Differential Geometry and Topology

Exercises, continued

1 Compactly supported cohomology and Poincaré duality

Exercise 1. Compute the de Rham cohomology of the punctured torus $\Sigma = T^2 \setminus \{x\}$ by the following steps.

1. Find $H^0_{dR}(\Sigma)$.
2. Find $H^0_c(\Sigma)$.
3. By Poincaré duality this gives $H^2_{dR}(\Sigma)$.
4. Let D be an open disk containing the point x, so $T^2 = \Sigma \cup D$. Given that $H^0_{dR}(T^2) \cong \mathbb{R} \cong H^2_{dR}(T^2)$, $H^1_{dR}(T^2) \cong \mathbb{R}^2$, use the Mayer-Vietoris sequence to compute $H^1_{dR}(\Sigma)$.

Exercise 2.

Let us prove, under certain assumptions, the *Künneth formula for compactly supported cohomology*:

$$H^*_c(M \times N) \cong H^*_c(M) \otimes H^*_c(N).$$

1. Let M and N be smooth manifolds, and let $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ be the projection maps. Show that the cross product map ψ defined by

   $$\psi(\omega \otimes \eta) = \pi_1^*\omega \wedge \pi_2^*\eta$$

   is well-defined as a map $\Omega^k_c(M) \otimes \Omega^l_c(N) \to \Omega^{k+l}_c(N)$, and that this induces a well-defined map

   $$\psi : H^*_c(M) \otimes H^*_c(N) \to H^*_c(M \times N).$$

2. If M and N are orientable and both have finite good covers, show that the Künneth formula for compactly supported cohomology is a direct consequence of Poincaré duality and the usual Künneth formula.
3. Under the weaker assumption that M has a finite good cover (and no assumption on orientability of M and N), use Mayer–Vietoris and induction to prove that $\psi : H^*_c(M) \otimes H^*_c(N) \to H^*_c(M \times N)$ is an isomorphism.

2 Hodge theory

Exercise 3. Let $\langle , \rangle$ be an inner product on a k-dimensional vector space V, Denote again by $\langle , \rangle$ the extension of the inner product to the vector spaces $\Lambda^p(V)$. Let $\Omega \in \Lambda^k(V)$ be a volume form normalized so that $\langle \Omega, \Omega \rangle = 1$, and let $*$ be the Hodge star w.r.t. $\langle , \rangle$ and Ω. Show that $\langle \omega, \eta \rangle = \langle *\omega, *\eta \rangle$ for all $\omega, \eta \in \Lambda^p(V)$.

Exercise 4. Let S^2 be the unit sphere in $\mathbb{R}^3$, with standard spherical coordinates (θ, ϕ) where θ is the angle measured from the z axis, and ϕ is the angle in the xy-plane measured from the x-axis. Consider the local chart on $U = S^2 \setminus \{(0, 0, 1), (0, 0, -1)\}$ given by $\theta \in (0, \pi), \phi \in [0, 2\pi]$.

1. Show that on this local chart for U, the metric induced from the ambient Euclidean metric in $\mathbb{R}^3$ is $g(\theta, \phi) = d\theta \otimes d\theta + \sin^2 \theta d\phi \otimes d\phi$.

   (In other words use the map $f : (0, \pi] \times [0, 2\pi] \to S^2 \subset \mathbb{R}^3$ given by these spherical coordinates to pull back the Euclidean metric on $\mathbb{R}^3$, i.e. $f^*(dx \otimes dx + dy \otimes dy + dz \otimes dz)$.)

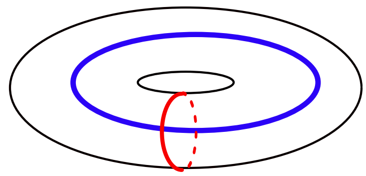
2. Find an expression for the normalized volume form Ω on $S^2 \setminus \{(0, 0, 1)\}$ which has the same orientation as $d\theta \wedge d\phi$.

   (In other words, $\Omega = f(\theta, \phi) d\theta \wedge d\phi$ for some positive function $f(\theta, \phi)$, which you work out by solving $\langle \Omega, \Omega \rangle = 1$. You should get $\Omega = \frac{1}{\sin \theta} d\theta \wedge d\phi$.)

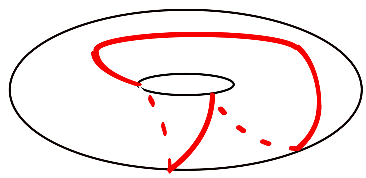
3. Find the explicit expressions on this local chart for the Hodge star operators $*$, codifferential operators δ, and Laplace-Beltrami operators $\triangle$ with respect to g and Ω.

3 Poincaré duals, intersection numbers, Euler characteristic

Exercise 5. Let T^2 be the torus depicted below, E its equator (blue) and M its meridian (red). Compute $I(E,E)$, $I(M,M)$, and $I(M,E)$. Conclude that E and M are not isotopic to each other.



Exercise 6. Let S be the curve in T^2 depicted in red below. Choose an orientation for S and compute $I(S,S)$. Let E and M be the equator and meridian of the previous exercise. Compute $I(S,E)$ and $I(S,M)$ with respect to any orientation of E and M. Conclude that S is not isotopic to either E or M.



Exercise 7. Let S_1 and S_2 be two compact oriented submanifolds of $\mathbb{R}^n$ of complementary dimension, with Poincaré duals η_{S_1} and η_{S_2}. Explain why $\int_M \eta_{S_1} \wedge \eta_{S_2} = 0$.

Exercise 8. Prove that the Euler characteristic of the product of two compact, oriented manifolds is the product of their Euler characteristics.

Exercise 9. Let $\triangle \subset S^2 \times S^2$ be the diagonal, which itself is isomorphic to a sphere. Show that there is no isotopy $\Phi : S^2 \times S^2 \to S^2 \times S^2$ such that $\Phi(\triangle) \cap \triangle = \emptyset$.

Exercise 10. More generally, for $k > 0$ even, let S^k be the k-dimensional sphere. Show that there is no isotopy $\Phi : S^k \times S^k$ such that $\Phi(\triangle) \cap \triangle = \emptyset$.