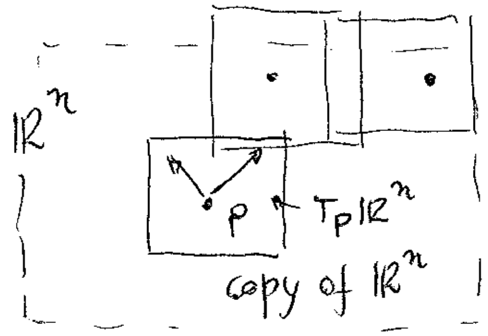
Lectures on DIFFERENTIAL GEOMETRY AND TOPOLOGY

Mauro Spera

Lecture VI

* Tangent vectors and vector fields

Let $p \in \mathbb{R}^n$. Let $T_p\mathbb{R}^n$ denote a copy of $\mathbb{R}^n$, thinking of its elements as "applied" vectors at p, and call them tangent vectors at p. The vector space $T_p\mathbb{R}^n$ itself is called tangent space to $\mathbb{R}^n$ at p.

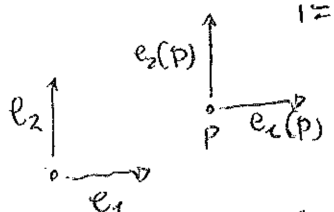


In the sequel, a more formal definition will be given

Given real, smooth functions $X_i = X_i(p)$, the map $X : \mathbb{R}^n \to \mathbb{R}^n$ given by

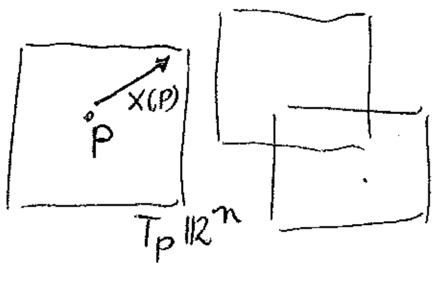
$$X(p) := \sum_{i=1}^{n} X_i(p)\, e_i(p) \qquad \in T_p\mathbb{R}^n$$

$e_i(p) \equiv e_i$ applied at p
↑ canonical basis



actually: $X : p \longmapsto X(p)$, $p \in \mathbb{R}^n$, $X(p) \in T_p\mathbb{R}^n = \mathbb{R}^n$

is called a (smooth) vector field on $\mathbb{R}^n$



The union

$$T\mathbb{R}^n := \bigcup_{p \in \mathbb{R}^n} T_p\mathbb{R}^n$$

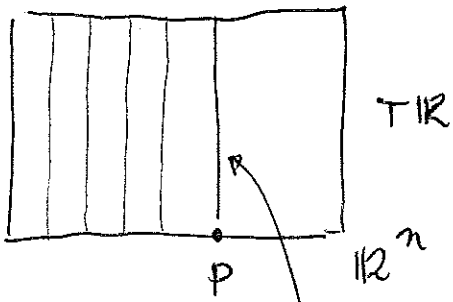
actually, the disjoint union of copies of $\mathbb{R}^n$ labelled by $p \in \mathbb{R}^n$

is called tangent bundle of $\mathbb{R}^n$ (or associated to $\mathbb{R}^n$)
fibrato

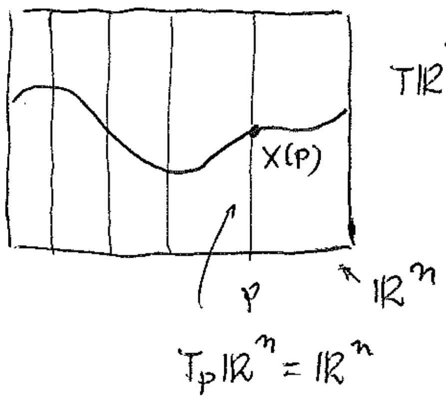
in italiano: fibrato tangente

the various $T_p\mathbb{R}^n$ constitute the fibres of the tangent bunde

A vector field is a section of the tangent bundle



$T_p\mathbb{R}^n$ = fibre of $T\mathbb{R}^n$ at p
the "vertical" $\mathbb{R}^n$: typical fibre



depiction of a vector field

$$\mathbb{R}^n \ni p \longmapsto X(p) \in \underbrace{T_p\mathbb{R}^n}_{\text{fibre at } p} = \mathbb{R}^n$$

Now — and this is a crucial point — let us interpret tangent vectors as directional derivatives:

$$\boxed{X(p) \longleftrightarrow \sum_{i=1}^{n} X_i(p) \underbrace{\frac{\partial}{\partial x^i}\Big|_p}_{\equiv\ \partial_i|_p}}$$

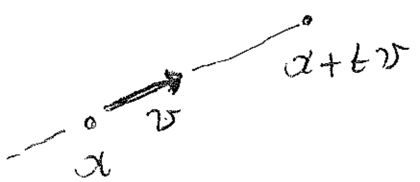
i^{th} partial derivative

directional derivative (applied to a generic smooth function) along the vector $X(p) = \begin{pmatrix} X_1(p) \\ \vdots \\ X_1(p) \end{pmatrix}$

Recall, from analysis

$$\frac{\partial f}{\partial v}(x) = \sum_{i=1}^{n} v_i \frac{\partial f}{\partial x^i}(x) \qquad v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

$$\frac{\partial f}{\partial v}(x) = \frac{d f(x+tv)}{dt}\Big|_{t=0} = \sum_{i=1}^{n} v_i \frac{\partial f}{\partial x^i}(x) \qquad \text{(chain rule)}$$



Shortly:

$$v \longleftrightarrow v \cdot \nabla = \sum_i v_i \frac{\partial}{\partial x^i} \qquad \nabla = \begin{pmatrix} \frac{\partial}{\partial x^1} \\ \vdots \\ \frac{\partial}{\partial x^n} \end{pmatrix}$$

tangent vector

directional derivative

to be applied to a smooth function

★ In particular, we have the following interpretation of $(e_1 \dots e_n)$ ← canonical basis

$$\boxed{e_i \longleftrightarrow \frac{\partial}{\partial x^i}}$$ ← partial derivative operator with respect to the i-th coordinate

* Cotangent vectors and differential forms

Let $p \in \mathbb{R}^n$. Let us denote by $T_p^* \mathbb{R}^n$ a replica of $(\mathbb{R}^n)^*$ (dual of $\mathbb{R}^n$), thinking of its elements (dual vectors, or covectors) as being "applied" at p. The vector space $T_p^* \mathbb{R}^n$ is called cotangent space of $\mathbb{R}^n$ at p

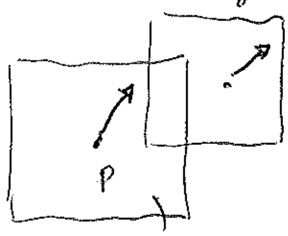
spazio cotangente

A differential 1-form ω is a map (smooth) $\omega : \mathbb{R}^n \to (\mathbb{R}^n)^*$

$$\omega = \omega(p) := \sum_{i=1}^{n} \underset{\underset{C^\infty(\mathbb{R}^n)}{\cap}}{\omega_i(p)} \; \underset{\underset{e_i^* \text{ applied at } p}{|||}}{e_i^*(p)} \qquad \in T_p^* \mathbb{R}^n$$

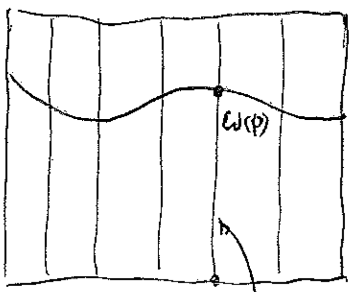
The set $T^* \mathbb{R}^n = \bigcup_{p \in \mathbb{R}^n} T_p^* \mathbb{R}^n$ is called cotangent bundle of $\mathbb{R}^n$.

fibrato cotangente



$T_p^* \mathbb{R}^n$

In a similar vein, differential 1-forms are the sections of the cotangent bundle



$T^* \mathbb{R}^n$

p $\quad \mathbb{R}^n$

$T_p^* \mathbb{R}^n$ fibre at p

portrait of a differential 1-form

$\omega : p \to \omega(p) \in T_p^* \mathbb{R}^n$

We wish to give an interpretation of the dual basis $(e^*_1 \dots e^*_n)$.

Upon recalling that $\frac{\partial x^i}{\partial x^j} = \delta_{ij}$, we find

$$\boxed{e^*_j \mapsto dx^j}$$ (differential of the j-th coordinate function)

Indeed, recall that $df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i$ $\quad p \mapsto df|_p \in T^*_p\mathbb{R}^n$ — this we already know!

and, consistently,

$$dx^j = \sum_{i=1}^{n} \frac{\partial x^j}{\partial x^i} dx^i = \sum_{i=1}^{n} \delta_{ij}\, dx^i = dx^j$$

From now on we shall replace e_i by $\frac{\partial}{\partial x^i}$ $\left(\frac{\partial}{\partial x^i}\Big|_p \right)$

$e_i = e_i(p) = e_i$ (constant)

and e^*_i by dx^i

$e^*_i = e^*_i(p) = e^*_i$ (constant)

0-forms

We have the following fundamental formula: $\forall$ $X \in \mathcal{X}(\mathbb{R}^n)$ ← vector fields on $\mathbb{R}^n$, $f \in \Lambda^0(\mathbb{R}^n) \equiv C^\infty(\mathbb{R}^n)$

$$\boxed{\underbrace{X(f)}_{\in \Lambda^0(\mathbb{R}^n)} = \underbrace{df(X)}_{\in \Lambda^0(\mathbb{R}^n)}} \equiv (df, X)$$

[namely, at every point p, $X(f)(p) = df|_p(X(p))$]

(use tensorial notation; differential at p evaluated on $X(p)$)

Proof: In components: $X = \sum_j b^j \frac{\partial}{\partial x^j}$, $df = \sum_i \frac{\partial f}{\partial x^i} dx^i$

$X(f)(p) = \sum b^i(p) \frac{\partial f}{\partial x^i}(p)$ and $df(X) = \left(\sum_i \frac{\partial f}{\partial x^i} dx^i\right)\left(\sum_j b^j \frac{\partial}{\partial x^j}\right)$

$$= \sum_{i,j} \frac{\partial f}{\partial x^i} b^j \underbrace{dx^i\left(\frac{\partial}{\partial x^j}\right)}_{=\delta_{ij}} = \sum_i \frac{\partial f}{\partial x^i} b^i \qquad \square$$

More generally, given $X \in \mathcal{X}(\mathbb{R}^n)$, $X = \sum_j b^j \frac{\partial}{\partial x^j}$

$\omega \in \Lambda^1(\mathbb{R}^n)$, $\omega = \sum_i a_i \, dx^i$

covector

vector

then

$$\omega(X) = \left(\sum_j b^j \frac{\partial}{\partial x^j} , \sum_i a_i \, dx^i \right) = \sum_{i,j} b^j a_i \underbrace{\frac{\partial x^i}{\partial x^j}}_{= \delta_{ij}}$$

$$= \sum_i a_i b^i$$

Let us redo the calculation using Einstein's notation

$$X = \underbrace{b^j \frac{\partial}{\partial x^j}}_{\text{sum omitted}} \qquad \omega = a_i \, dx^i \qquad \begin{array}{l} b^j = b^j(p) \\ a_i = a_i(p) \end{array}$$

$$\omega(a) = \dots = a_i b^i \qquad \text{(a smooth function)}$$

and, in particular, $X(f) = b^j \frac{\partial f}{\partial x^j}$

$$df(X) = \left(\frac{\partial f}{\partial x^i} dx^i \right) \left(b^j \frac{\partial}{\partial x^j} \right) = \dots b^j \frac{\partial f}{\partial x^j}$$