✶ Examples

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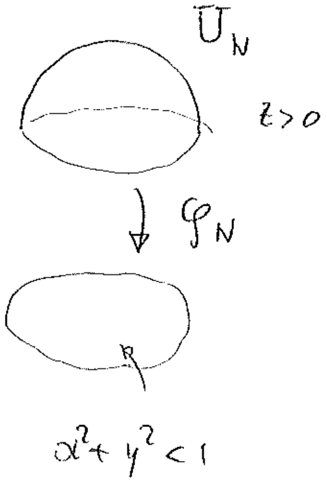
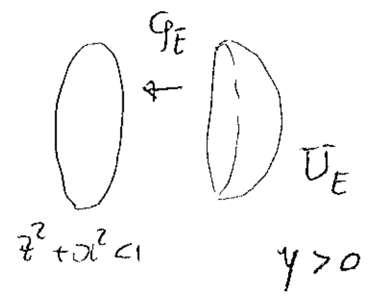
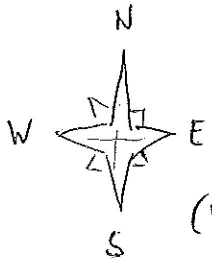


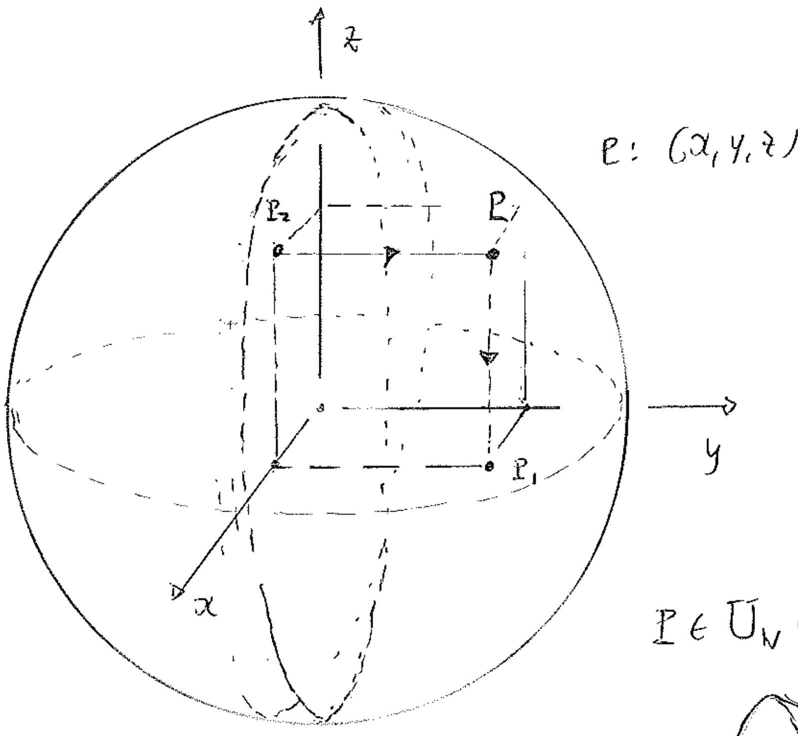




(rosa dei venti)

compass-card

bussola



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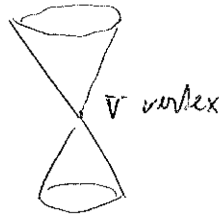

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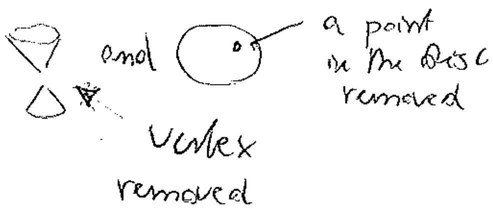

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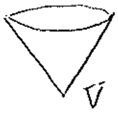
2'. 

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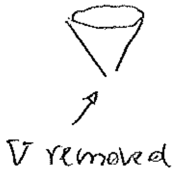
2''. 

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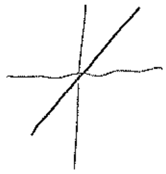
3. A remark on the concept of differentiable structure

$M_1 = (\mathbb{R}, t)$ $\qquad \varphi_1(t) = t$



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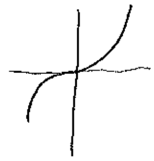

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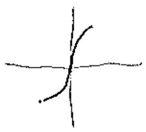

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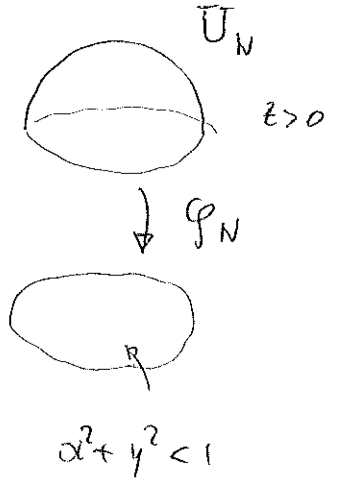
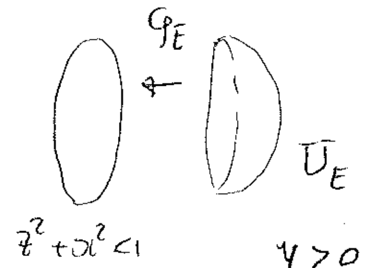
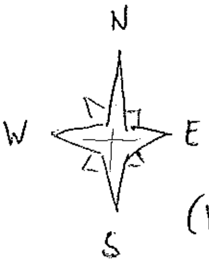


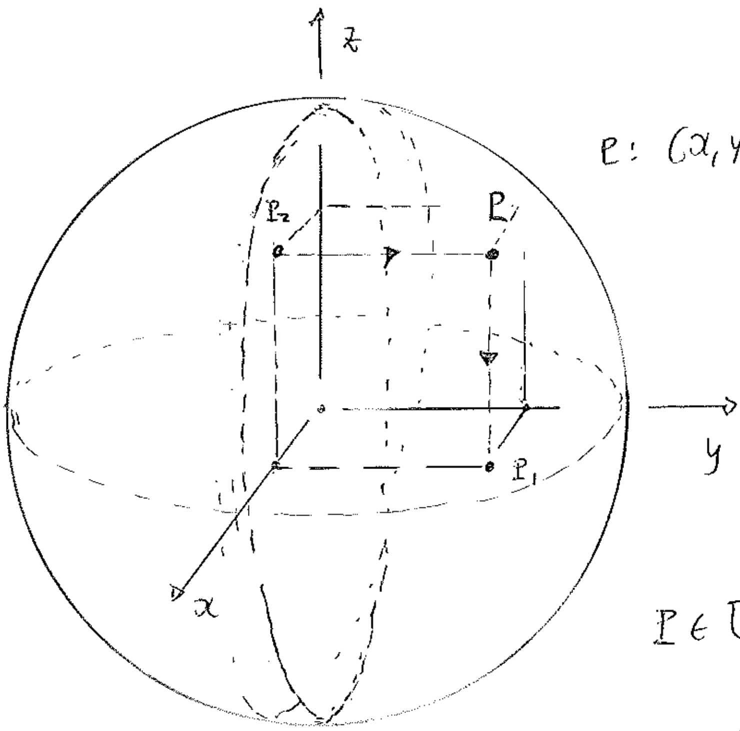




(rosa dei venti)

compass-card

bussola



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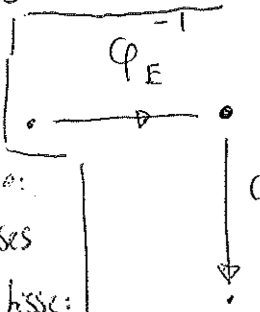

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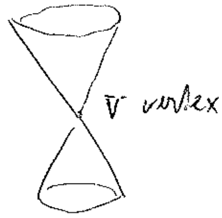

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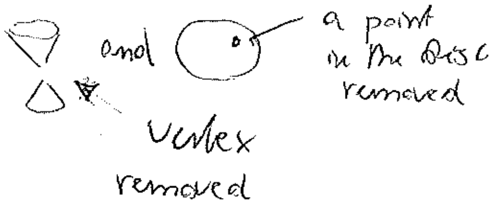

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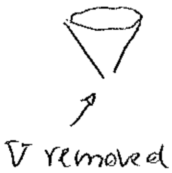
2''. 

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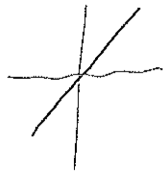

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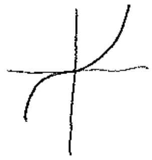

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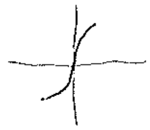

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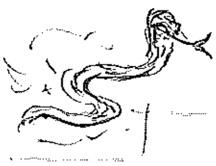

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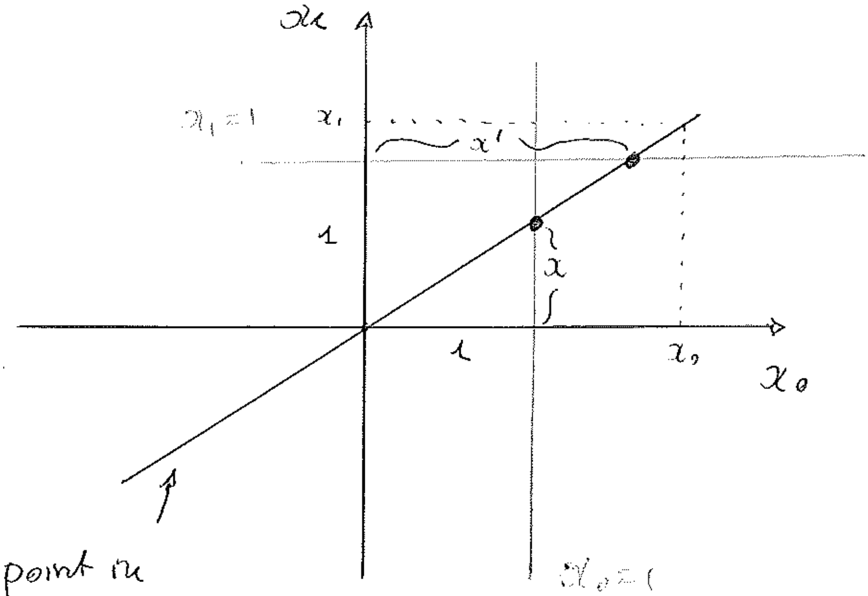

a point in $\mathbb{P}^1(\mathbb{R})$

inhomogeneous (affine) coordinate

$$\mathcal{U}_0 \ni [\underset{\neq 0}{x_0}, x_1] \xmapsto{\varphi_0} \Big(1, \underbrace{\frac{x_1}{x_0}}_{}\Big) \equiv x, \qquad x = \frac{x_1}{x_0}$$

removed

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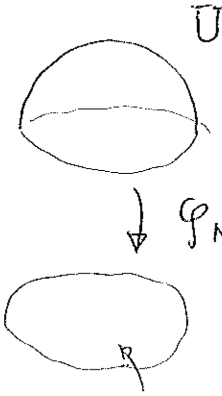

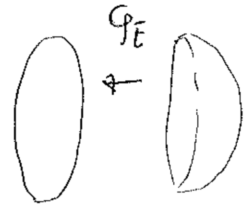

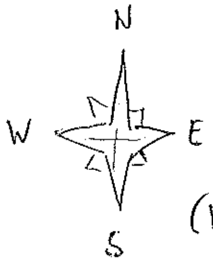

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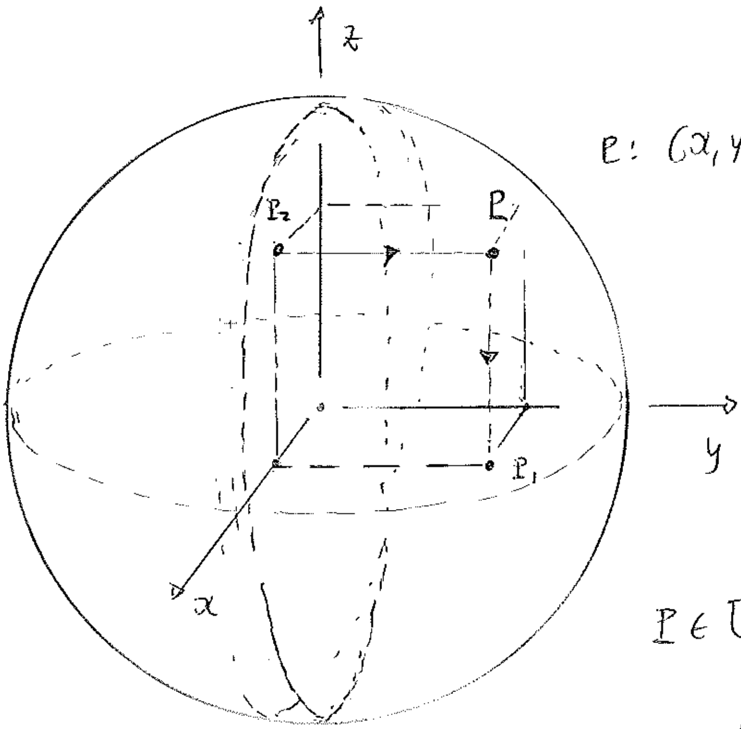

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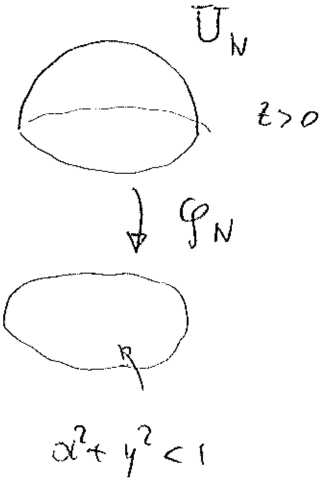

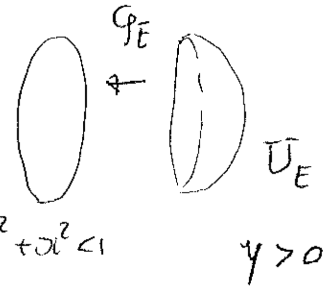

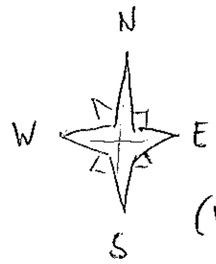

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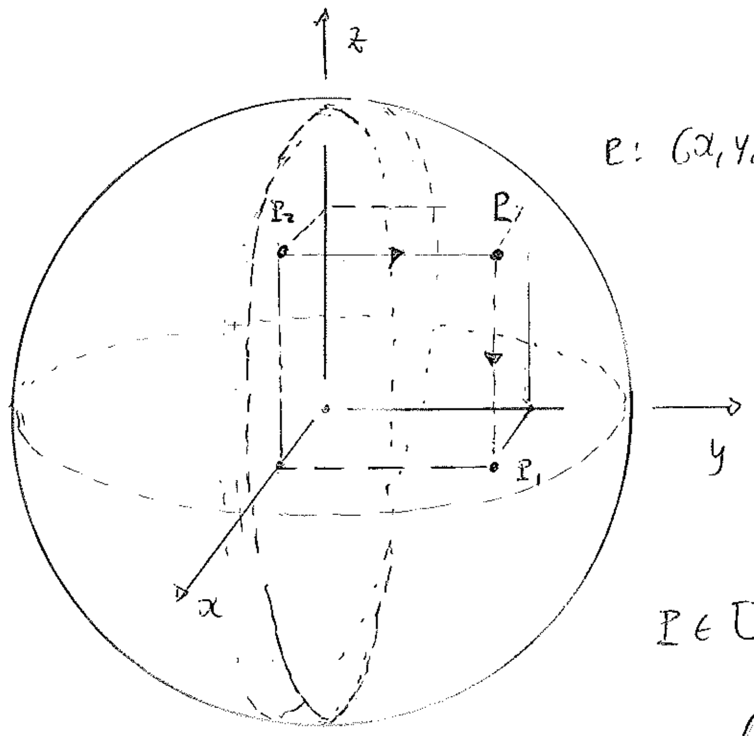

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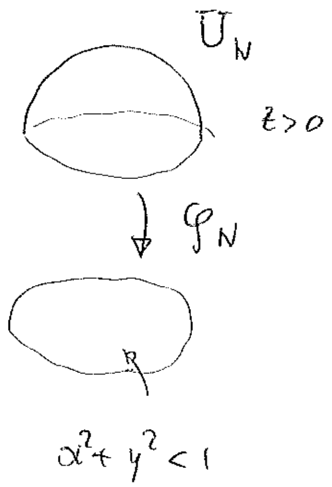

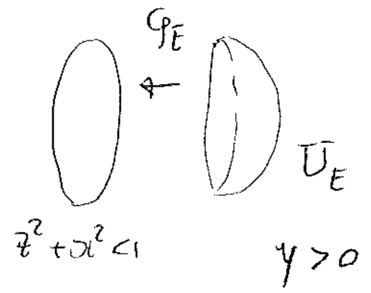

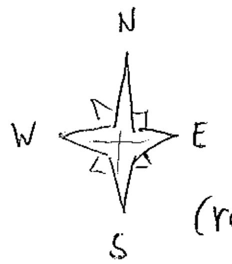

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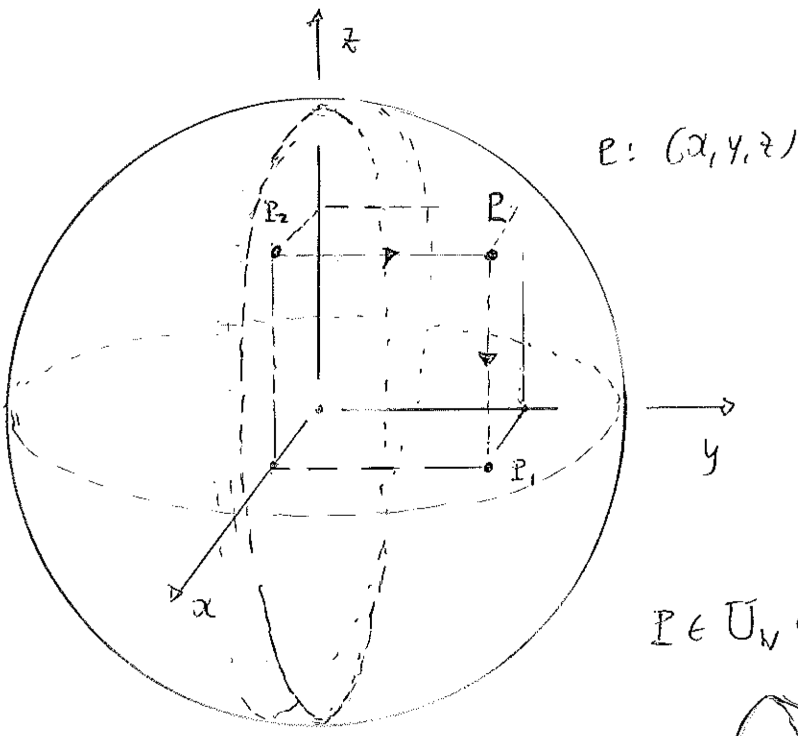

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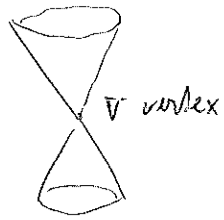

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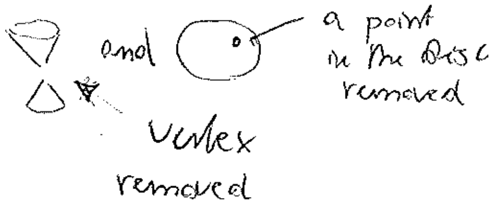

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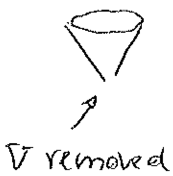
2''. 

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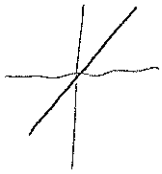

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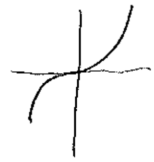

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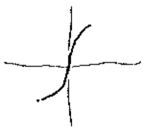

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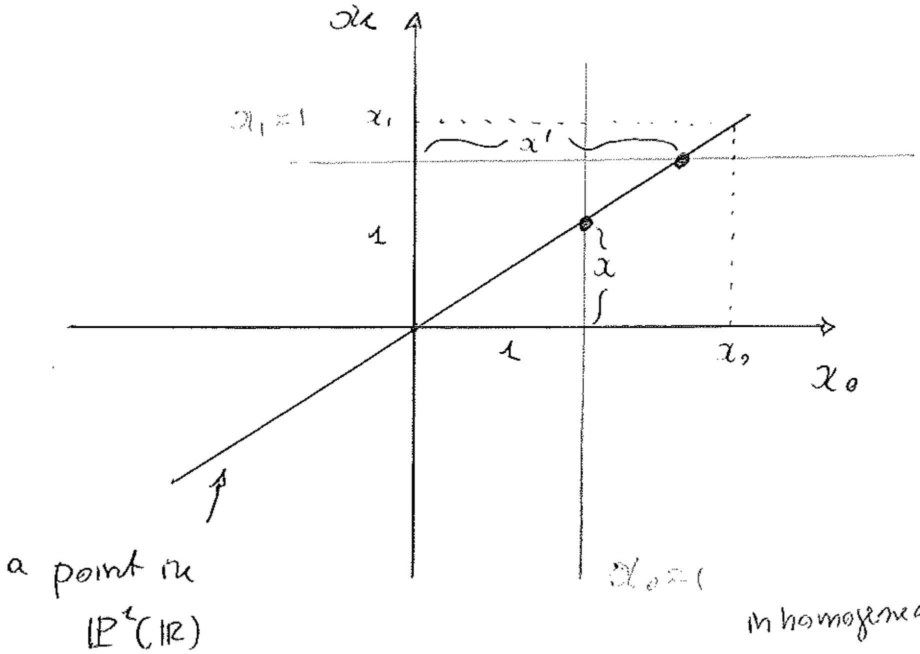


inhomogeneous (affine) coordinate

$$\mathcal{U}_0 \ni [\underset{\neq 0}{x_0}, x_1] \xmapsto{\varphi_0} \left(1, \overset{x}{\overset{\parallel}{\frac{x_1}{x_0}}}\right) \equiv x$$

removed

$$\mathcal{U}_1 \ni [x_0, \underset{\neq 0}{x_1}] \xmapsto{\varphi_1} \left(\underbrace{\frac{x_0}{x_1}}_{x'}, 1\right) \equiv x'$$

in $\mathcal{U}_0 \cap \mathcal{U}_1$, $x_0 \neq 0$, $x_1 \neq 1$

$$x \xmapsto{\varphi_1 \circ \varphi_0^{-1}} x' = \frac{1}{x}$$

$x' = \frac{1}{x}$

smooth with smooth inverse

5. <u>Projective</u> <u>spaces</u> (real & complex)

$$\underset{\mathbb{C}}{\mathbb{P}^n(\mathbb{R})} \equiv \underset{\mathbb{C}^{n+1}}{\mathbb{P}(\mathbb{R}^{n+1})}$$

$$\varphi_i([x_0 \dots x_n]) = \left(\frac{x_0}{x_i}, \dots \underbrace{\widehat{\frac{x_i}{x_i}}}_{\substack{=1 \\ \text{omitted}}}, \dots \frac{x_n}{x_i}\right) \in \begin{matrix}\mathbb{R}^n \\ \mathbb{C}^n\end{matrix}$$

homogeneous coordinates

φ_i defined on

$$\mathcal{U}_i = \{[x],\ x_i \neq 0\}$$

Let us calculate the <u>transition maps</u>, for $\mathcal{U}_i \cap \mathcal{U}_j \neq \phi$
(i.e. $x_i \neq 0$, $x_j \neq 0$)

jth position, ith-position

$$\boxed{\varphi_j \circ \varphi_i^{-1}(x_1 \dots x_n) = \left(\frac{x_1}{x_j} \dots \underbrace{\widehat{\frac{x_j}{x_j}}}_{=1,\ \text{omitted}}, \frac{1}{x_j}, \dots \frac{x_n}{x_j}\right)}$$

in detail:

insert at ith-pos.

$$(x_1 \dots x_n) \overset{\varphi_i^{-1}}{\longmapsto} [x_1 \dots 1 \dots x_n] \in \mathbb{P}^n$$

$$\left(= \left[\frac{X_1}{X_i}, \dots \frac{X_i}{X_i} \dots \frac{X_n}{X_i}\right]\right)$$

X_i : coordinates in $\mathbb{R}^{n+1}$ ($\mathbb{C}^{n+1}$)

ith-pos.

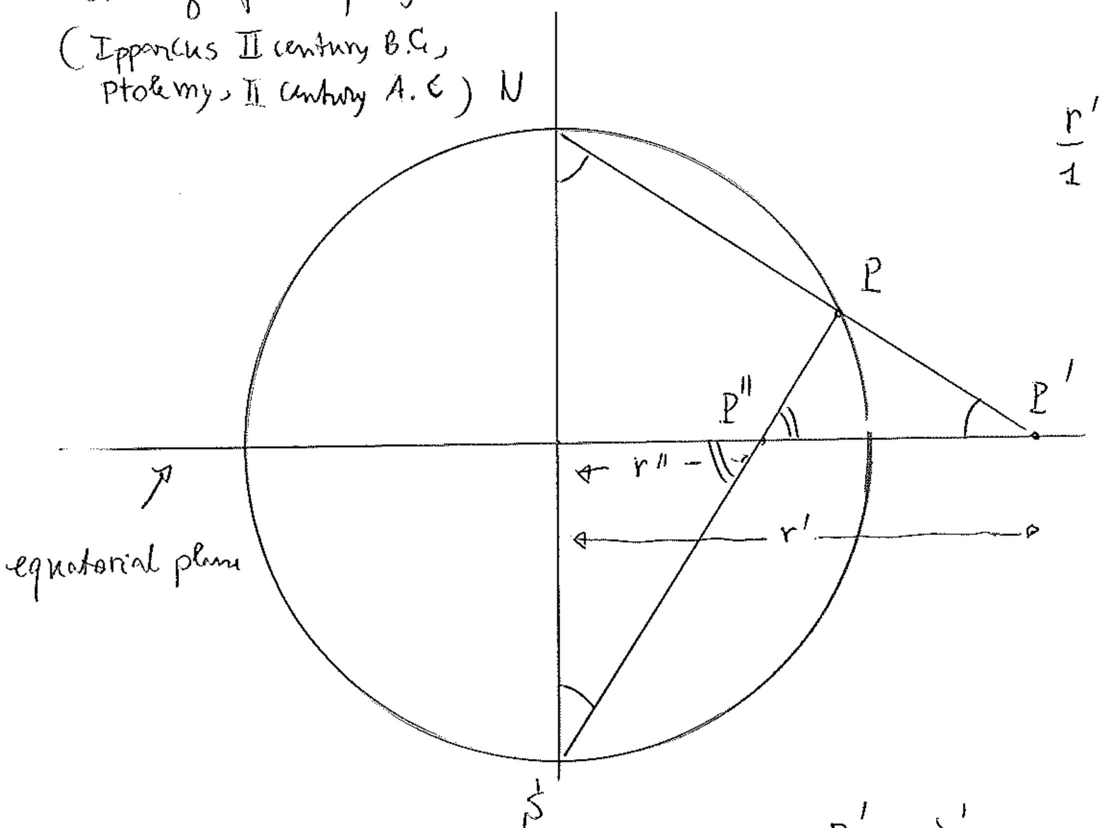
$$\overset{\varphi_j}{\longmapsto} \left(\frac{x_1}{x_j} \dots \underbrace{\widehat{\frac{x_j}{x_j}}}_{=1}, \dots \frac{1}{x_j}, \dots \frac{x_n}{x_j}\right)$$

(at jth position)
<u>remove</u> <u>it</u>

These maps are smooth (and holomorphic in the complex case)

5'. The Riemann sphere (the complex projective line as a Riemann surface)

stereographic projections
(Ipparcus II century B.C., Ptolemy, II century A.C)



$$\frac{r'}{1} = \frac{1}{r''}$$

$$r' = \frac{1}{r''}$$

$P' \leftrightarrow \zeta'$ $\qquad r' = |\zeta'|$

$P'' \leftrightarrow \zeta''$ $\qquad r'' = |\zeta''|$

$U_N = S^2 \setminus \{N\}$ 

$U_S = S^2 \setminus \{S\}$ 

$S^2 = U_N \cup U_S$

$\varphi_N : P \mapsto P'$

$\varphi_S : P \mapsto P''$

If $P \notin \{N, S\}$ they are both defined

$$\varphi_N \circ \varphi_S^{-1} : \begin{array}{ccccc} P'' & \xrightarrow{\varphi_S^{-1}} & P & \xrightarrow{\varphi_N} & P' \\ \cap & & & & \cap \\ \mathbb{C} & & & & \mathbb{C} \end{array}$$

equatorial plane $\quad \zeta'' \longmapsto \zeta'$ $\qquad \zeta' = \dfrac{1}{\zeta''}$

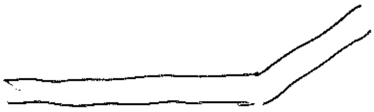
for suitable orientations (see further on)

real form

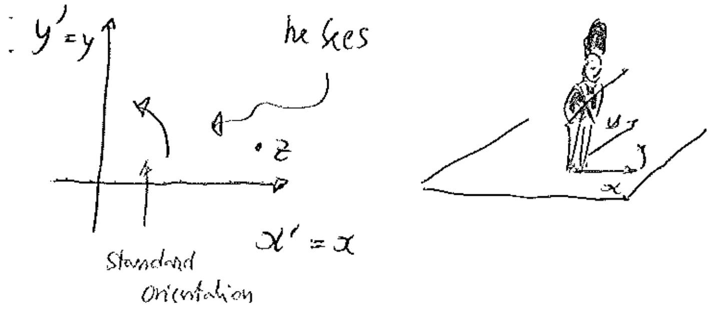
$$x' + iy' = \frac{1}{x'' + iy''} = \frac{x'' - iy''}{x''^2 + y''^2} \Rightarrow$$

$$\begin{cases} x' = \dfrac{x''}{x''^2 + y''^2} \\ y' = \dfrac{-y''}{x''^2 + y''^2} \end{cases}$$

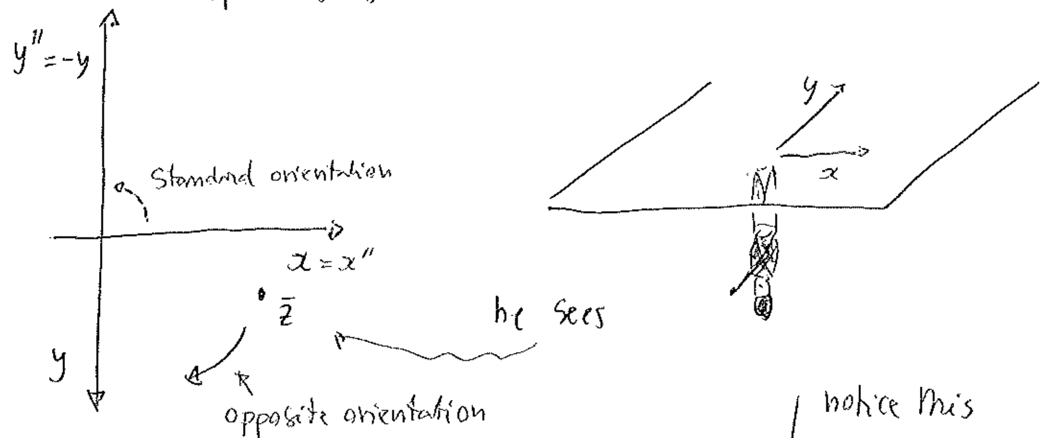
Remark on orientation



take two copies of the equatorial plane $\cong \mathbb{C}$



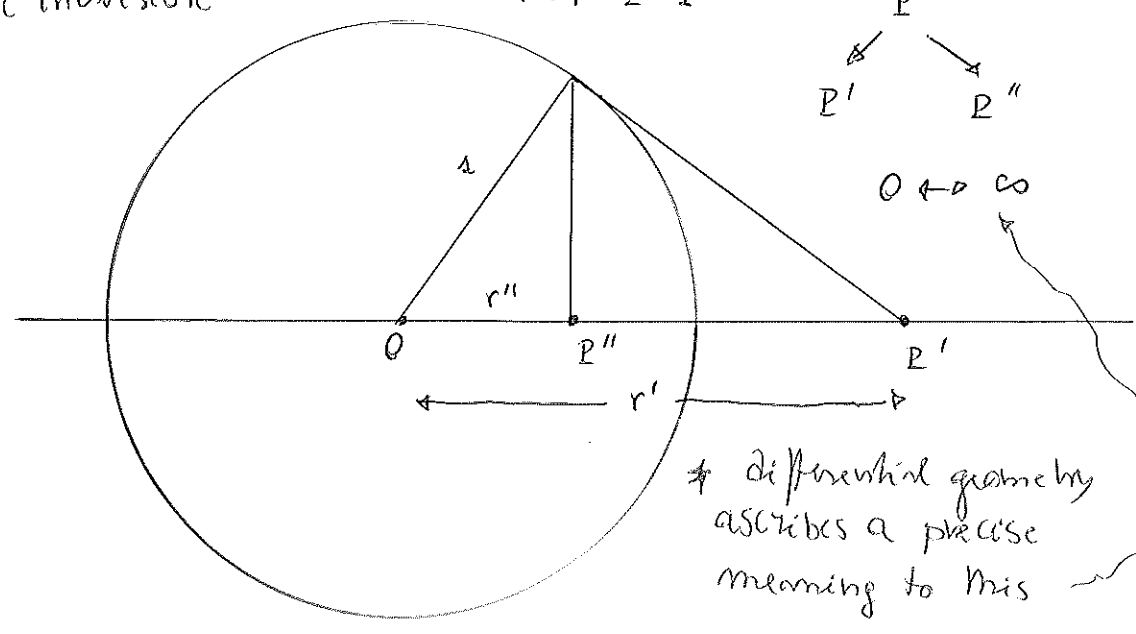
* conjugation is involved



notice this

$$\underbrace{x'' + iy''}_{\zeta''} = \frac{1}{\underbrace{x' + iy'}_{\zeta'}} = \frac{x' - iy'}{x'^2 + y'^2}$$

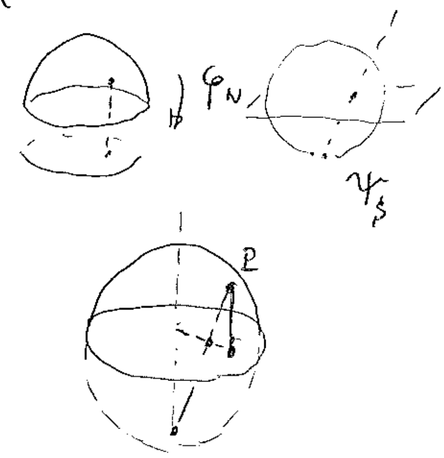
Further remark: P' and P'' are related by a circular inversion $\quad r' \cdot r'' = 1$



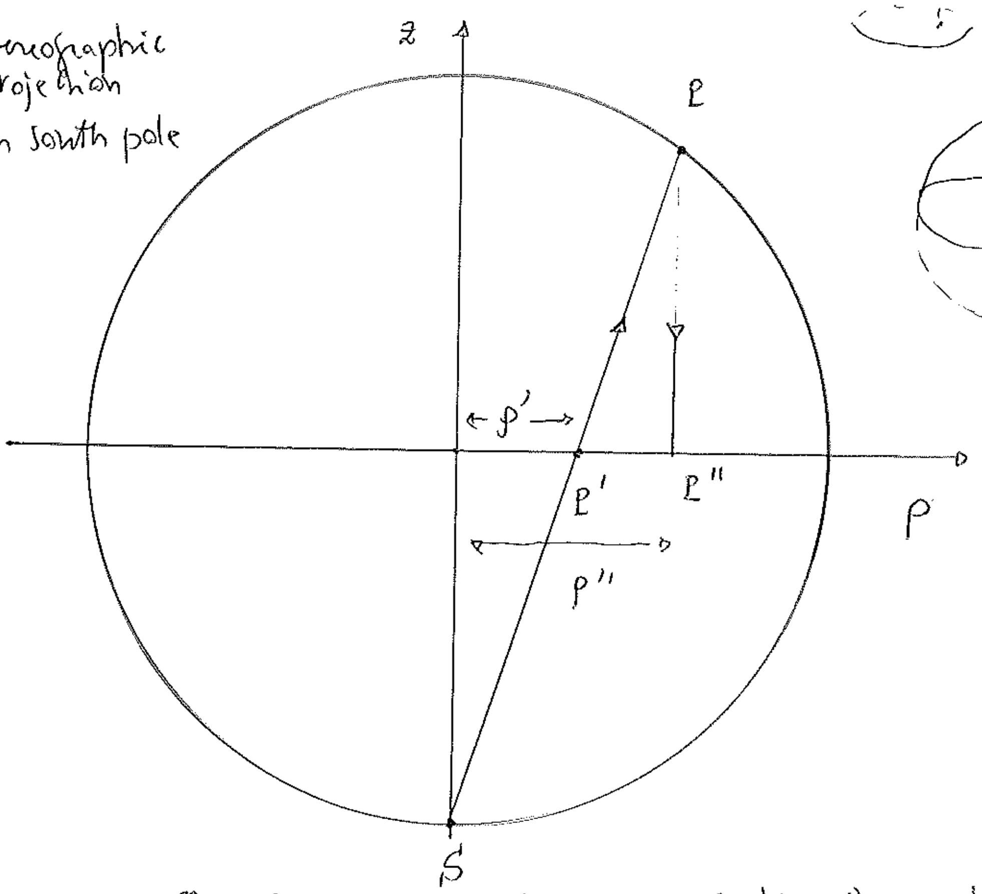
* differential geometry ascribes a precise meaning to this

6. Notice that we exhibited two atlases for the sphere. Let us check that they are compatible; it is then enough to show that the two charts below are compatible

$\varphi_N : U_N \to \mathbb{R}^2$ considered previously



ψ_S : stereographic projection from south pole



This is geometrically clear. Find the relationship between ρ' and ρ'' ....

They define the same differentiable structure