ATTENZIONE: queste note sostituiscono:
Def 5.3, Def 5.8, lemma 5.9, proposizione 5.10, teorema 5.11, corollario 5.13
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Definition 3.1.4 *(i) A theory T is a collection of formulas with the property $T \vdash \varphi \Rightarrow \varphi \in T$ (a theory is closed under derivability).*
(ii) A set Γ such that $T = \{\varphi | \Gamma \vdash \varphi\}$ is called an axiom set of the theory T. The elements of Γ are called axioms.
(iii) T is called a Henkin theory if for each sentence $\exists x\varphi(x)$ there is a constant c such that $\exists x\varphi(x) \to \varphi(c) \in T$ (such a c is called a witness for $\exists x\varphi(x)$).

Note that $T = \{\sigma | \Gamma \vdash \sigma\}$ is a theory. For, if $T \vdash \varphi$, then $\sigma_1, \ldots, \sigma_k \vdash \varphi$ for certain σ_i with $\Gamma \vdash \sigma_i$.

$$\begin{array}{c} \mathcal{D}_1 \;\; \mathcal{D}_2 \;\ldots\; \mathcal{D}_k \\ \sigma_1 \;\; \sigma_2 \;\ldots\; \sigma_k \\ \mathcal{D} \\ \hline \varphi \end{array}$$

From the derivations $\mathcal{D}_1, \ldots, \mathcal{D}_k$ of $\Gamma \vdash \sigma_1, \ldots, \Gamma \vdash \sigma_k$ and $\mathcal{D}$ of $\sigma_1, \ldots, \sigma_k \vdash \varphi$ a derivation of $\Gamma \vdash \varphi$ is obtained, as indicated.

Definition 3.1.5 *Let T and T' be theories in the languages L and L'.*
(i) T' is an extension of T if $T \subseteq T'$,
(ii) T' is a conservative extension of T if $T' \cap L = T$ (i.e. all theorems of T' in the language L are already theorems of T).

Example of a conservative extension: Consider propositional logic P' in the language L with $\to$, $\wedge$, $\bot$, $\leftrightarrow$, $\neg$. Then exercise 2, section 1.6, tells us that P' is conservative over P.

Our first task is the construction of *Henkin extensions* of a given theory T, that is to say: extensions of T which are Henkin theories.

Definition 3.1.6 *Let T be a theory with language L. The language L^* is obtained from L by adding a constant c_φ for each sentence of the form $\exists x\varphi(x)$, a constant c_φ . T^* is the theory with axiom set*
$T \cup \{\exists x\varphi(x) \to \varphi(c_\varphi) |\ \exists x\varphi(x)$ in $L\}$

Lemma 3.1.7 *T^* is conservative over T.*

Proof. (a) Let $\exists x\varphi(x) \to \varphi(c)$ be one of the new axioms. Suppose $\Gamma, \exists x\varphi(x) \to \varphi(c) \vdash \psi$, where ψ does not contain c and where Γ is a set of *formulas* none of which contains the constant c. We show $\Gamma \vdash \psi$ in a number of steps.

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1. $\Gamma \vdash (\exists x\varphi(x) \to \varphi(c)) \to \psi$,
2. $\Gamma \vdash (\exists x\varphi(x) \to \varphi(y)) \to \psi$, where y is a variable that does not occur in the associated derivation. 2 follows from 1 by Theorem 2.8.3.
3. $\Gamma \vdash \forall y[(\exists x\varphi(x) \to \varphi(y)) \to \psi]$. This application of $(\forall I)$ is correct, since c did not occur in Γ.
4. $\Gamma \vdash \exists y(\exists x\varphi(x) \to \varphi(y)) \to \psi$, (cf. example of section 2.9).
5. $\Gamma \vdash (\exists x\varphi(x) \to \exists y\varphi(y)) \to \psi$, (section 2.9 exercise 7).
6. $\vdash \exists x\varphi(x) \to \exists y\varphi(y)$.
7. $\Gamma \vdash \psi$, (from 5,6).

(b) Let $T^* \vdash \psi$ for a $\psi \in L$. By the definition of derivability $T \cup \{\sigma_1, \ldots, \sigma_n\} \vdash \psi$, where the σ_i are the new axioms of the form $\exists x\varphi(x) \to \varphi(c)$. We show $T \vdash \psi$ by induction on n. For $n = 0$ we are done. Let $T \cup \{\sigma_1, \ldots, \sigma_{n+1}\} \vdash \psi$. Put $\Gamma' = T \cup \{\sigma_1, \ldots, \sigma_n\}$, then $T', \sigma_{n+1} \vdash \psi$ and we may apply (a). Hence $T \cup \{\sigma_1, \ldots, \sigma_n\} \vdash \psi$. Now by induction hypothesis $T \vdash \psi$. □

Although we have added a large number of witnesses to T, there is no evidence that T^* is a Henkin theory, since by enriching the language we also add new existential statements $\exists x\tau(x)$ which may not have witnesses. In order to overcome this difficulty we iterate the above process countably many times.

Lemma 3.1.8 *Define $T_0 := T; T_{n+1} := (T_n)^*; T_\omega := \cup\{T_n | n \geq 0\}$. Then T_ω is a Henkin theory and it is conservative over T.*

Proof. Call the language of T_n (resp. T_ω) L_n (resp. L_ω).

(i) T_n is conservative over T. Induction on n.

(ii) T_ω is a theory. Suppose $T_\omega \vdash \sigma$, then $\varphi_0, \ldots, \varphi_n \vdash \sigma$ for certain $\varphi_0, \ldots, \varphi_n \in T_\omega$. For each $i \leq n$ $\varphi_i \in T_{m_i}$ for some m_i. Let $m = max\{m_i | i \leq n\}$. Since $T_k \subseteq T_{k+1}$ for all k, we have $T_{m_i} \subseteq T_m (i \leq n)$. Therefore $T_m \vdash \sigma$. T_m is (by definition) a theory, so $\sigma \in T_m \subseteq T_\omega$.

(iii) T_ω is a Henkin theory. Let $\exists x\varphi(x) \in L_\omega$, then $\exists x\varphi(x) \in L_n$ for some n. By definition $\exists x\varphi(x) \to \varphi(c) \in T_{n+1}$ for a certain c. So $\exists x\varphi(x) \to \varphi(c) \in T_\omega$.

(iv) T_ω is conservative over T. Observe that $T_\omega \vdash \sigma$ if $T_n \vdash \sigma$ for some n and apply (i). □

As a corollary we get: T_ω is consistent if T is so. For suppose T_ω inconsistent, then $T_\omega \vdash \perp$. As T_ω is conservative over T (and $\perp \in L$) $T \vdash \perp$. Contradiction.

Our next step is to extend T_ω as far as possible, just as we did in propositional logic (1.5.7). We state a general principle:

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Lemma 3.1.9 (Lindenbaum) *Each consistent theory is contained in a maximally consistent theory.*

Proof. We give a straightforward application of *Zorn's Lemma*. Let T be consistent. Consider the set A of all consistent extensions T' of T, partially ordered by inclusion. Claim: A has a maximal element.

1. Each chain in A has an upper bound. Let $\{T_i | i \in I\}$ be a chain. Then $T' = \cup T_i$ is a consistent extension of T containing all T_i's (Exercise 2). So T' is an upper bound.
2. Therefore A has a maximal element T_m (Zorn's lemma).
3. T_m is a maximally consistent extension of T. We only have to show: $T_m \subseteq T'$ and $T' \in A$, then $T_m = T'$. But this is trivial as T_m is maximal in the sense of $\subseteq$. Conclusion: T is contained in the maximally consistent theory T_m. □

Note that in general T has many maximally consistent extensions. The above existence is far from unique (as a matter of fact the proof of its existence essentially uses the axiom of choice). Note, however, that if the language is countable, one can mimick the proof of 1.5.7 and dispense with Zorn's Lemma.

We now combine the construction of a Henkin extension with a maximally consistent extension. Fortunately the property of being a Henkin theory is preserved under taking a maximally consistent extension. For, the language remains fixed, so if for an existential statement $\exists x\varphi(x)$ there is a witness c such that $\exists x\varphi(x) \to \varphi(c) \in T$, then trivially, $\exists x\varphi(x) \to \varphi(c) \in T_m$. Hence

Lemma 3.1.10 *An extension of a Henkin theory with the same language is again a Henkin theory.*

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