^2(\mathbb{Z})$ norm:

$$\|f\|^2 = \langle f, f \rangle = \sum_{n=-\infty}^{+\infty} |f[n]|^2.$$

The space $\mathbf{l}^2(\mathbb{Z})$ of finite energy sequences is therefore a Hilbert space. The Cauchy-Schwarz inequality (A.7) proves that

$$\left| \sum_{n=-\infty}^{+\infty} f[n]\, g^*[n] \right| \leq \left(\sum_{n=-\infty}^{+\infty} |f[n]|^2 \right)^{1/2} \left(\sum_{n=-\infty}^{+\infty} |g[n]|^2 \right)^{1/2}.$$

Example 4

Example A.4 Over analog signals $f(t)$ and $g(t)$, an inner product can be defined by

$$\langle f, g \rangle = \int_{-\infty}^{+\infty} f(t)\, g^*(t)\, dt.$$

The resulting norm is

$$\|f\| = \left(\int_{-\infty}^{+\infty} |f(t)|^2\, dt \right)^{1/2}.$$

The space $\mathbf{L}^2(\mathbb{R})$ of finite energy functions is thus also a Hilbert space. In $\mathbf{L}^2(\mathbb{R})$, the Cauchy-Schwarz inequality (A.7) is

$$\left| \int_{-\infty}^{+\infty} f(t)\, g^*(t)\, dt \right| \leq \left(\int_{-\infty}^{+\infty} |f(t)|^2\, dt \right)^{1/2} \left(\int_{-\infty}^{+\infty} |g(t)|^2\, dt \right)^{1/2}.$$

Two functions f_1 and f_2 are equal in $\mathbf{L}^2(\mathbb{R})$ if

$$\|f_1 - f_1\|^2 = \int_{-\infty}^{+\infty} |f_1(t) - f_2(t)|^2\, dt = 0,$$

which means that $f_1(t) = f_2(t)$ for almost all $t \in \mathbb{R}$.

Bases of Hilbert spaces

Orthonormal Basis

A family $\{e_n\}_{n\in\mathbb{N}}$ of a Hilbert space $\mathbf{H}$ is orthogonal if for $n \neq p$,

$$\langle e_n, e_p\rangle = 0.$$

If for $f \in \mathbf{H}$ there exists a sequence $a[n]$ such that

$$\lim_{N\to+\infty} \|f - \sum_{n=0}^{N} a[n]\, e_n\| = 0,$$

then $\{e_n\}_{n\in\mathbb{N}}$ is said to be an *orthogonal basis* of $\mathbf{H}$. The orthogonality implies that necessarily $a[n] = \langle f, e_n\rangle / \|e_n\|^2$, and we write

$$f = \sum_{n=0}^{+\infty} \frac{\langle f, e_n\rangle}{\|e_n\|^2}\, e_n. \qquad \textbf{(A.8)}$$

A Hilbert space that admits an orthogonal basis is said to be *separable*.

The basis is *orthonormal* if $\|e_n\| = 1$ for all $n \in \mathbb{N}$. Computing the inner product of $g \in \mathbf{H}$ with each side of (A.8) yields a Parseval equation for orthonormal bases:

$\langle f, g\rangle = \langle g, f\rangle^*$

$$\langle f, g\rangle = \sum_{n=0}^{+\infty} \langle f, e_n\rangle\, \langle g, e_n\rangle^*. \qquad \textbf{(A.9)}$$

Bases of Hilbert space

When $g = f$, we get an energy conservation called the *Plancherel formula*:

$$\|f\|^2 = \sum_{n=0}^{+\infty} |\langle f, e_n \rangle|^2. \qquad \textbf{(A.10)}$$

The Hilbert spaces $\ell^2(\mathbb{Z})$ and $\mathbf{L}^2(\mathbb{R})$ are separable. For example, the family of translated Diracs $\{e_n[k] = \delta[k-n]\}_{n \in \mathbb{Z}}$ is an orthonormal basis of $\ell^2(\mathbb{Z})$. Chapters 7 and 8 construct orthonormal bases of $\mathbf{L}^2(\mathbb{R})$ with wavelets, wavelet packets, and local cosine functions.

Riesz basis

Link to the discrete domain: the existance of a Riesz bases provides a *discretization theorem*

Definition: A family of vectors is a Riesz basis of a space H if

1. *it is linearly independent*
2. *there exist A,B>0 such that*

$$\forall y \in H \qquad \exists \lambda[n]: \qquad y = \sum_{n=0}^{+\infty} \lambda[n] e_n$$

$$\frac{1}{B}\|y\|^2 \le \sum_{n=0}^{+\infty} |\lambda[n]|^2 \le \frac{1}{A}\|y\|^2$$

The existance of a Riesz basis for V_0 provides ***a discretization theorem. There exists A and B such that any $f \in V_0$ can be uniquely decomposed into***

$$\forall f(t) \in V_0 \rightarrow f(t) = \sum_n a[n] \vartheta(t-n) \quad (7.9)$$

$$A\|f\|^2 \le \sum_n |a[n]|^2 \le B\|f\|^2 \qquad (7.10)$$

(7.4) $\forall j \in Z, f(t) \in V_j \Leftrightarrow f\left(\frac{t}{2}\right) \in V_{j+1} \rightarrow \left\{ \frac{1}{\sqrt{2^j}} \vartheta\left(\frac{t-2^j n}{2^j}\right) \right\}_{n \in Z}$ is a Riesz basis for V_j

Riesz basis

- **Proposition 7.1** A family $\{\vartheta(t-n)\}_{n\in Z}$ is a Riesz basis of the space V_0 it generates if and only if there are A>0 and B>0 such that

(7.11)
$$\forall\omega\in[-\pi,\pi],\frac{1}{B}\le\sum_{k=-\infty}^{+\infty}\left|\vartheta(\omega-2k\pi)\right|^2\le\frac{1}{A}$$

- Proof
- $$\forall f\in V_0\rightarrow f(t)=\sum_{n=-\infty}^{+\infty}a[n]\vartheta(t-n)$$ taking the FT of both sides (7.12)

$$\hat{f}(\omega)=\hat{a}(\omega)\hat{\vartheta}(\omega)$$

Since a[n] is a Fourier series

$$\hat{a}(\omega)=\sum_{n=-\infty}^{+\infty}a[n]\mathrm{e}^{-j\omega n}$$ and is 2π periodic, hence

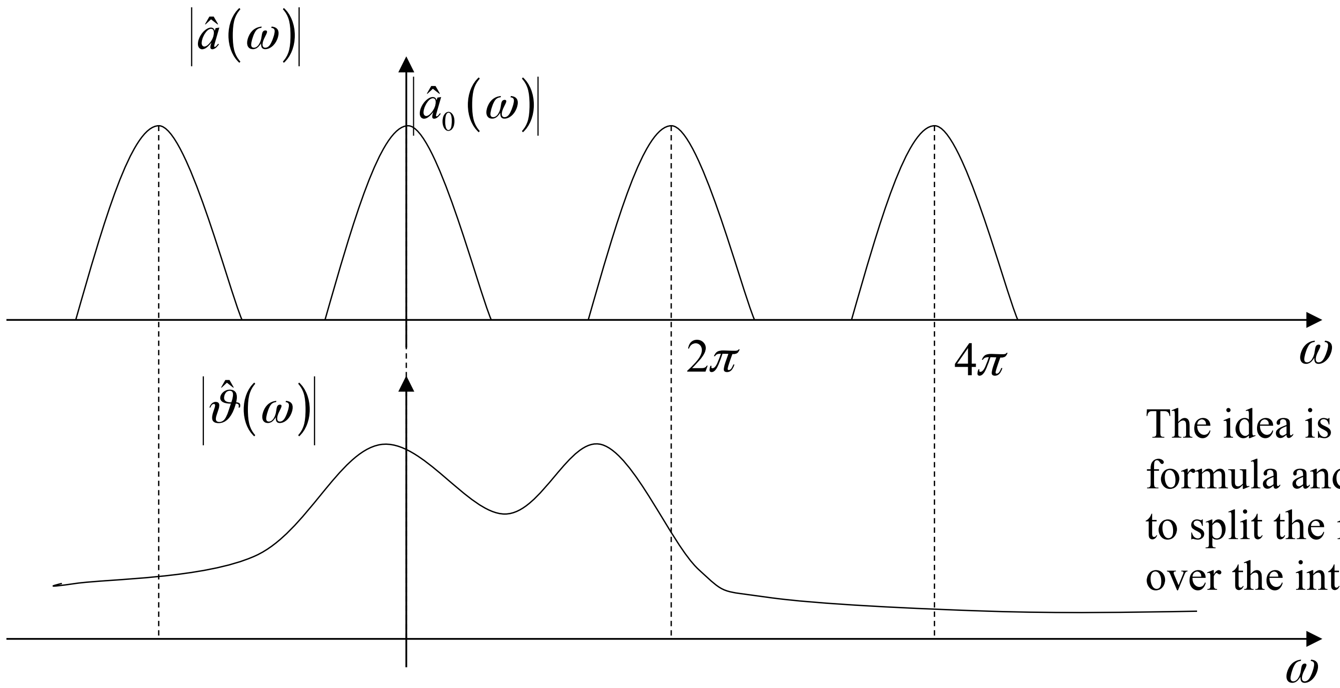
$$\hat{a}(\omega)=\sum_{n=-\infty}^{+\infty}\hat{a}(\omega-n2\pi)=\hat{a}(\omega)*\sum_{n=-\infty}^{+\infty}\delta(\omega-n2\pi)$$

Intuition (1) $\int_{-\infty}^{+\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{f}(\omega)|^2 \, d\omega.$

$$\hat{a}(\omega) = \sum_{n=-\infty}^{+\infty} \hat{a}(\omega - n2\pi) = \hat{a}(\omega) * \sum_{n=-\infty}^{+\infty} \delta(\omega - n2\pi)$$

- Applying the definition of norm (Plancherel)

$$\left\|\hat{f}(\omega)\right\|^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|\hat{f}(\omega)\right|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|\hat{a}(\omega)\right|^2 \left|\hat{\vartheta}(\omega)\right|^2 d\omega \qquad (1)$$



The idea is to exploit the Plancherel's formula and the fact that a(ω) is periodic to split the integral into sums of integrals over the interval 0-2π.

Proof (1)

- Using Plancherel formula and the fact that a(ω) is periodic (see Mallat version 2009 page 67)

$$\left\|\hat{f}(\omega)\right\|^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|\hat{f}(\omega)\right|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|\hat{a}(\omega)\right|^2\left|\hat{\vartheta}(\omega)\right|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|\hat{a}(\omega)\hat{\vartheta}(\omega)\right|^2 d\omega =$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|\sum_k \hat{a}(\omega-2n\pi)\hat{\vartheta}(\omega)\right|^2 d\omega =$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left|\hat{a}(\omega)\right|^2 \sum_{k=-\infty}^{+\infty}\left|\hat{\vartheta}(\omega-2k\pi)\right|^2 d\omega$$

since a(ω) is periodic, taking the integral over subsequent intervals amounts only to "shifting" the second function. The first, a(ω), remains the same so it can be taken out of the sum.

Proof (2)

- Norm

$$\left\|\hat{f}(\omega)\right\|^2 = \frac{1}{2\pi}\int_0^{2\pi}\left|\hat{a}(\omega)\right|^2 \sum_{k=-\infty}^{+\infty}\left|\hat{\vartheta}(\omega - 2k\pi)\right|^2 d\omega$$

$$\forall \omega \in [-\pi, \pi] \qquad \frac{1}{B} \le \sum_{k=-\infty}^{+\infty}\left|\vartheta(\omega - 2k\pi)\right|^2 \le \frac{1}{A} \text{ then} \qquad (2)$$

$$\left\|f(t)\right\|^2 \le \frac{1}{A}\frac{1}{2\pi}\int_0^{2\pi}\left|a(\omega)\right|^2 d\omega = \frac{1}{A}\sum_{n=-\infty}^{+\infty}\left|a[n]\right|^2 \rightarrow$$

$$A\left\|f(t)\right\|^2 \le \sum_{n=-\infty}^{+\infty}\left|a[n]\right|^2$$

$$\sum_{k=-\infty}^{+\infty}\left|\vartheta(\omega - 2k\pi)\right|^2 \qquad (1)$$

To note: (1) is a function of omega thus the condition (2) means that the pointwise sum of the values of the translates of the function in omega is finite.

For A=B=1 the basis is orthonormal and (2) takes the definition of "partition of unity". This is the case for the scaling function.

Proof (3)

- Similarly

$$B\left\|f(t)\right\|^2 \geq \sum_{n=-\infty}^{+\infty}\left|a[n]\right|^2$$

- Thus

(7.15)
$$A\left\|f(t)\right\|^2 \leq \sum_{n=-\infty}^{+\infty}\left|a[n]\right|^2 \leq B\left\|f(t)\right\|^2$$

In summary, if θ(t-n) satisfies (7.11 Mallat 99) then (7.15) is satisfied. Then, θ(t-n) is a Riesz basis for V_0 and every function in V_0 can be expressed as in (7.12)

$$f(t) = \sum_{k=-\infty}^{+\infty} a[n]\vartheta(t-n) \qquad (7.12)$$

Scaling function

- The **scaling function** is obtained by the **orthogonalization of the Riesz basis**

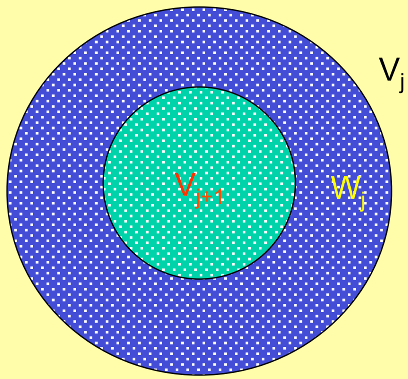
Theorem 7.1

Let V_j be a multiresolution approximation and φ be the scaling function whose FT is

$$\hat{\varphi}(\omega) = \frac{\hat{\vartheta}(\omega)}{\left(\sum_{k=-\infty}^{+\infty} \left|\hat{\vartheta}(\omega + 2k\pi)\right|^2\right)^{1/2}}$$

Let us denote

$$\varphi_{j,n}(t) = \frac{1}{\sqrt{2^j}} \varphi\left(\frac{t - 2^j n}{2^j}\right)$$



The family $\{\varphi_{j,n}\}_{n \text{ in } Z}$ is an orthonormal basis of V_j for all j in Z

Proof

Proof[1]. To construct an orthonormal basis, we look for a function $\phi \in \mathbf{V}_0$. It can thus be expanded in the basis $\{\theta(t-n)\}_{n\in\mathbf{Z}}$:

$$\phi(t) = \sum_{n=-\infty}^{+\infty} a[n]\,\theta(t-n),$$

which implies that

$$\hat{\phi}(\omega) = \hat{a}(\omega)\,\hat{\theta}(\omega),$$

where $\hat{a}$ is a 2π periodic Fourier series of finite energy. To compute $\hat{a}$ we express the orthogonality of $\{\phi(t-n)\}_{n\in\mathbf{Z}}$ in the Fourier domain. Let $\bar{\phi}(t) = \phi^*(-t)$. For any $(n,p) \in \mathbb{Z}^2$,

$$\langle \phi(t-n), \phi(t-p) \rangle = \int_{-\infty}^{+\infty} \phi(t-n)\,\phi^*(t-p)\,dt \qquad (1)$$

$$= \phi \star \bar{\phi}(p-n)\,. \qquad (7.18)$$

Hence $\{\phi(t-n)\}_{n\in\mathbf{Z}}$ is orthonormal if and only if $\phi \star \bar{\phi}(n) = \delta[n]$. Computing the Fourier transform of this equality yields

$$\sum_{k=-\infty}^{+\infty} |\hat{\phi}(\omega + 2k\pi)|^2 = 1. \qquad (7.19)$$

Indeed, the Fourier transform of $\phi \star \bar{\phi}(t)$ is $|\hat{\phi}(\omega)|^2$, and we proved in (3.3) that sampling a function periodizes its Fourier transform. The property (7.19) is verified if we choose

$$\hat{a}(\omega) = \left(\sum_{k=-\infty}^{+\infty} |\hat{\theta}(\omega + 2k\pi)|^2 \right)^{-1/2}.$$

Proposition 7.1 proves that the denominator has a strictly positive lower bound, so $\hat{a}$ is a 2π periodic function of finite energy. ■

Thus here we apply the same idea as in the previous proof: relying on Plancherel formula and explicitating the fact that the function is periodic in the Fourier domain. Thus, replacing the result in (1) we get the orthogonalization formula.

Approximation

- The orthogonal projection of f onto V_j is obtained as an expansion in the scaling orthogonal basis

$$P_{V_j} f = \sum_{n=-\infty}^{+\infty} \langle f, \varphi_{j,n} \rangle \varphi_{j,n}$$

- The inner products $a_j[n]$ are the projection coefficients at scale 2^j

$$a_j[n] = \langle f, \varphi_{j,n} \rangle = \int_{-\infty}^{+\infty} f(t) \frac{1}{\sqrt{2^j}} \varphi\left(\frac{t-2^j n}{2^j}\right) = f * \bar{\varphi}_j(2^j n)$$

$$\bar{\varphi}_j(t) = \frac{1}{\sqrt{2^j}} \varphi\left(-\frac{t}{2^j}\right)$$

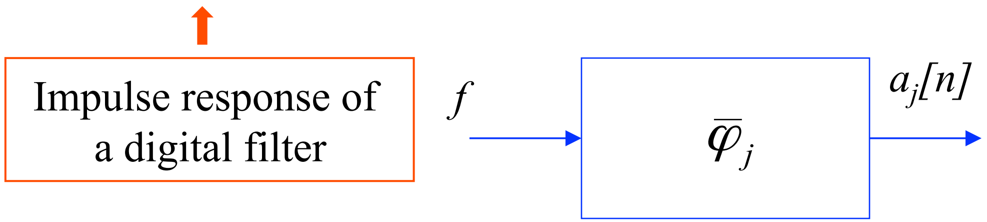
  - As proved in what above, the normalization factor at the denominator ensures that

$$\hat{\varphi}(\omega) = \frac{\hat{\vartheta}(\omega)}{\left(\sum_{k=-\infty}^{+\infty} \left|\hat{\vartheta}(\omega + 2k\pi)\right|^2\right)^{1/2}}$$

$$\sum_{k=-\infty}^{\infty} \left|\hat{\varphi}(\omega + 2k\pi)\right|^2 = 1$$ ***partition of unity***

Approximation

$$a_j[n] = f * \bar{\varphi}_j(2^j n)$$



only one sample out out of 2^j is retained → downsampling

– The energy of φ_j is mostly concentrated in $[-\pi/2^j, \pi/2^j]$ which corresponds to low pass filtering

- The *signal approximation* is obtained by convolving f with a *low-pass filter* and downsampling by 2 -> any scaling function corresponds to a *conjugate mirror filter*
- A multiresolution is *completely characterized* by the scaling function

Wavelet representation

- Summarizing

$$A^d_{2^j} f = PV_j f = \sum_{n=-\infty}^{+\infty} \langle f, \varphi_{j,n} \rangle \varphi_{j,n}$$

discrete approximation at resolution j

$$a_j[n] = \langle f, \varphi_{j,n} \rangle$$

discrete approximation coefficients at resolution j

$$d_{2^j} f = PW_j f = \sum_{n=-\infty}^{+\infty} \langle f, \psi_{j,n} \rangle \psi_{j,n}$$

details at resolution j

$$d_j[n] = \langle f, \psi_{j,n} \rangle$$

wavelet coefficients at resolution j

$$\left\{ A^d_{2^J} f, \left\{ d_{2^j} f \right\}_{1 \le j \le J} \right\}$$

wavelet representation

Analytic versus real wavelets

- Real wavelets are used to detect sharp signal transitions
- Analytic wavelets can measure the time evolution of a frequency gradient, as they allow to separate the phase and amplitude information
- Fourier analogy
  - DCT: real basis functions
  - DFT: complex basis functions

  DCT describes all (symmetrized) signals as a linear combination of cosinusoids such that the phase information is lost.

  On the contrary, complex exponentials preserve the information about the phase

Analytic signals

- A function $f_a(x) \in L^2(R)$ is said to be analytic if its Fourier transform is zero for negative frequencies

$$\hat{f}_a(\omega) = 0 \quad if \quad \omega < 0$$

- An analytic function is necessarily complex but it is completely characterized by its real part Re[$f_a(\omega)$]

$$\hat{f}(\omega) = \frac{\hat{f}_a(\omega) + \hat{f}^*_a(-\omega)}{2} \leftrightarrow \hat{f}_a(\omega) = \begin{cases} 2\hat{f}(\omega) \quad if \quad \omega \geq 0 \\ 0 \quad if \quad \omega < 0 \end{cases}$$

Discrete analytic part

Discrete Analytic Part The analytic part $f_a[n]$ of a discrete signal $f[n]$ of size N is also computed by setting to zero the negative frequency components of its discrete Fourier transform. The Fourier transform values at $k = 0$ and $k = N/2$ must be carefully adjusted so that $\mathrm{Real}[f_a] = f$:

$$\hat{f}_a[k] = \begin{cases} \hat{f}[k] & \text{if } k = 0, N/2 \\ 2\,\hat{f}[k] & \text{if } 0 < k < N/2 \\ 0 & \text{if } N/2 < k < N \end{cases} . \tag{4.48}$$

We obtain $f_a[n]$ by computing the inverse discrete Fourier transform.

Example

Example 4.8 The Fourier transform of

Real function

$$f(t) = a\cos(\omega_0 t + \phi) = \frac{a}{2}\Big(\exp[i(\omega_0 t + \phi)] + \exp[-i(\omega_0 t + \phi)]\Big)$$

is

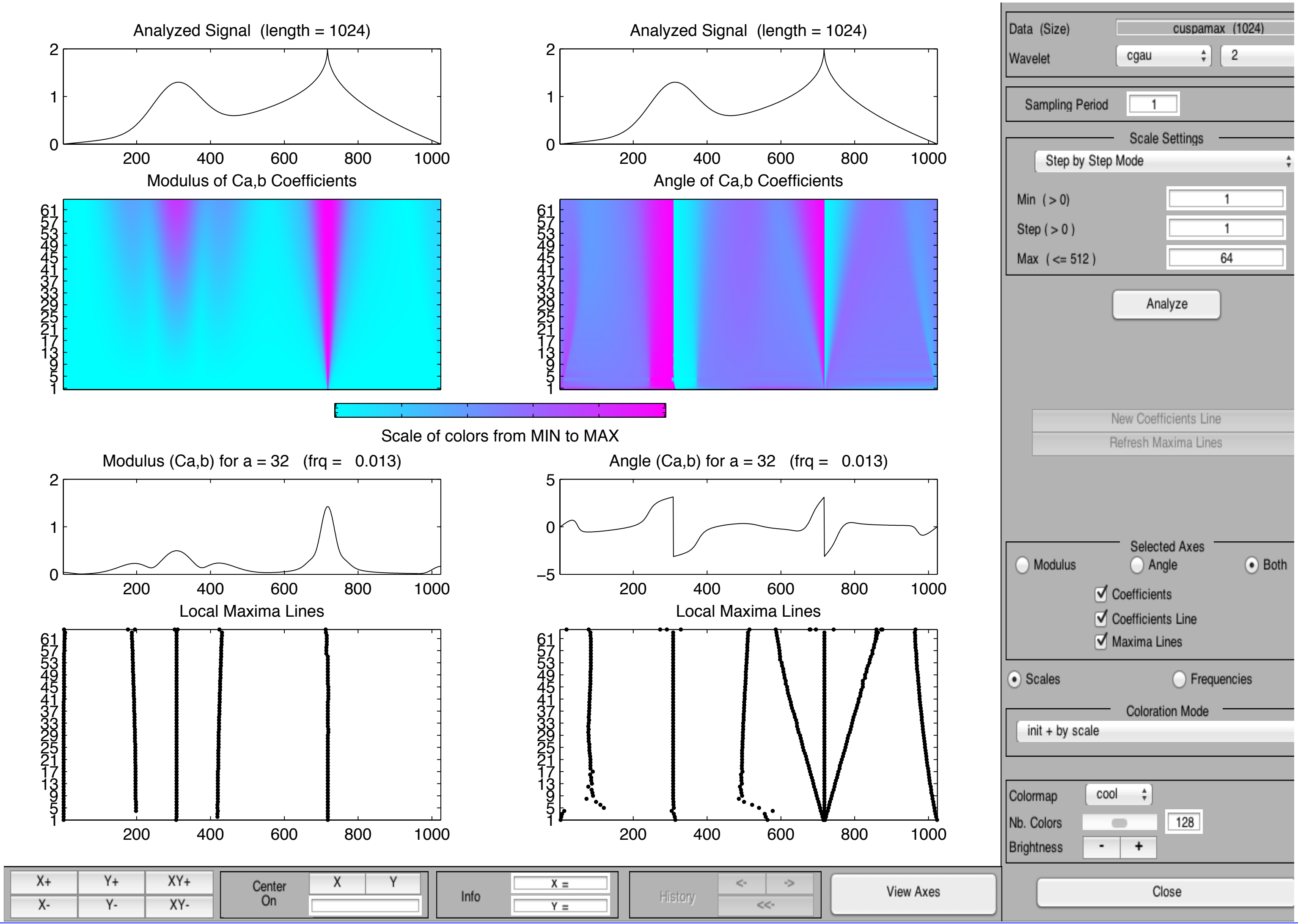
$$\hat{f}(\omega) = \pi a\Big(\exp(i\phi)\,\delta(\omega - \omega_0) + \exp(-i\phi)\,\delta(\omega + \omega_0)\Big).$$

The Fourier transform of the analytic part computed with (4.47) is $\hat{f}_a(\omega) = 2\pi a\exp(i\phi)\,\delta(\omega - \omega_0)$ and hence

$$f_a(t) = a\exp[i(\omega_0 t + \phi)]. \tag{4.49}$$

Complex function

Example: analytic wavelet



Wavelets and multiresolution representations

Scaling equation

- A multiresolution approximation is completely characterized by the function φ that generates the orthonormal bases for each V_j

→ We study the properties of φ which guarantee that all the spaces V_j satisfy all conditions of a multiresolution approximation.

→ It is proved that **any scaling function corresponds to a discrete filter called conjugate mirror filter**

- Procedure
  1. Link φ to the corresponding discrete filter $h[n]$
  2. Determine the properties of $h[n]$ such that φ is a scaling function

Scaling equation

- From multiresolution conditions follows

$$V_j \subset V_{j-1}$$

$$\frac{1}{\sqrt{2}}\varphi\left(\frac{t}{2}\right) \subset V_1 \subset V_0$$

$$\frac{1}{\sqrt{2}}\varphi\left(\frac{t}{2}\right) = \sum_{n=-\infty}^{+\infty} h[n]\varphi(t-n) \qquad (1)$$

$$h[n] = \left\langle \frac{1}{\sqrt{2}}\varphi\left(\frac{t}{2}\right), \varphi(t-n) \right\rangle$$

f(t)

- The **scaling equation** relates a dilation of φ by 2 to its integer translations.
- The sequence *h[n]* will be interpreted as a discrete filter

Scaling equation

- Taking the F-trasform of (1)

convolution product

$$\Im\left\{\frac{1}{\sqrt{2}}\varphi\left(\frac{t}{2}\right)\right\} = \Im\left\{\sum_{n=-\infty}^{+\infty} h[n]\varphi(t-n)\right\} \rightarrow$$

$$\hat{\phi}(2\omega) = \frac{1}{\sqrt{2}}\hat{h}(\omega)\hat{\phi}(\omega) \qquad (2)$$

– where

$$\hat{h}(\omega) = \sum_{n=-\infty}^{+\infty} h[n]e^{-j\omega n}$$

- Next step is thus the expression of ^φ(ω) as a product of dilations of ^h(ω).
  - For any p≥0, (2) implies

$$\hat{\phi}\left(2^{-p+1}\omega\right) = \frac{1}{\sqrt{2}}\hat{h}\left(2^{-p}\omega\right)\hat{\phi}\left(2^{-p}\omega\right) \Longleftrightarrow \hat{\phi}\left(\frac{\omega}{2^{p-1}}\right) = \frac{1}{\sqrt{2}}\hat{h}\left(\frac{\omega}{2^{p}}\right)\hat{\phi}\left(\frac{\omega}{2^{p}}\right)$$

Scaling equation

Iterating (2):

$$\hat{\Phi}(2\omega) = \frac{1}{\sqrt{2}}\hat{h}(\omega)\hat{\Phi}(\omega) \rightarrow$$

$$\hat{\Phi}(\omega) = \frac{1}{\sqrt{2}}\hat{h}\left(\frac{\omega}{2}\right)\hat{\Phi}\left(\frac{\omega}{2}\right), \rightarrow \hat{\Phi}\left(\frac{\omega}{2}\right) = \frac{1}{\sqrt{2}}\hat{h}\left(\frac{\omega}{4}\right)\hat{\Phi}\left(\frac{\omega}{4}\right) \rightarrow .....\hat{\Phi}\left(2^{-p+1}\omega\right) = \hat{h}\left(2^{-p}\omega\right)\hat{\Phi}\left(2^{-p}\omega\right)$$

replacing in the expression above for all values of p up to P:

$$\hat{\Phi}(\omega) = \left(\frac{1}{\sqrt{2}}\right)^2 \hat{\Phi}\left(\frac{\omega}{4}\right)\hat{h}\left(\frac{\omega}{4}\right)\hat{h}\left(\frac{\omega}{2}\right)$$

....

$$\hat{\Phi}(\omega) = \prod_{p=1}^{P}\frac{\hat{h}\left(2^{-p}\omega\right)}{\sqrt{2}}\hat{\Phi}(2^{-P}\omega)$$

If $\hat{\varphi}(\omega)$ is continuous at ω=0 then

$$\lim_{P\rightarrow+\infty}\left(\hat{\Phi}\left(2^{-p}\omega\right)\right) = \hat{\Phi}(0) \rightarrow$$

$$\hat{\Phi}(\omega) = \prod_{p=1}^{+\infty}\frac{\hat{h}\left(2^{-p}\omega\right)}{\sqrt{2}}\hat{\Phi}(0)$$

→ Next step: find the necessary and sufficient conditions on ^h(ω) to guarantee that this infinite product is the F-transform of a scaling function

Conjugate Mirror Filters

Teorem 7.2 (Mallat&Meyer)

Let φ *in* $L^2(R)$ be an integrable scaling function. The F-series of *h[n]* satisfies

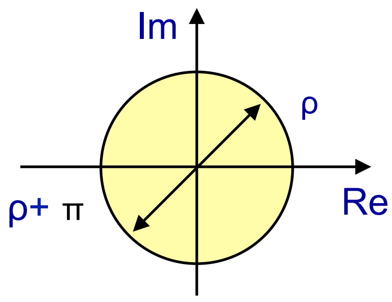
(2) $$\forall\omega \qquad \left|\hat{h}(\omega)\right|^2 + \left|\hat{h}(\omega+\pi)\right|^2 = 2 \qquad and \quad \hat{h}(0) = \sqrt{2}$$ CMF

Conversely, if $h\hat{}(\omega)$ is 2π periodic and continuously differentiable in a neighborhood of $\omega=0$, if it satisfies (2) and if

$$\inf_{\omega\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]} \left|\hat{h}(\omega)\right| > 0$$

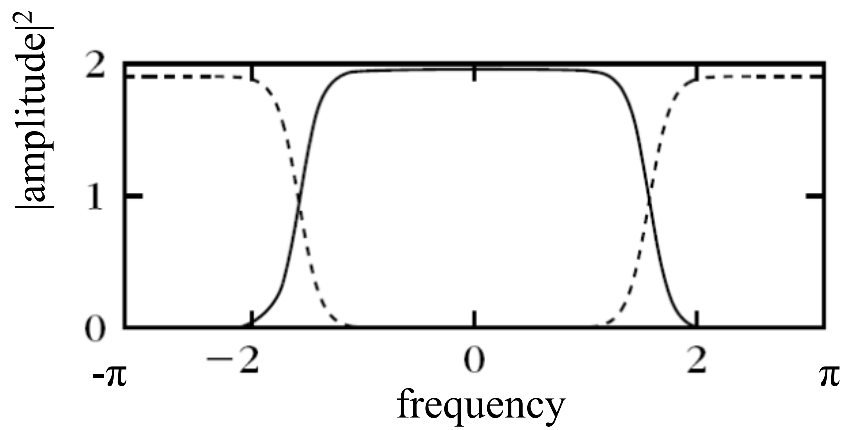
It does not vanish at ω=0

Then, $\hat{\Phi}(\omega) = \prod_{p=1}^{+\infty} \frac{\hat{h}\left(2^{-p}\omega\right)}{\sqrt{2}} \hat{\Phi}(0)$ is the F-transform of a scaling function.



This theorem provides the conditions under which the **discrete filter h[n] generates a scaling function** or, equivalently, a multiresolution representation frame.

CMF property



The solid line gives $|\hat{h}(\omega)|^2$ on $[-\pi,\pi]$ for a cubic spline multiresolution. The dotted line corresponds to $|\hat{g}(\omega)|^2$, namely the corresponding band-pass filter.

Conjugate mirror filters

Table 7.1 Conjugate Mirror Filters $h[n]$ for Linear Splines $m = 1$ and Cubic Splines $m = 3$

	n	$h[n]$
$m=1$	0	0.817645956
	1, −1	0.397296430
	2, −2	−0.069101020
	3, −3	−0.051945337
	4, −4	0.016974805
	5, −5	0.009990599
	6, −6	−0.003883261
	7, −7	−0.002201945
	8, −8	0.000923371
	9, −9	0.000511636
	10, −10	−0.000224296
	11, −11	−0.000122686
$m=3$	0	0.766130398
	1, −1	0.433923147
	2, −2	−0.050201753
	3, −3	−0.110036987
	4, −4	0.032080869
$m=3$	5, −5	0.042068328
	6, −6	−0.017176331
	7, −7	−0.017982291
	8, −8	0.008685294
	9, −9	0.008201477
	10, −10	−0.004353840
	11, −11	−0.003882426
	12, −12	0.002186714
	13, −13	0.001882120
	14, −14	−0.001103748
	15, −15	−0.000927187
	16, −16	0.000559952
	17, −17	0.000462093
	18, −18	−0.000285414
	19, −19	−0.000232304
	20, −20	0.000146098

Note: The coefficients below 10^{-4} are not given.

What about wavelets? QUI

- Orthonormal wavelets carry the details needed to increase the resolution of a signal approximation.
- The approximations of f at scales 2^j and $2^{(j+1)}$ are respectively equal to its orthogonal projections in V_j and V_{j+1}
- We know that V_{j+1} is included in V_j
- Let W_{j+1} be the *orthogonal complement* of V_{j+1} in V_j

$$V_{j-1} = V_j \oplus W_j$$

- The orthogonal projection of f on V_j can be decomposed as follows

$$PV_{j-1}f = PV_j f + PW_j f$$

- The complement $PW_{j+1}f$ provides the details that appear at scale j but disappear at the next coarser scale.
- Next theorem will show that basis for W_j can be constructed by scaling and translating a wavelet ψ

Corresponding orthogonal wavelet family

- Theorem 7.3 [Mallat&Meyer]

Let φ be a scaling function and h the corresponding CMF. Let Ψ be such that

$$\hat{\Psi}(\omega) = \frac{1}{\sqrt{2}}\hat{g}\left(\frac{\omega}{2}\right)\hat{\Phi}\left(\frac{\omega}{2}\right)$$

with

$$\hat{g}(\omega) = e^{-j\omega}\hat{h}^{*}(\omega+\pi)$$

The frequency modulation changes the low-pass filter h[n] to the band-pass filter g[n]
The phase modulation introduces a unitary step delay

Let us denote

$$\psi_{j,n}(t) = \frac{1}{\sqrt{2^j}}\psi\left(\frac{t-2^j n}{2^j}\right)$$

For any scale, $\{\Psi_{j,n}\}_{j \text{ in } Z}$ is an orthonormal basis for W_j.

For all j, $\{\psi_{j,n}\}_{j,n\in\mathbb{Z}^2}$ is an orthonormal basis for L^2.

Signal domain

$$\hat{g}(\omega) = e^{-j\omega}\hat{h}^{*}(\omega+\pi) \leftrightarrow g(z) = z^{-1}h(-z^{-1}) \leftrightarrow g[n] = (-1)^{1-n}h[1-n]$$

Proof

$$h(\omega) \to h[n]$$

$$h(\omega+\pi) \to (-1)^n h[n]$$

$$h^*(\omega+\pi) = h(-(\omega+\pi)) \to (-1)^{-n} h[-n]$$

$$e^{-j\omega} h^*(\omega+\pi) \to (-1)^{1-n} h[1-n]$$

1. Frequency modulation changes each-other sample sign
2. Phase reverse changes n in –n
3. Phase modulation introduces a unit delay

$$h(\omega) = \ldots + h[0] + h[1]e^{-j\omega} + h[2]e^{-2j\omega} + h[3]e^{-3j\omega} + \ldots..$$

$$h(\omega+\pi) = \ldots + h[0] + h[1]e^{-j(\omega+\pi)} + h[2]e^{-2j(\omega+\pi)} + h[3]e^{-3j(\omega+\pi)} + \ldots.. =$$

$$= \ldots + h[0] - h[1]e^{-\omega} + h[2]e^{-2j\omega} - h[3]e^{-3j\omega} + \ldots.. = \sum_n (-1)^n h[n] e^{-j\omega n}$$

$$h^*(\omega+\pi) = \sum_n (-1)^n h[n] e^{j\omega n} = \sum_n (-1)^{-n} h[-n] e^{-j\omega n} = \Im\left\{(-1)^{-n} h[-n]\right\}$$

$$e^{-j\omega} h^*(\omega+\pi) \to (-1)^{1-n} h[1-n]$$

Corresponding orthogonal wavelet family

– Lemma 7.1. The family $\{\psi_{j,n}\}_{n \text{ in } Z}$ is an orthonormal basis for W_j iif

$$\left|\hat{g}(\omega)\right|^2 + \left|\hat{g}(\omega+\pi)\right|^2 = 2$$

and

$$\hat{g}(\omega)\hat{h}^*(\omega) + \hat{g}(\omega+\pi)\hat{h}^*(\omega+\pi) = 2$$

– Furthermore

$$V_{j-1} = V_j + W_j \rightarrow \frac{1}{\sqrt{2}}\psi\left(\frac{t}{2}\right) \in W_1 \subset V_0$$

since $\{\varphi(t-n)\}_{n\in Z}$ is an ortonormal basis of $V_0 \rightarrow$

$$\frac{1}{\sqrt{2}}\psi\left(\frac{t}{2}\right) = \sum_{n=-\infty}^{+\infty} g[n]\varphi(t-n) \qquad \text{with}$$

$$g[n] = \left\langle \frac{1}{\sqrt{2}}\psi\left(\frac{t}{2}\right), \varphi(t-n) \right\rangle$$

- The orthogonal wavelets **carry the details lost going from scale j to scale j+1**
- Wavelets are the **basis functions for W_j**
- The details at scale j are obtained by **projecting the signal onto the wavelet family $\psi_{j,n}$**

Summary

- Approximation function at scale 2^j: $$P_{V_j} f = \sum_{n=-\infty}^{+\infty} \langle f, \varphi_{j,n} \rangle \varphi_{j,n}$$

- Details ("residual" functions) at scale 2^j: $$P_{W_j} f = \sum_{n=-\infty}^{+\infty} \langle f, \psi_{j,n} \rangle \psi_{j,n}$$

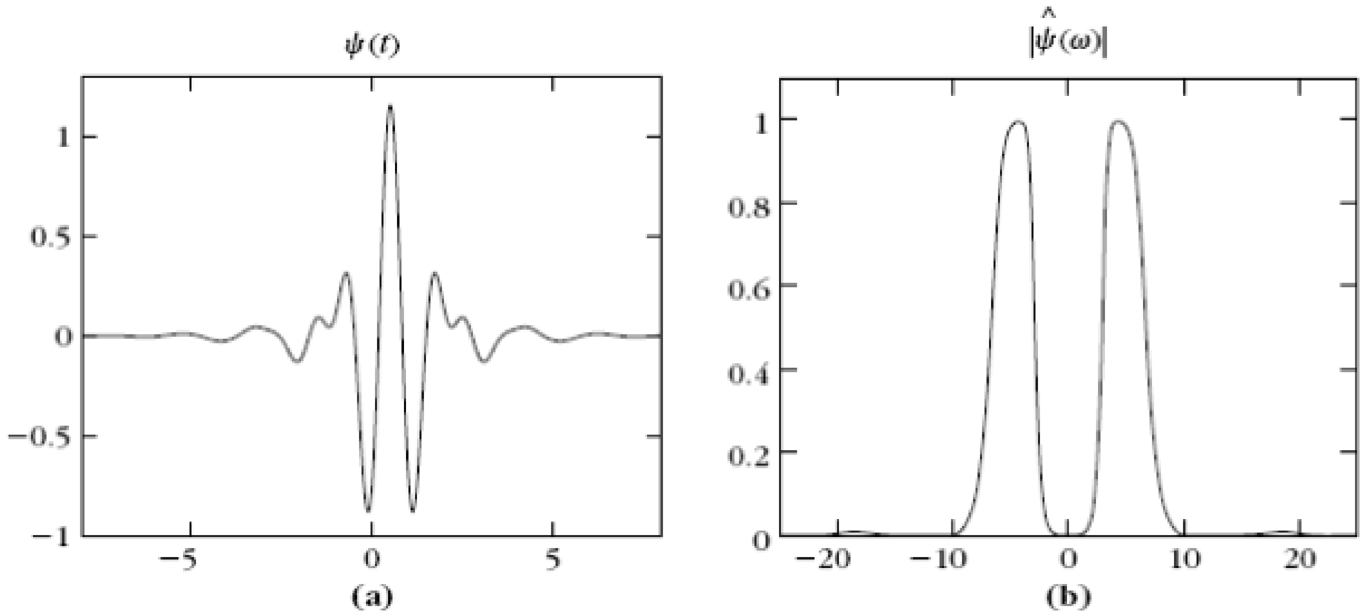
- Wavelet representation: $$f = \sum_{j=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \langle f, \psi_{j,n} \rangle \psi_{j,n}$$

- If the basis is orthogonal, the scaling function characterizes the multi-resolution completely

Scaling function φ -> h[n] -> g[n] -> wavelet ψ

Example

- Battle-Lemarié cubic spline wavelet and its spectrum



Example

- Property: for any ψ that can generate an orthonormal family, one can verify that

$$\forall \omega \in \mathbb{R} - \{0\}, \quad \sum_{j=-\infty}^{+\infty} \left| \hat{\psi}\left(2^j \omega\right) \right|^2 = 1$$

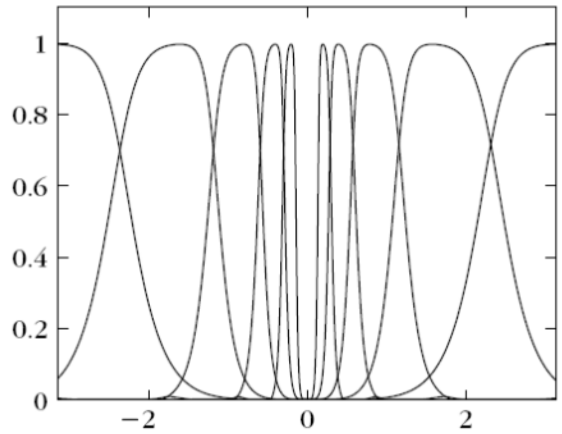


FIGURE 7.6

Graph of $|\hat{\psi}(2^j\omega)|^2$ for the cubic spline Battle-Lemarié wavelet, with $1 \leq j \leq 5$ and $\omega \in [-\pi, \pi]$.

Example

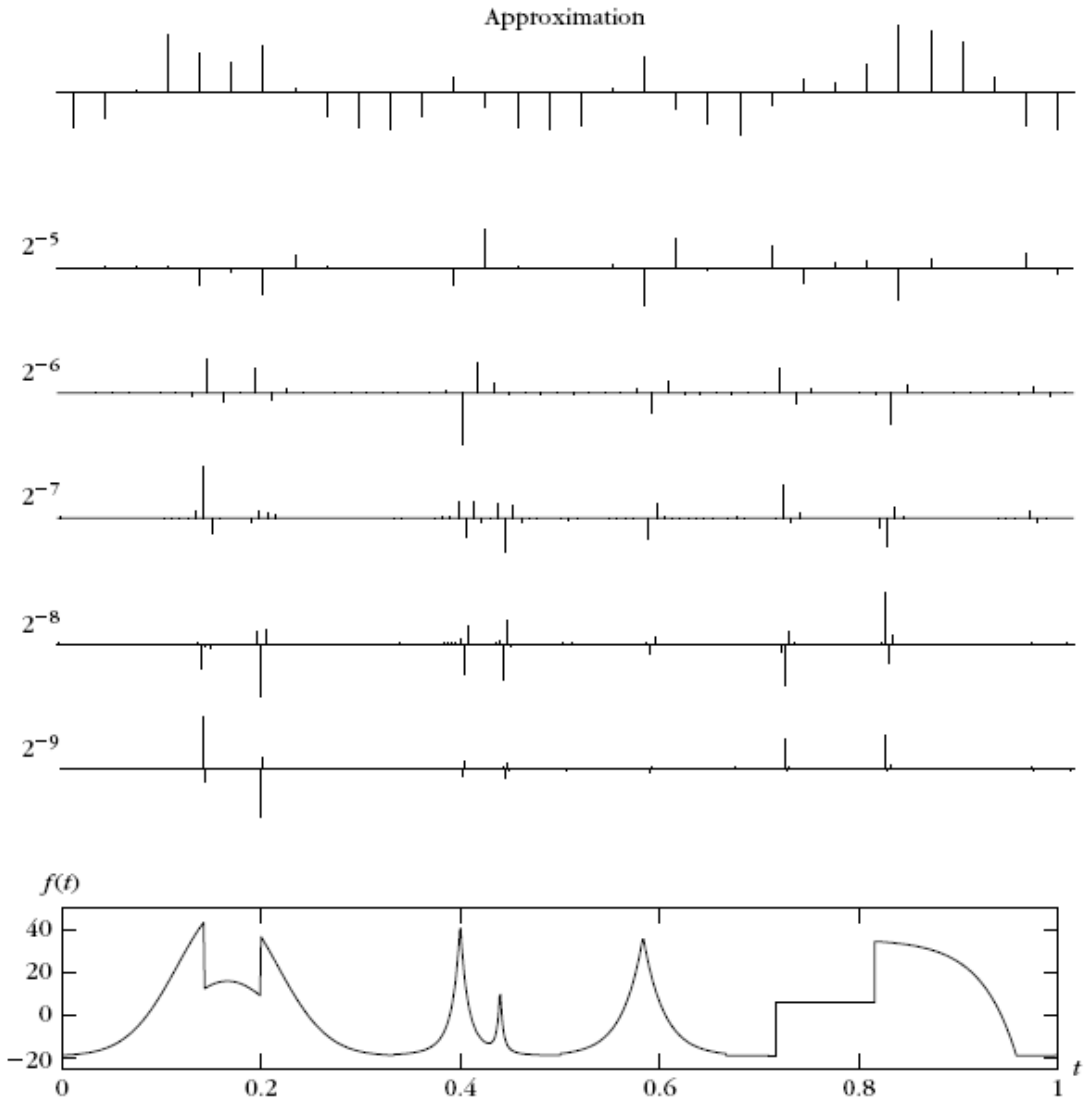


FIGURE 7.7

Wavelet coefficients $d_j[n] = \langle f, \psi_{j,n} \rangle$ calculated at scales 2^j with the cubic spline wavelet. Each up or down Dirac gives the amplitude of a positive or negative wavelet coefficient. At the top is the remaining coarse-signal approximation $a_J[n] = \langle f, \phi_{J,n} \rangle$ for $J = -5$.

Warning

- Each CMF generates a wavelet orthonormal bases
- Does any **wavelet orthonormal bases** correspond to a multiresolution approximation and CMF? It depends on the support:
  - **If ψ has compact support than it corresponds to a multiresolution approximation** [Lemarié]
  - However, there exists "pathological" wavelets that decay as $|t|^{-1}$ that cannot be derived from any multiresolution approximation

Classes of wavelet bases

- Wavelets are interesting for applications for their ability to represent signals with **few non zero coefficients**
- The best basis for an application is the one that maximizes the number of zero or close to zero coefficients. This depends on
  - The regularity of f
  - The number of vanishing moments of the wavelet
  - The size of its support
- The constraints on the wavelet translate to **design rules for the filter g[n], thus h[n]**
  - Thus, we need conditions on $\hat{h}(\omega)$

Wavelet properties

- Vanishing moments
  - The wavelet has p vanishing moments if

$$\int_{-\infty}^{+\infty} t^k \psi(t)dt = 0 \qquad \text{for} \qquad 0 \le k < p \qquad (3)$$

  - The number of vanishing moments is equal to the multiplicity of zeros of h^(ω) in π or, equivalently, the number of vanishing derivatives of ^ψ in zero

- Theorem 7.4: Vanishing moments

Let φ and ψ be a scaling function and a wavelet that generate an orthonormal basis. Suppose that $|\psi(t)|=O((1+t^2)^{-p/2-1})$ *and* $|\varphi(t)|=O((1+t^2)^{-p/2-1})$. *The four following statements are equivalent*

1. *The wavelet ψ has p vanishing moments*
2. $\hat{\psi}(\omega)$ *and its first p-1 derivatives are zero at ω=0*
3. *^h(ω) and its first p-1 derivatives are zero at ω=π*
4. *for any* $0 \le k < p$ $\quad q_k(t) = \sum_{n=-\infty}^{+\infty} n^k \varphi(t-n)$ is a polynomial of degree k

hints of the proof

- Point 1. The decay of $|\varphi(t)|$ *and* $|\psi(t)|$ *imply that* $|\hat{\varphi}(\omega)|$ *and* $|\hat{\psi}(\omega)|$ *are p-times differentiable*
- Point 2. The k-th order derivative of $\hat{\psi}^{(k)}(\omega)$ is the F-transform of $(-it)^k \psi(t)$
  thus

$$\hat{\psi}^{(k)}(0) = \int_{-\infty}^{+\infty} (-it)^k \psi(t)\, dt. \qquad (4)$$

  (4) is equivalent to (3), which proves 2.
- Point 3.

$$\hat{\Psi}(\omega) = \frac{1}{\sqrt{2}} \hat{g}\left(\frac{\omega}{2}\right) \hat{\Phi}\left(\frac{\omega}{2}\right) \qquad \hat{g}(\omega) = e^{-j\omega} \hat{h}^{*}(\omega + \pi) \qquad \text{thus}$$

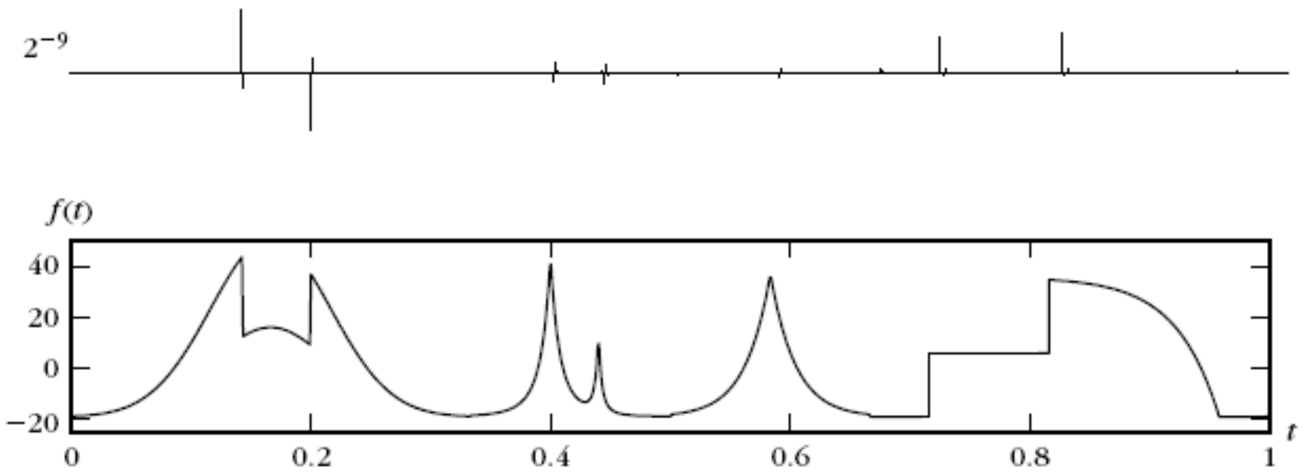
$$\hat{\Psi}(2\omega) = \frac{1}{\sqrt{2}} \hat{g}(\omega) \hat{\Phi}(\omega) = e^{-i\omega} \hat{h}^{*}(\omega + \pi) \hat{\Phi}(\omega)$$

  since $\hat{\Phi}(0) \neq 0$ by differentiating this expression we prove that 2. is equivalent to 3.
- Finally, it is proved that 4. is equivalent to 1. and viceversa.

hints of the proof

Let us now prove that (4) implies (1). Since ψ is orthogonal to $\{\phi(t-n)\}_{n\in\mathbb{Z}}$, it is also orthogonal to the polynomials q_k for $0 \leqslant k < p$. This family of polynomials is a basis of the space of polynomials of degree at most $p-1$. Thus, ψ is orthogonal to any polynomial of degree $p-1$ and in particular to t^k for $0 \leqslant k < p$. This means that ψ has p vanishing moments.

A wavelet with p vanishing moments **kills polynomials up to degree p**



Wavelet properties

- Support
  - The larger the support, the more the singularities will spread along scales: it should be **as short as possible**

  BUT a wavelet with *p* vanishing moments will have a support at least *2p-1* -> trade-off

- **Proposition 7.2:** *Compact Support*. The scaling function has a compact support if and only if *h* has a compact support and their supports are equal. If the support of *h* and φ is $[N_1,N_2]$, then the support of ψ is $[(N_1-N_2+1)/2, (N_1-N_2+1)/2]$.

$$\hat{\Psi}(\omega) = \frac{1}{\sqrt{2}}\hat{g}\left(\frac{\omega}{2}\right)\hat{\Phi}\left(\frac{\omega}{2}\right)$$

$$\hat{g}(\omega) = e^{-j\omega}\hat{h}^*(\omega+\pi)$$

⇨

Product in Fourier ➔convolution in time
g[n] has the same support of h[n]

The relation between the supports of the wavelet and the basis function comes from the properties of the convolution applied to the shrinked function (the support of ψ(t/2) is the same as that of φ(t) thus the support of ψ(t), that is a shrinked version, is the half.

Proof

Proof[1]. If ϕ has a compact support, since

$$h[n] = \frac{1}{\sqrt{2}} \left\langle \phi\left(\frac{t}{2}\right), \phi(t-n) \right\rangle,$$

we derive that h also has a compact support. Conversely, the scaling function satisfies

$$\frac{1}{\sqrt{2}} \phi\left(\frac{t}{2}\right) = \sum_{n=-\infty}^{+\infty} h[n]\, \phi(t-n). \tag{7.79}$$

If h has a compact support then one can prove [144] that ϕ has a compact support. The proof is not reproduced here.

To relate the support of ϕ and h, we suppose that $h[n]$ is non-zero for $N_1 \leq n \leq N_2$ and that ϕ has a compact support $[K_1, K_2]$. The support of $\phi(t/2)$ is $[2K_1, 2K_2]$. The sum at the right of (7.79) is a function whose support is $[N_1 + K_1, N_2 + K_2]$. The equality proves that the support of ϕ is $[K_1, K_2] = [N_1, N_2]$.

Support of the wavelet

Let us recall from (7.73) and (7.72) that

$$\frac{1}{\sqrt{2}}\psi\left(\frac{t}{2}\right) = \sum_{n=-\infty}^{+\infty} g[n]\,\phi(t-n) = \sum_{n=-\infty}^{+\infty} (-1)^{1-n}\,h[1-n]\,\phi(t-n).$$

If the supports of ϕ and h are equal to $[N_1, N_2]$, the sum in the right-hand side has a support equal to $[N_1 - N_2 + 1, N_2 - N_1 + 1]$. Hence ψ has a support equal to $[(N_1 - N_2 + 1)/2, (N_2 - N_1 + 1)/2]$. ■

If h has a finite impulse response in $[N_1, N_2]$, Proposition 7.2 proves that ψ has a support of size $N_2 - N_1$ centered at $1/2$. To minimize the size of the support, we must synthesize conjugate mirror filters with as few non-zero coefficients as possible.

Properties

- Support
  - To minimize the size of the support of the wavelet, we must synthesize conjugate mirror filters with *as few nonzero coefficients as possible*
  - However, the constraints imposed on orthogonal wavelets imply that if *the wavelet* has p vanishing moments, then its support is at least of size $2p-1$ → trade off
  - **Daubechies wavelets** are optimal in the sense that they have a **minimum size support for a given number of vanishing moments**
    - If f has **few isolated singularities** and is very regular between singularities, we must choose a wavelet with **many** vanishing moments to produce a large number of small wavelet coefficients $\langle f, \psi_{j,n} \rangle$. If the density of singularities increases, it might be better to decrease the size of its support at the cost of reducing the number of vanishing moments. Indeed, **wavelets that overlap the singularities create high-amplitude coefficients**.
- Regularity
  - The regularity or *smoothness* has mostly a cosmetic influence on the error introduced by *quantizing or thresholding* the coefficients. Such operation introduces a noise which is less visible if it is smooth. Better quality is reached with smoother wavelets
    - The Haar wavelet is not a good choice

Popular wavelet families

- Shannon, Meyer, Haar, and Battle-Lemarié Wavelets
  - Starting point

$$\hat{\Psi}(\omega) = \frac{1}{\sqrt{2}}\hat{g}\left(\frac{\omega}{2}\right)\hat{\Phi}\left(\frac{\omega}{2}\right) \qquad \hat{g}(\omega) = e^{-j\omega}\hat{h}^{*}(\omega+\pi)$$

$$\hat{\Psi}(2\omega) = \frac{1}{\sqrt{2}}\hat{g}(\omega)\hat{\Phi}(\omega) = e^{-i\omega}\hat{h}^{*}(\omega+\pi)\hat{\Phi}(\omega)$$

$$\Downarrow$$

$$\hat{\psi}(\omega) = \frac{1}{\sqrt{2}}\hat{g}\left(\frac{\omega}{2}\right)\hat{\phi}\left(\frac{\omega}{2}\right) = \frac{1}{\sqrt{2}}\exp\left(\frac{-i\omega}{2}\right)\hat{h}^{*}\left(\frac{\omega}{2}+\pi\right)\hat{\phi}\left(\frac{\omega}{2}\right). \qquad (7.82)$$

Shannon wavelets: real and complex

Shannon Wavelet

The Shannon wavelet is constructed from the Shannon multiresolution approximation, which approximates functions by their restriction to low-frequency intervals. It corresponds to $\hat{\phi} = \mathbf{1}_{[-\pi,\pi]}$ and $\hat{h}(\omega) = \sqrt{2}\,\mathbf{1}_{[-\pi/2,\pi/2]}(\omega)$ for $\omega \in [-\pi, \pi]$. We derive from (7.82) that

$$\hat{\psi}(\omega) = \begin{cases} \exp(-i\omega/2) & \text{if } \omega \in [-2\pi, -\pi] \cup [\pi, 2\pi] \\ 0 & \text{otherwise,} \end{cases} \qquad \textbf{(7.83)}$$

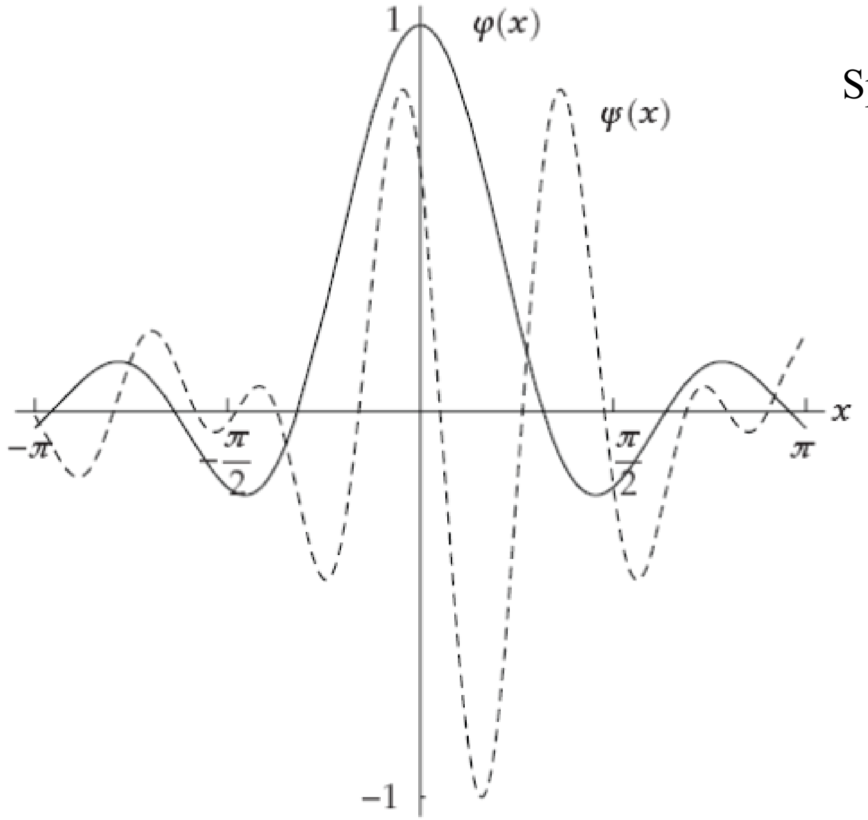
and thus,

Real SW
$$\psi(t) = \frac{\sin 2\pi(t-1/2)}{2\pi(t-1/2)} - \frac{\sin \pi(t-1/2)}{\pi(t-1/2)}.$$

This wavelet is $\mathbf{C}^\infty$ but has a slow asymptotic time decay. Since $\hat{\psi}(\omega)$ is zero in the neighborhood of $\omega = 0$, all its derivatives are zero at $\omega = 0$. Thus, Theorem 7.4 implies that ψ has an infinite number of vanishing moments.

Since $\hat{\psi}(\omega)$ has a compact support we know that $\psi(t)$ is $\mathbf{C}^\infty$. However, $|\psi(t)|$ decays only like $|t|^{-1}$ at infinity because $\hat{\psi}(\omega)$ is discontinuous at $\pm\pi$ and $\pm 2\pi$.

Real Shannon wavelets

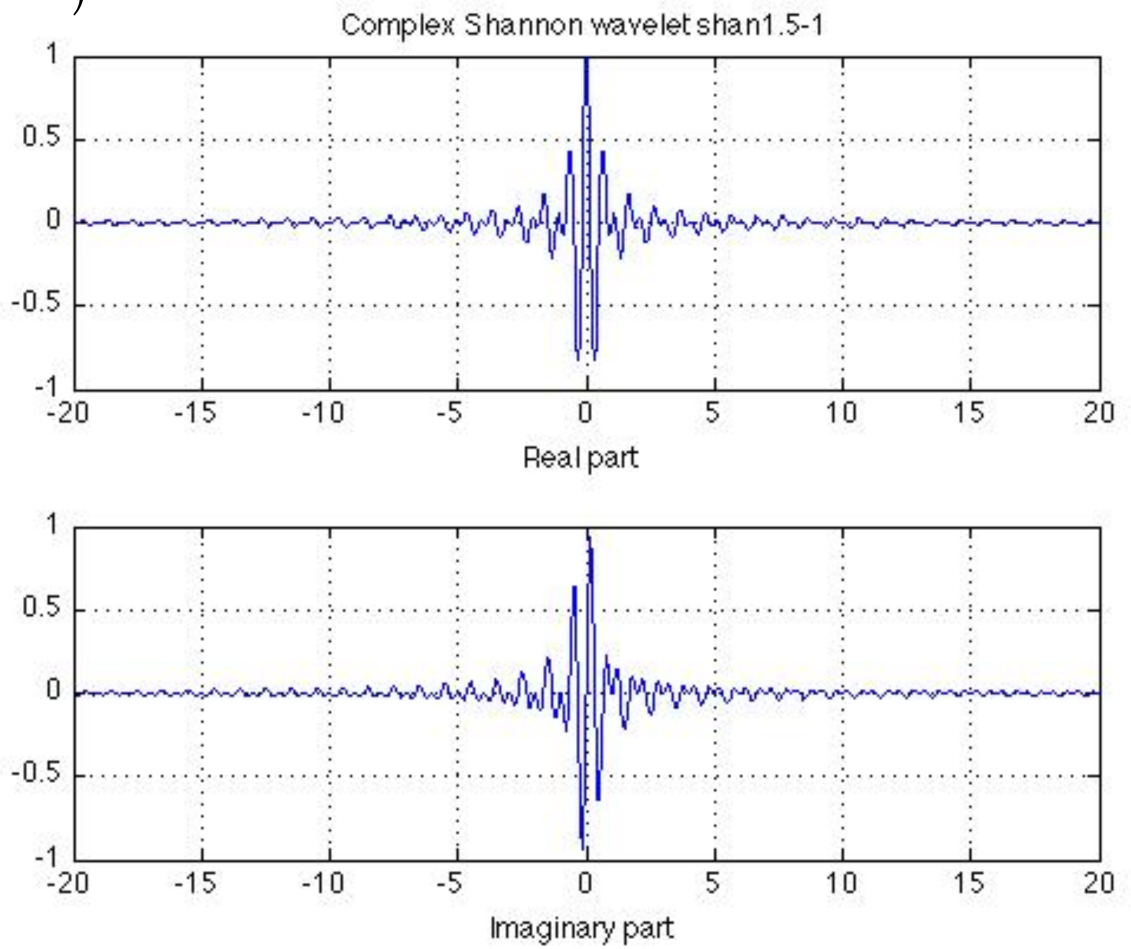


Spatial domain

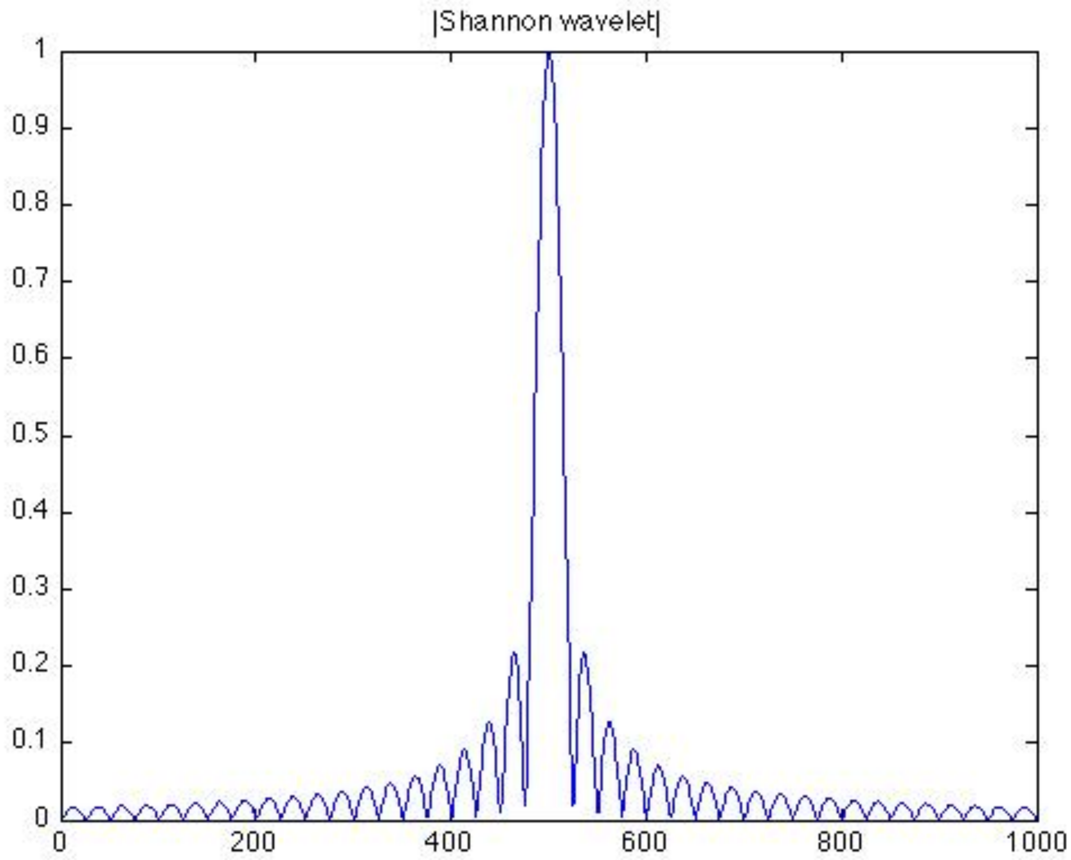
Shannon scaling function (continuous) and wavelet (dashed) lines.

Complex Shannon wavelet

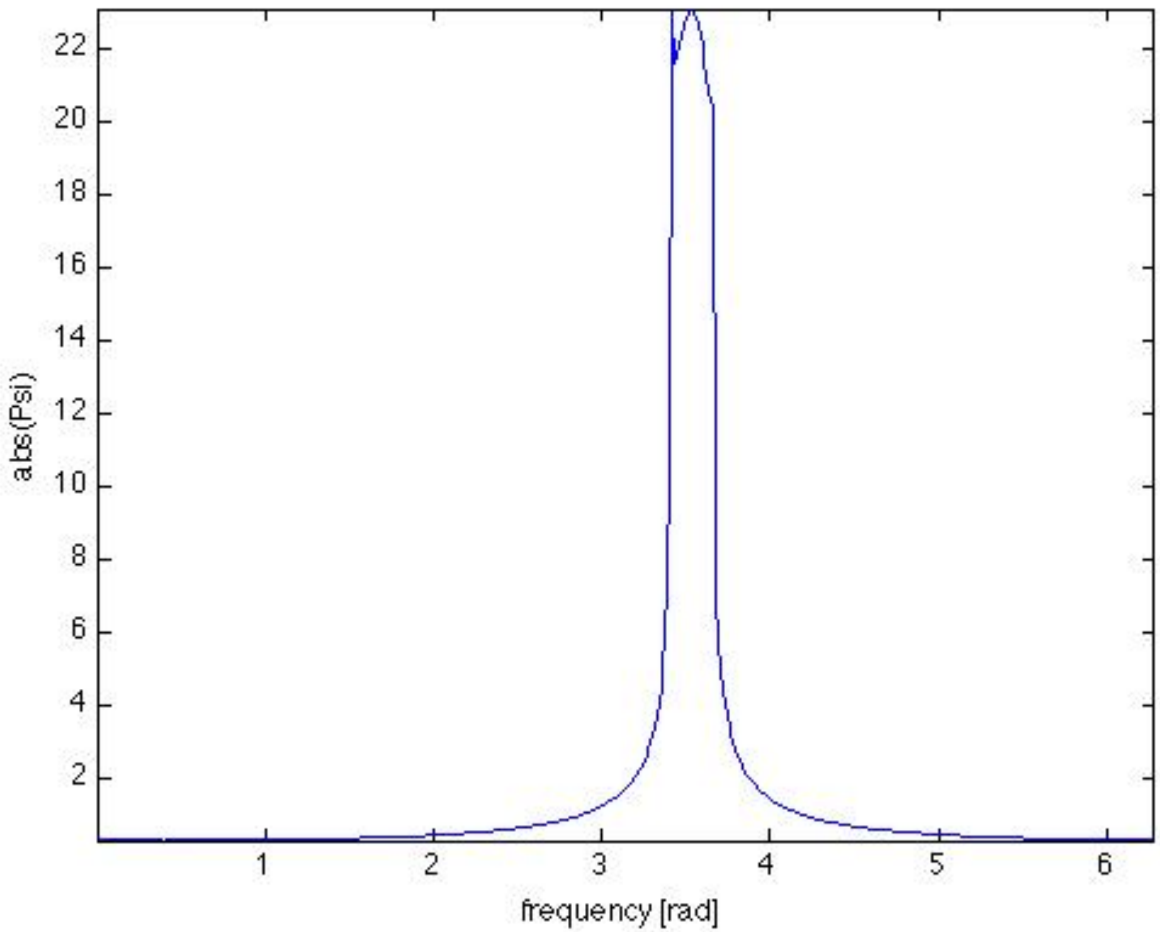
$$\psi(t) = \frac{\sin(t)}{t} \exp(-j2\pi t)$$



Shannon wavelet



Shannon wavelet



Meyer wavelets

Meyer Wavelets

A Meyer wavelet [375] is a frequency band-limited function that has a Fourier transform that is smooth, unlike the Fourier transform of the Shannon wavelet. This smoothness provides a much faster asymptotic decay in time. These wavelets are constructed with conjugate mirror filters $\hat{h}(\omega)$ that are $\mathbf{C}^n$ and satisfy

$$\hat{h}(\omega) = \begin{cases} \sqrt{2} & \text{if } \omega \in [-\pi/3, \pi/3] \\ 0 & \text{if } \omega \in [-\pi, -2\pi/3] \cup [2\pi/3, \pi]. \end{cases} \tag{7.84}$$

The only degree of freedom is the behavior of $\hat{h}(\omega)$ in the transition bands $[-2\pi/3, -\pi/3] \cup [\pi/3, 2\pi/3]$. It must satisfy the quadrature condition

$$|\hat{h}(\omega)|^2 + |\hat{h}(\omega + \pi)|^2 = 2, \tag{7.85}$$

and to obtain $\mathbf{C}^n$ junctions at $|\omega| = \pi/3$ and $|\omega| = 2\pi/3$, the n first derivatives must vanish at these abscissa. One can construct such functions that are $\mathbf{C}^\infty$.

The scaling function $\hat{\phi}(\omega) = \prod_{p=1}^{+\infty} 2^{-1/2}\, \hat{h}(2^{-p}\omega)$ has a compact support and one can verify that

$$\hat{\phi}(\omega) = \begin{cases} 2^{-1/2}\, \hat{h}(\omega/2) & \text{if } |\omega| \leqslant 4\pi/3 \\ 0 & \text{if } |\omega| > 4\pi/3. \end{cases} \tag{7.86}$$

Meyer wavelets

The resulting wavelet (7.82) is

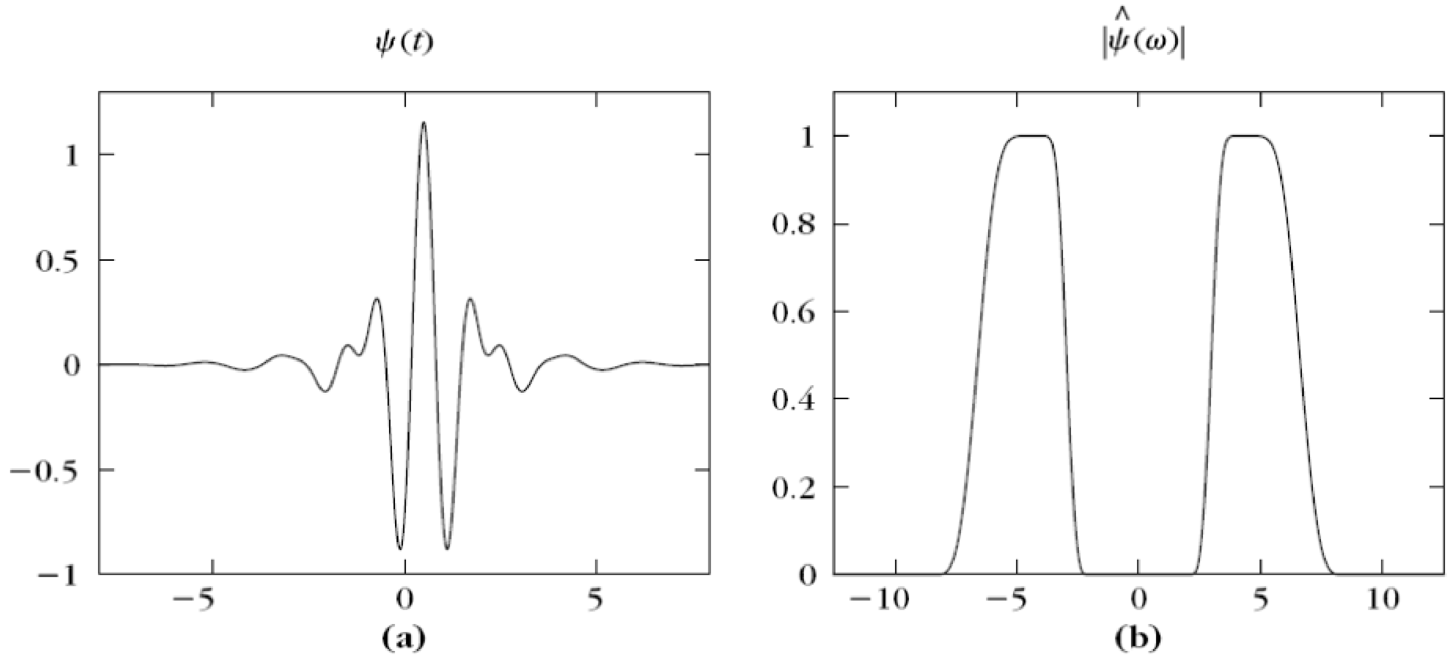
$$\hat{\psi}(\omega) = \begin{cases} 0 & \text{if } |\omega| \leqslant 2\pi/3 \\ 2^{-1/2}\,\hat{g}(\omega/2) & \text{if } 2\pi/3 \leqslant |\omega| \leqslant 4\pi/3 \\ 2^{-1/2}\,\exp(-i\omega/2)\,\hat{h}(\omega/4) & \text{if } 4\pi/3 \leqslant |\omega| \leqslant 8\pi/3 \\ 0 & \text{if } |\omega| > 8\pi/3. \end{cases} \quad \textbf{(7.87)}$$

The functions ϕ and ψ are $\mathbf{C}^\infty$ because their Fourier transforms have a compact support. Since $\hat{\psi}(\omega) = 0$ in the neighborhood of $\omega = 0$, all its derivatives are zero at $\omega = 0$, which proves that ψ has an infinite number of vanishing moments.

If $\hat{h}$ is $\mathbf{C}^n$, then $\hat{\psi}$ and $\hat{\phi}$ are also $\mathbf{C}^n$. The discontinuities of the $(n+1)^{\text{th}}$ derivative of $\hat{h}$ are generally at the junction of the transition band $|\omega| = \pi/3$, $2\pi/3$, in which case one can show that there exists A such that

$$|\phi(t)| \leqslant A\,(1+|t|)^{-n-1} \quad \text{and} \quad |\psi(t)| \leqslant A\,(1+|t|)^{-n-1}.$$

Meyer wavelet: example



Haar wavelets

Haar Wavelets

The Haar basis is obtained with a multiresolution of piecewise constant functions. The scaling function is $\phi = \mathbf{1}_{[0,1]}$. The filter $h[n]$ given in (7.46) has two nonzero coefficients equal to $2^{-1/2}$ at $n = 0$ and $n = 1$. Thus,

$$\frac{1}{\sqrt{2}}\psi\left(\frac{t}{2}\right) = \sum_{n=-\infty}^{+\infty} (-1)^{1-n}\, h[1-n]\,\phi(t-n) = \frac{1}{\sqrt{2}}\Big(\phi(t-1) - \phi(t)\Big),$$

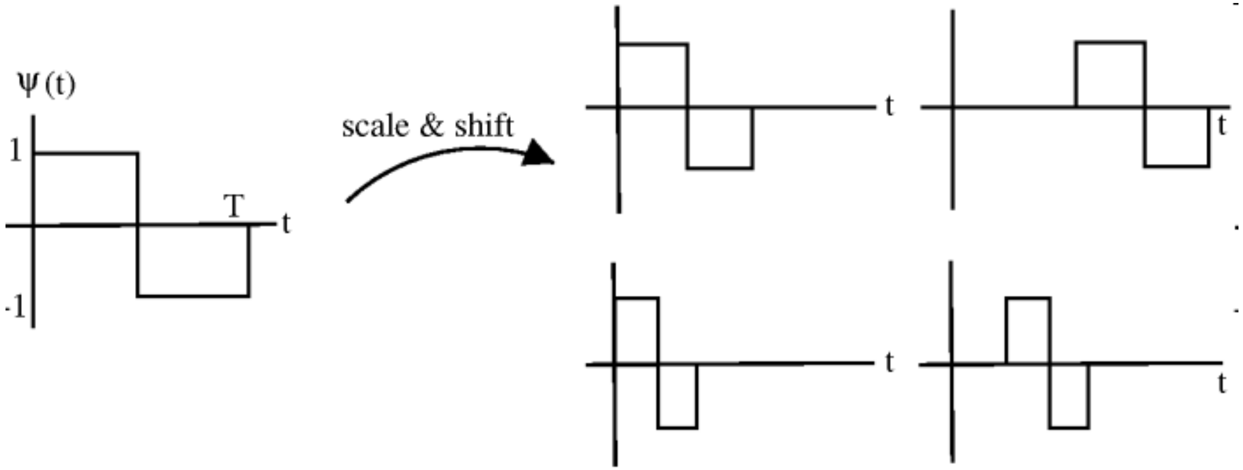
so

$$\psi(t) = \begin{cases} -1 & \text{if } 0 \leqslant t < 1/2 \\ 1 & \text{if } 1/2 \leqslant t < 1 \\ 0 & \text{otherwise.} \end{cases} \tag{7.90}$$

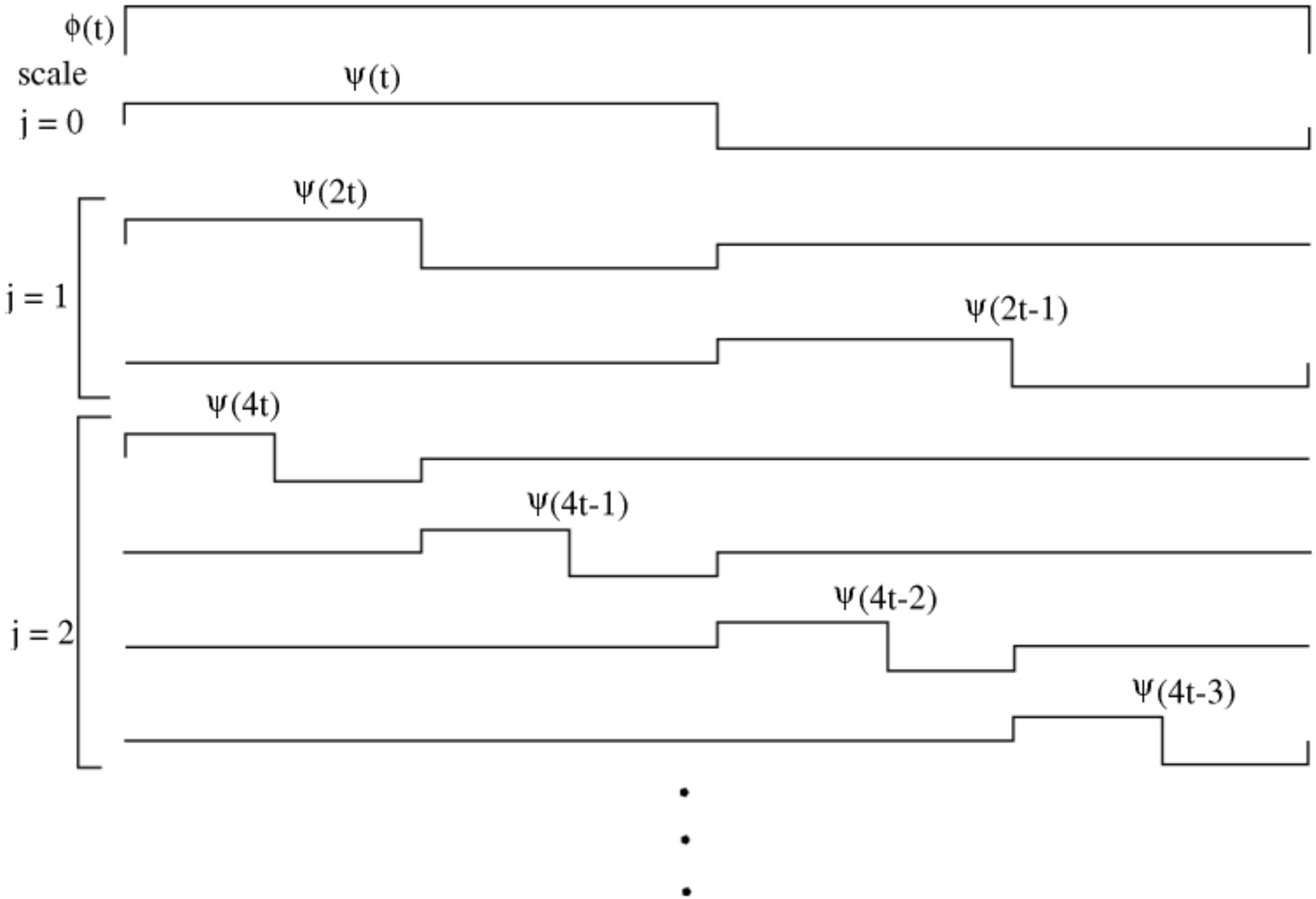
The Haar wavelet has the shortest support among all orthogonal wavelets. It is not well adapted to approximating smooth functions because it has only one vanishing moment.

reminder: $$\frac{1}{\sqrt{2}}\psi\left(\frac{t}{2}\right) = \sum_{n=-\infty}^{+\infty} g[n]\,\phi(t-n) = \sum_{n=-\infty}^{+\infty} (-1)^{1-n}\, h[1-n]\,\phi(t-n).$$

Haar wavelets



Haar wavelets



Battle-Lemarié wavelets

Battle-Lemarié Wavelets

Polynomial spline wavelets introduced by Battle [99] and Lemarié [345] are computed from spline multiresolution approximations. The expressions of $\hat{\phi}(\omega)$ and $\hat{h}(\omega)$ are given, respectively, by (7.18) and (7.48). For splines of degree m, $\hat{h}(\omega)$ and its first m derivatives are zero at $\omega = \pi$. Theorem 7.4 derives that ψ has $m+1$ vanishing moments. It follows from (7.82) that

$$\hat{\psi}(\omega) = \frac{\exp(-i\omega/2)}{\omega^{m+1}} \sqrt{\frac{S_{2m+2}(\omega/2+\pi)}{S_{2m+2}(\omega)\, S_{2m+2}(\omega/2)}}.$$

This wavelet ψ has an exponential decay. Since it is a polynomial spline of degree m, it is $m-1$ times continuously differentiable. Polynomial spline wavelets are less regular than Meyer wavelets but have faster time asymptotic decay. For m odd, ψ is symmetric about 1/2. For m even, it is antisymmetric about 1/2. Figure 7.5 gives the graph of the cubic spline wavelet ψ corresponding to $m=3$. For $m=1$, Figure 7.9 displays linear splines ϕ and ψ. The properties of these wavelets are further studied in [15, 106, 164].

Battle-Lemarié wavelets

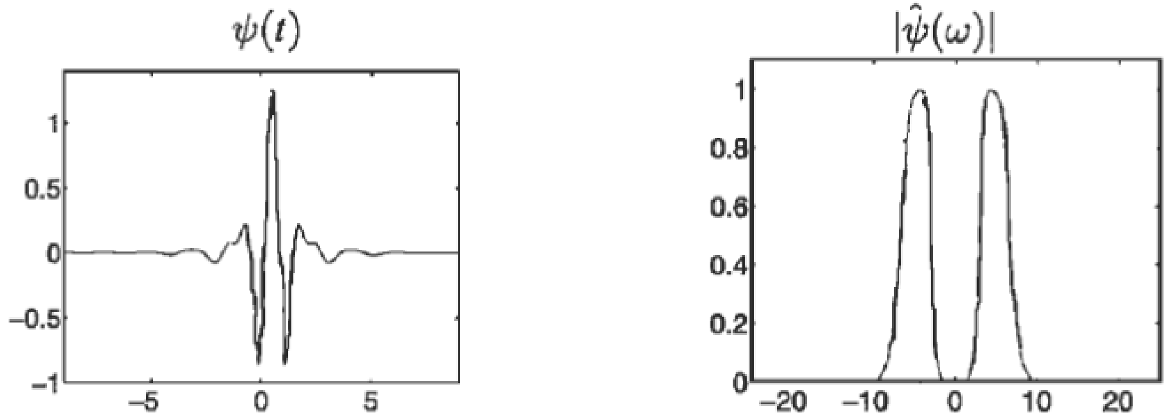


FIGURE 7.5 Battle-Lemarié cubic spline wavelet ψ and its Fourier transform modulus.

Battle-Lemarié: example

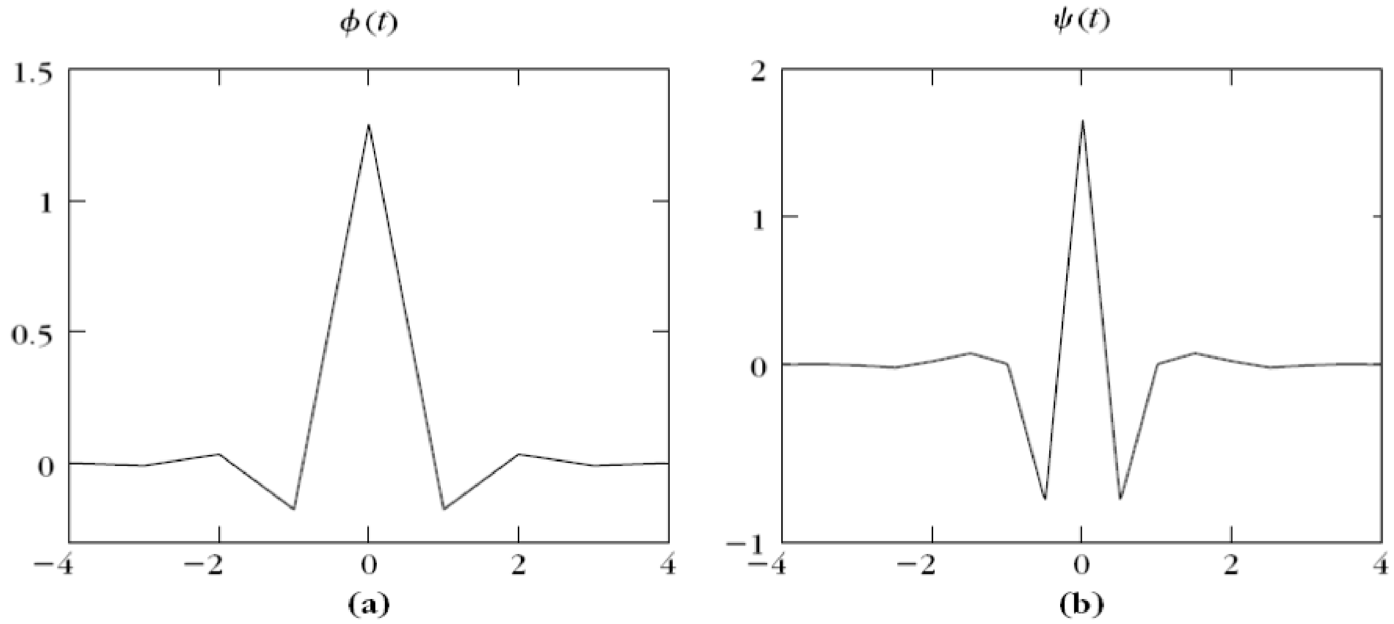


FIGURE 7.9

Linear spline Battle-Lemarié scaling function ϕ **(a)** and wavelet ψ **(b)**.

Daubechies compactly supported wavelets

7.2.3 Daubechies Compactly Supported Wavelets

Daubechies wavelets have a support of minimum size for any given number p of vanishing moments. Theorem 7.5 proves that wavelets of compact support are computed with finite impulse-response conjugate mirror filters h. We consider real causal filters $h[n]$, which implies that $\hat{h}$ is a trigonometric polynomial:

$$\hat{h}(\omega) = \sum_{n=0}^{N-1} h[n]\, e^{-in\omega}.$$

To ensure that ψ has p vanishing moments, Theorem 7.4 shows that $\hat{h}$ must have a zero of order p at $\omega = \pi$. To construct a trigonometric polynomial of minimal size, we factor $(1 + e^{-i\omega})^p$, which is a minimum-size polynomial having p zeros at $\omega = \pi$:

$$\hat{h}(\omega) = \sqrt{2}\left(\frac{1 + e^{-i\omega}}{2}\right)^p R(e^{-i\omega}). \quad \textbf{(7.91)}$$

The difficulty is to design a polynomial $R(e^{-i\omega})$ of minimum degree m such that $\hat{h}$ satisfies

$$|\hat{h}(\omega)|^2 + |\hat{h}(\omega + \pi)|^2 = 2. \quad \textbf{(7.92)}$$

As a result, h has $N = m + p + 1$ nonzero coefficients. Theorem 7.7 by Daubechies [194] proves that the minimum degree of R is $m = p - 1$.

Daubechies compactly supported wavelets

- **Theorem 7.7:** *Daubechies*. A real conjugate mirror filter h, such that $\hat{h}(\omega)$ has p zeroes at π, has at least $2p$ nonzero coefficients. Daubechies filters have $2p$ nonzero coefficients.
- **Theorem 7.9:** *Daubechies*. If ψ is a wavelet with p vanishing moments that generates an orthonormal basis of $\mathbf{L}^2(\mathbb{R})$, then it has a support of size larger than or equal to $2p+1$.

  A Daubechies wavelet has a *minimum-size support* equal to $[-p+1, p]$. The support of the corresponding scaling function is $[0, 2p-1]$.

Daubechies wavelets: example

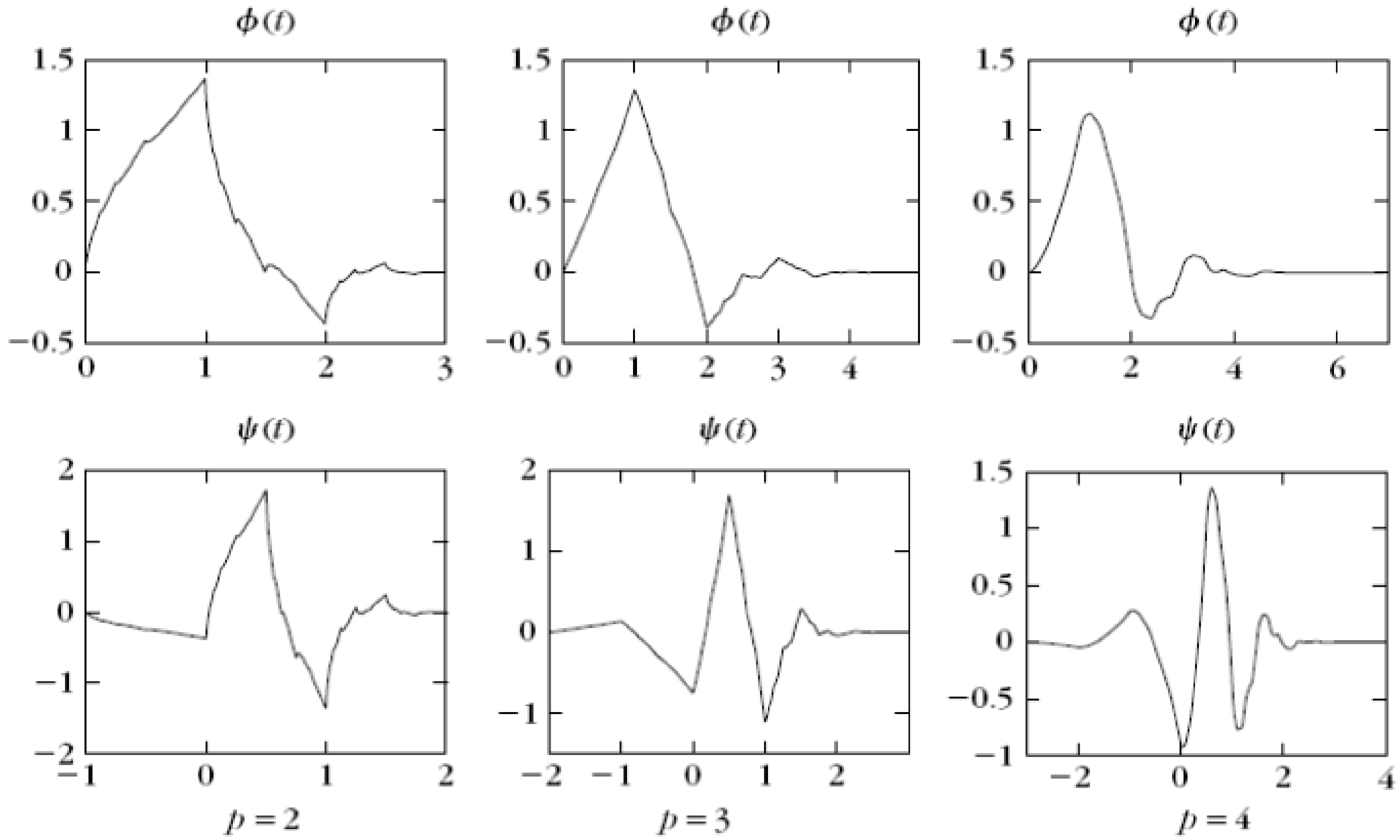


FIGURE 7.10

Daubechies scaling function ϕ and wavelet ψ with p vanishing moments.

Symlets

Symmlets

Daubechies wavelets are very asymmetric because they are constructed by selecting the minimum-phase square root of $Q(e^{-i\omega})$ in (7.97). One can show [51] that filters corresponding to a minimum-phase square root have their energy optimally concentrated near the starting point of their support. Thus, they are highly nonsymmetric, which yields very asymmetric wavelets.

To obtain a symmetric or antisymmetric wavelet, the filter h must be symmetric or antisymmetric with respect to the center of its support, which means that $\hat{h}(\omega)$ has a linear complex phase. Daubechies proved [194] that the Haar filter is the only real compactly supported conjugate mirror filter that has a linear phase. The Daubechies *symmlet* filters are obtained by optimizing the choice of the square root $R(e^{-i\omega})$ of $Q(e^{-i\omega})$ to obtain an almost linear phase. The resulting wavelets still have a minimum support $[-p+1, p]$ with p vanishing moments, but they are more symmetric, as illustrated by Figure 7.11 for $p=8$. The coefficients of the symmlet filters are in WAVELAB. Complex conjugate mirror filters with a compact support and a linear phase can be constructed [352], but they produce complex wavelet coefficients that have real and imaginary parts that are redundant when the signal is real.

Dubechies versus Symlets

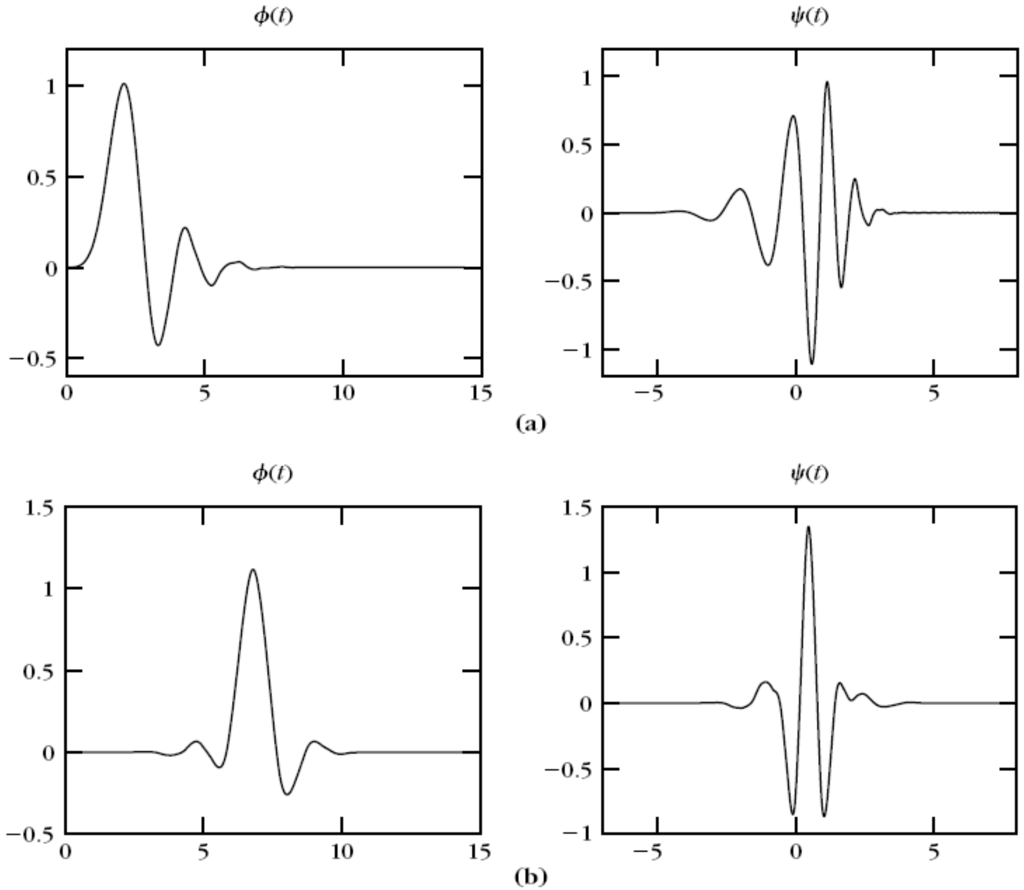


FIGURE 7.11

Daubechies **(a)** and symmlet **(b)** scaling functions and wavelets with $p = 8$ vanishing moments.

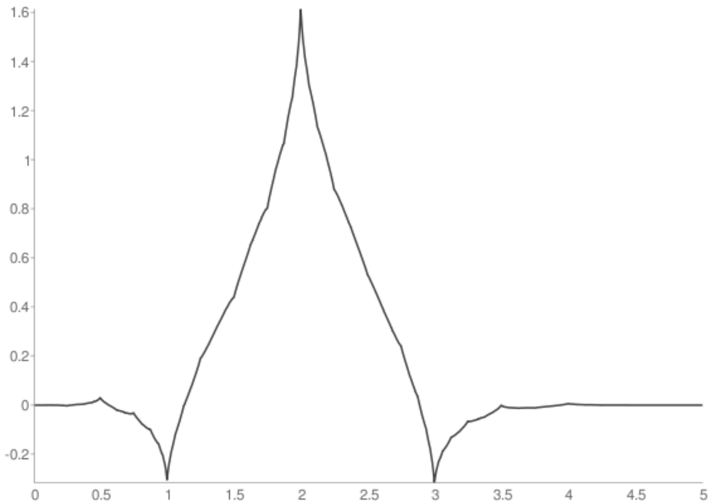
Coiflets

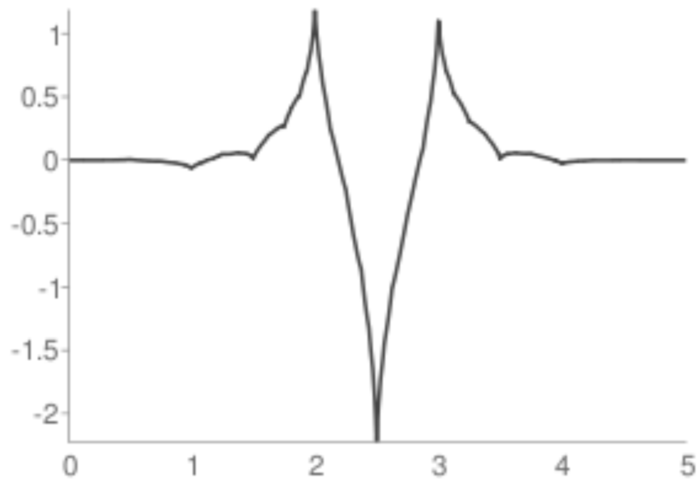
Coiflets

For an application in numerical analysis, Coifman asked Daubechies [194] to construct a family of wavelets ψ that have p vanishing moments and a minimum-size support, with scaling functions that also satisfy

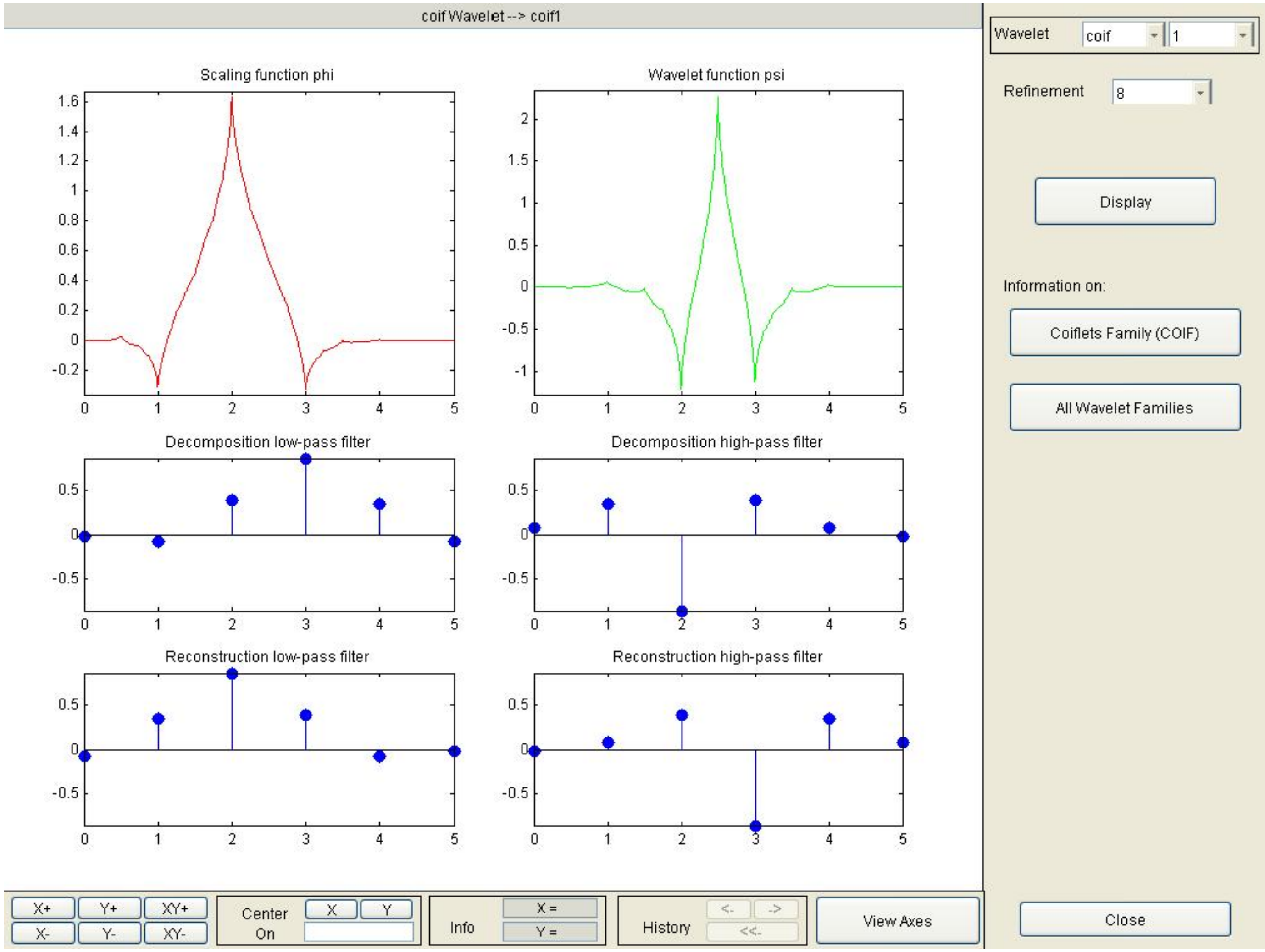
$$\int_{-\infty}^{+\infty} \phi(t)\, dt = 1 \quad \text{and} \quad \int_{-\infty}^{+\infty} t^k\, \phi(t)\, dt = 0 \quad \text{for } 1 \leqslant k < p. \tag{7.99}$$

p=1

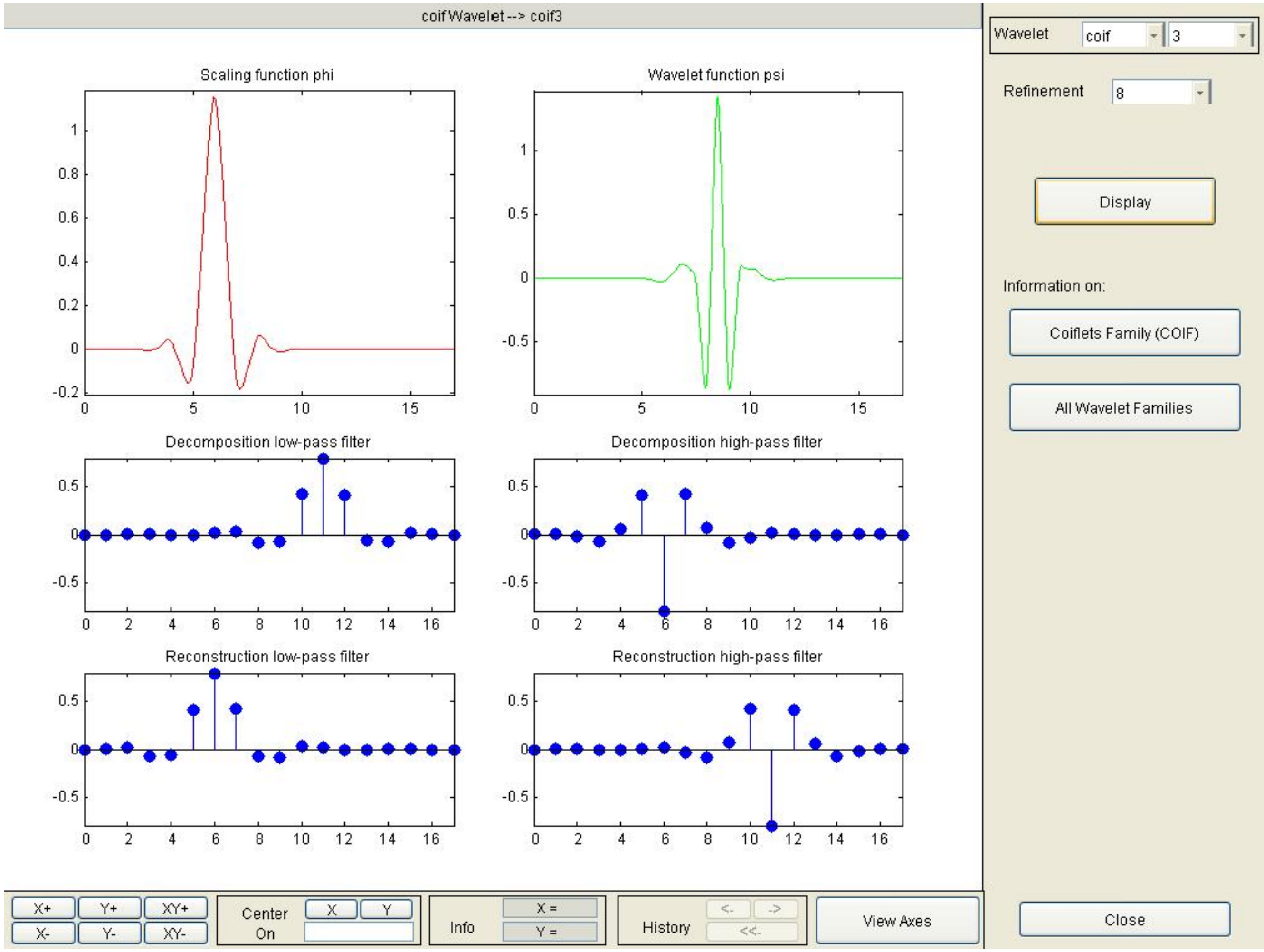




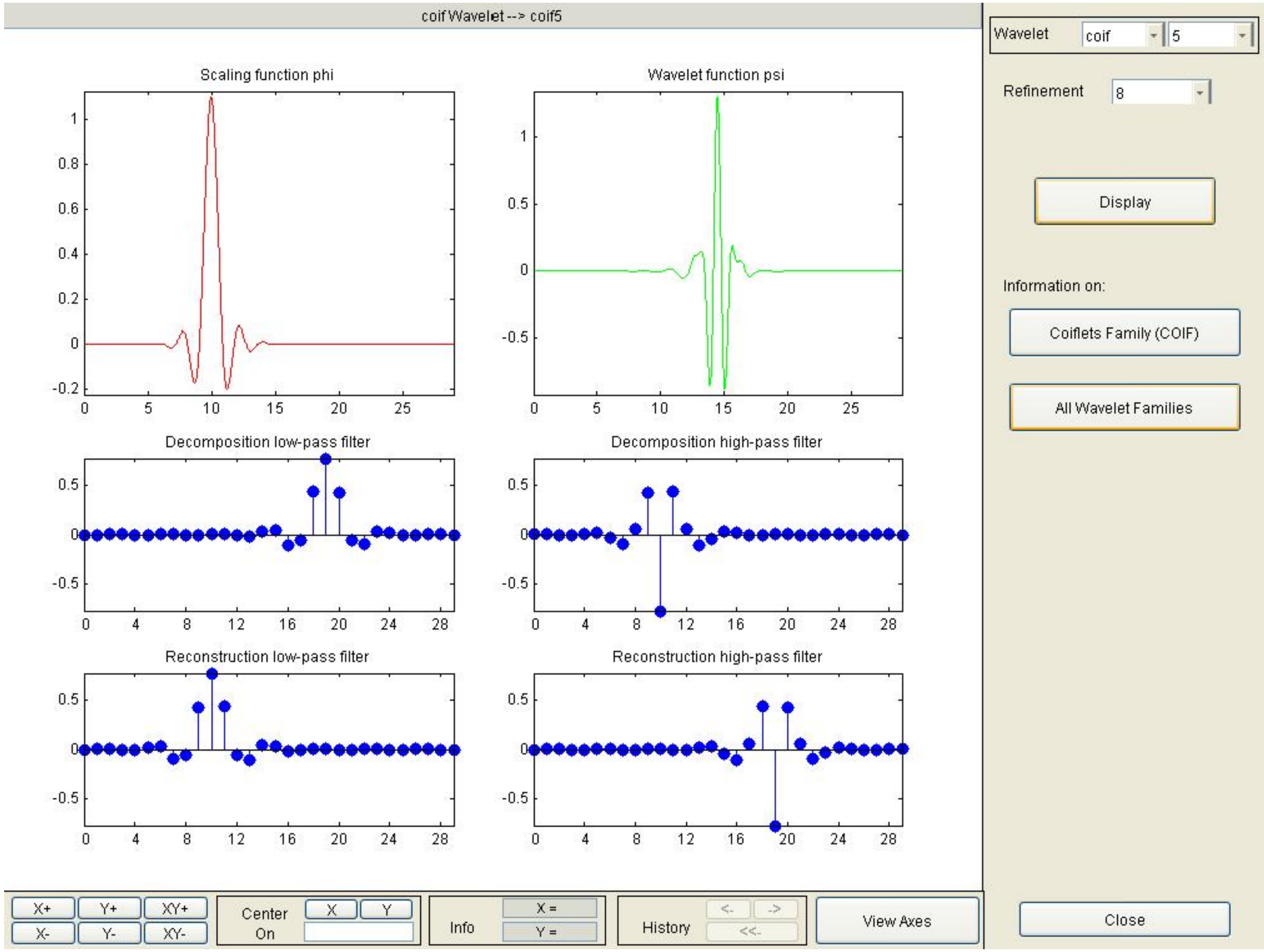
Coiflets, order=1



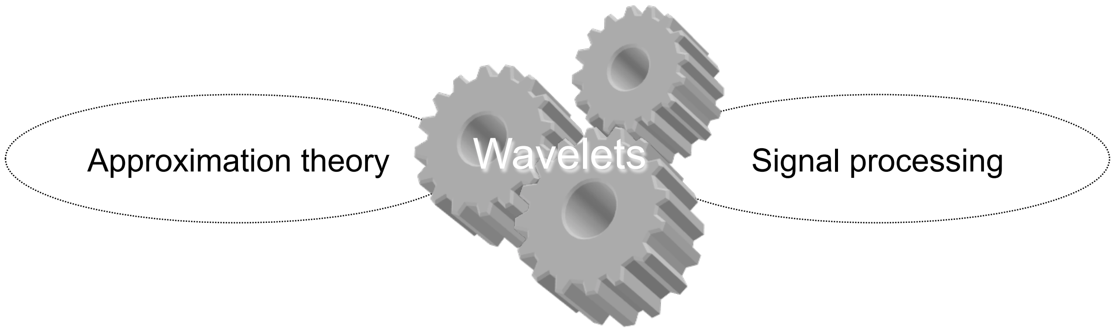
Coiflets, order=3



Coiflets:order=5



An approximation tour



- Linear approximation
  - Projects the signal f over M vectors of the ortho-normal basis B which are chosen *a-priori* among the basis B, say the first M

$$f_M = \sum_{n=0}^{M-1} \langle f, \phi_n \rangle \phi_n$$

  - Approximation error

$$\varepsilon[M] = \| f - f_M \|^2 = \sum_{n=M}^{+\infty} |\langle f, \phi_n \rangle|^2$$

choosing the first M vectors amounts to reconstruct f at a given resolution. The convergence properties similar as in the Fourier domain

- Non-linear approximations
  - The M vectors are chosen *a posteriori*

$$f_M = \sum_{n \in I_M} \langle f, \phi_n \rangle \phi_n$$

Approximation error

$$\varepsilon[M] = \| f - f_M \|^2 = \sum_{n \in I_M}^{+\infty} |\langle f, \phi_n \rangle|^2$$

The error can be minimized by choosing the vectors corresponding to the highest $\langle f, \phi_n \rangle$

In wavelet basis this amounts to an *adaptive approximation grid* whose *resolution is locally increased where the signal is irregular!*

Adaptive basis choice

- Instead of choosing the basis a-priori, one could choose the *best* basis, depending on the signal
- The basis is chosen to minimize the non linear approximation error of f
- Same problem as the choice of the *optimal basis* for stimulus representation in visual perception
- The optimal basis could be chosen for *classes of signals*, considered as random processes
  - Gaussian processes → Karunen Loeve transform (KLT)
    - Diagonalization of the covariance matrix which removes the inter-dependencies among the samples and results in a set of independent coefficients (i.e. redundancy has been removed)
  - Other kind of processes → no golden rule
    - Images are not Gaussian and not stationary
    - In some cases wavelets do better

Adaptive basis

- Wavelet packets
  - The subband tree is progressively split according to the optimization of a cost function (i.e. rate/distortion)
- Matching pursuit
  - Vectors are progressively selected from a dictionary, while optimizing the signal approximation at each step

- Key issue: a good basis should be able to provide a good description (approximation properties) of the signal while being concise (sparseness properties)
  - Classical approaches: approximation theory, information theory, estimation in noise...
  - Perception based approaches: bring humans into the loop

Wavelet Packets