Functional Analysis

Second part - a.y. 2013-14

G. Orlandi

2-dec-2013 (2 hrs). The space $\mathcal{L}(E, F)$ of bounded linear operators between two Banach spaces E, F. Operator norm $||T||_{\mathcal{L}} = \sup\{||Tv||_F,\ ||v||_E \leq 1\}$. If $T \in \mathcal{L}(E)$ is injective and surjective, then $T^{-1} \in \mathcal{L}(E)$ by the open mapping theorem). Various notions of convergence for a sequence of operators $T_n \in \mathcal{L}(E, F)$ to $T \in \mathcal{L}(E, F)$: uniform ($||T_n - T||_{\mathcal{L}} \to 0$), strong ($T_n v \to Tv$ in $F\ \forall\ v \in F$), weak ($< \phi, T_n v > \to < \phi, Tv > \forall\, v \in F, \forall\, \phi \in F'$). The Weierstrass criterion for uniform convergence of series of operators. Neumann series: for $T \in \mathcal{L}(E)$ and $||T||_{\mathcal{L}} < 1$, $(I - T)$ is invertible (and $(I - T)^{-1} \in \mathcal{L}(E)$). Moreover, $(I - T)^{-1} = \sum_{n=0}^{+\infty} T^n$. It follows that the subset of invertible operators is *open* in $\mathcal{L}(E)$: if T is invertible then for any $S \in \mathcal{L}(E)$ such that $||S|| < ||T^{-1}||^{-1}$, the operator $T + S$ is invertible. Adjoint operator $T^* \in \mathcal{L}(F', E')$. It is defined by the identity $< T^*\phi, v > = < \phi, Tv >$ for any $v \in F$, $\phi \in F'$. It holds $\|T^*\| = \|T\|$, as a consequence of Hahn-Banach. In case $E = F = H$ a Hilbert space, from the identification $H \equiv H'$ given by the Riesz representation theorem, one considers $T, T^* \in \mathcal{L}(H)$. If $T = T^*$ the operator is called *self-adjoint* or *symmetric*. Examples: Hilbert-Schmidt integral operators on $L^2(\Omega)$, Fredholm integral operators on $C^0([a, b])$.

4-dec-2013 (2hrs). Elements of spectral theory for $T \in \mathcal{L}(E)$. Resolvent set $\rho(T) \subset \mathbb{C}$: we have $\lambda \in \rho(T)$ if $(\lambda I - T)^{-1} \in \mathcal{L}(E)$. The resolvent set is open in $\mathbb{C}$. Moreover, if $|\lambda| > \|T\|$ then $\{\lambda \in \mathbb{C},\ |\lambda| > ||T||\} \subset \rho(T)$. Actually, denoting $r(T) = \limsup_n(||T^n||)^{1/n} \leq ||T||$ the spectral radius of T, we have $\{\lambda \in \mathbb{C},\ |\lambda| > r\} \subset \rho(T)$. Spectrum $\sigma(T) = \mathbb{C} \setminus \rho(T)$ of $T \in \mathcal{L}(E)$: it is a closed set contained in $B(0, ||T||) \subset \mathbb{C}$. Let $\lambda \in \sigma(T)$: If $\ker(T - \lambda I) \neq 0$ then λ is an eigenvalue of T, and belongs to the point spectrum. Otherwise, λ belongs to the continuous spectrum ($\ker(T - \lambda I) = 0$ but $(T - \lambda I)$ is not surjective). In particular, the map $(T - \lambda I)^{-1}$ may be defined either in a dense or in a proper closed subspace of E, and may be either bounded or unbounded.

Examples: the right shift τ_r in ℓ^1 (or ℓ^2), or the diagonal operator $T_\alpha : \{x_n\} \mapsto \{\alpha_n x_n\}$ where $0 \neq \alpha_n \to 0$. In both cases 0 belongs to the continuous spectrum.

Moreover, $\{\alpha_n\} \subset \sigma(T_\alpha)$ is the point spectrum of T_α, while the point spectrum of τ_r is empty.

The multiplication operator $Tu(x) = x \cdot u(x)$ on $C^0([a,b])$. For any $\lambda \in \mathbb{R}$, $\ker(T - \lambda I) = 0$, hence there are no eigenvalues. Moreover, for $\lambda \notin [a,b]$, $(T-\lambda I)^{-1}v(x) = (x-\lambda)^{-1}v(x)$ is well-defined for any $v \in C^0([a,b])$ and is bounded, i.e. $\lambda \in \rho(T)$, while for $a \leq \lambda \leq b$ $(T-\lambda I)^{-1}$ is defined on the dense subspace $\{v \in C^0([a,b]), \ \ v(\lambda) = 0\}$, and it is unbounded. In particular, $\sigma(T) = [a,b]$ is the continuous spectrum of T.

The resolvent operator $R_\lambda = (T - \lambda I)^{-1}$ of $T \in \mathcal{L}(E)$, with $\lambda \in \rho(T)$. Resolvent equation $R_\lambda - R_\mu = (\lambda - \mu)R_\lambda R_\mu$: it yields $\frac{dR_\lambda}{d\lambda} = R_\lambda^2$, that is $\lambda \mapsto R_\lambda$ is a holomorphic function, whose singularities are in $\sigma(T)$. In particular, the Cauchy integral formula (and the calculus of residues) involving R_λ and a given holomorphic function $f(z)$ allows to consistently define $f(T)$ (in particular, if $f(z) = e^z$, we obtain a formula for $\exp(T)$, while if $f(z) = 1$ we derive some information on the Jordan blocks of T).

The space $\mathcal{K}(E,F) \subset \mathcal{L}(E,F)$ of compact operators. Uniform limits of compact operators in $\mathcal{L}(E,F)$ are compact, i.e. $\mathcal{K}(E,F)$ is closed in $\mathcal{L}(E,F)$. Operators whose range is finite dimensional are compact: they are called *finite rank* operators. Limits of sequences of finite rank operators are compact.

Finite rank approximation for $T \in \mathcal{K}(E,H)$, with H a Hilbert space: given $v_1, ..., v_N \in H$ a ϵ-net for $T(B_E)$, set $V_N =$span$\langle v_1, ..., v_N \rangle$ and $T_N = P_N \cdot T$, where P_N is the orthogonal projection on V_N. We have that T_N has finite rank and $||T_N - T||_{\mathcal{L}(E,H)} \leq 2\epsilon$.

Examples: $T : (x_n)_n \mapsto (a_n x_n)_n$, where $a_n \to 0$, is compact on ℓ^1 (resp. ℓ^2) as uniform limit of the finite rank operators $T_N : (x_n)_n \mapsto (\sigma_N a_n x_n)_n$, where $\sigma_N = 1$ for $n \leq N$ and $\sigma_N = 0$ for $n > N$.

5-dec-2013 (2hrs). Some further properties of compact operators: A compact operator (right- or left-) composed with a bounded operator is compact. In particular, $\mathcal{K}(E) \equiv \mathcal{K}(E,E)$ is a bilateral ideal of $\mathcal{L}(E)$. The identity map is compact if and only if E is finite dimensional. Any injective $T \in \mathcal{K}(E)$ doesn't admit a bounded inverse, unless E is finite dimensional. If E is reflexive (e.g. a Hilbert space), T is compact if and only if for any $v_n \rightharpoonup v$ weakly in E it holds $Tv_n \to Tv$ strongly in E. In particular, $T(\overline{B_E}) = \overline{T(B_E)}$. If $T \in \mathcal{K}(H)$ then $T^* \in \mathcal{K}(H)$ and conversely.

Some examples of compact operators: integral (kernel-based) operator of Fredholm-Volterra type on $C^0([a,b])$. More generally, operators based on integral kernels on a compact metric measure space (Mercer kernels): they are used in the statistical learning framework. Hilbert-Schmidt operators: if $K \in L^2([a,b] \times [a,b])$ then if $(Tx)(s) = \int_a^b K(s,t)x(t)\,dt$ we have $||T||_{\mathcal{L}} \leq ||K||_{L^2}$. Given a Hilbert basis (i.e. a complete orthonormal system) $\{\phi_n\}$ of $L^2([a,b])$, set $\psi_{nm}(s,t) = \phi_n(s)\phi_m(t)$: the elements ψ_{nm} are a Hilbert basis of $L^2([a,b] \times [a,b])$. Expanding $K(s,t) = \sum_{n=1}^\infty \sum_{m=1}^\infty k_{nm}\psi_{nm}(s,t)$, and setting respectively

$$K_N(s,t) = \sum_{n=1}^N \sum_{m=1}^N k_{nm}\psi_{nm}(s,t) \quad \text{ and } (T_N x)(s) = \int_a^b K_N(s,t)x(t)\,dt,$$

we have $||T_N - T||_{\mathcal{L}} \leq ||K_N - K||_2 \to 0$, hence $T \in \mathcal{K}(L^2([a,b]))$ as limit of finite rank operators.

The compact embedding $i : W^{1,p}([a,b]) \to C^0([a,b])$, $p > 1$; given a family of maps $u \in W^{1,p}([a,b])$ with equibounded norm, i.e. $\|u\|_p + \|u'\|_p \leq M$, we show that this family is equibounded in $C^0([a,b])$ and (uniformly) equicontinuous, hence by Ascoli-Arzela it is relatively compact in $C^0([a,b])$: decompose $u = v + c$, where $c = \frac{1}{b-a}\int_a^b u(x)\,dx$. We have, by Hölder inequality,

$$|c| \leq \frac{1}{|b-a|^{1/p}}\|u\|_p \leq \frac{1}{|b-a|^{1/p}}M, \quad |v(x)| \leq \int_{x_0}^x |u'(t)|\,dt \leq |b-a|^{1/p}\|u'\|_p \leq |b-a|^{1/p}M\,,$$

where $x_0 \in [a,b]$ is such that $v(x_0) = \frac{1}{b-a}\int_a^b v(x)\,dx = 0$. We deduce the uniform bound $\|u\|_\infty \leq CM$, where C depends only on $[a,b]$. To prove equicontinuity, observe that

$$|u(x) - u(y)| \leq \int_x^y |u'(t)|\,dt \leq |x-y|^\alpha \|u'\|_p \leq |x-y|^\alpha M, \qquad \text{where } \alpha = 1 - \frac{1}{p}\,,$$

hence the maps u are equi-Hölder continuous.

Spectrum of a compact operator: $0 \in \sigma(T)$ and $\sigma(T) \setminus \{0\}$, if non empty, is made of at most countably many eigenvalues λ_n, with $\lambda_n \to 0$ as $n \to +\infty$ (the fact that $0 \neq \lambda_n \in \sigma(T)$ is an eigenvalue follows from the Fredholm Alternative, point 3.). The corresponding eigenspaces $\ker(\lambda_n I - T) \neq 0$ are finite-dimensional (point 1. of Fredholm Alternative). If the operator is self-adjoint on a Hilbert space, then the eigenvalues are real, and $\max|\lambda_n| = \|T\|_{\mathcal{L}}$ (see Hilbert-Schmidt Spectral Theorem).

11-dec-2013 (2 hrs). The Fredholm Alternative for operators of the type $A = I - T$, with $T \in \mathcal{K}(H)$, H a Hilbert space:

1. $\ker A$ is finite dimensional,
2. the range $R(A)$ is closed, hence there holds the orthogonal direct sum decomposition $H = R(A) \oplus \ker A^* = R(A^*) \oplus \ker A$, where $A^* = I - T^*$.
3. $\ker A = 0 \Leftrightarrow R(A) = H$,
4. $\dim\ker A = \dim\ker A^* < +\infty$.

We have proved point 1., 2. and 3., following essentially references [1] and [3].

The Fredholm alternative gives a procedure to solve equation of the type $Au = u - Tu = f$, with T compact. First of all, solve the associated adjoint homogeneous equation, i.e. the fixed point equation $v = T^*v$. If the solution is trivial, then $Au = f$ admits a unique solution for any datum $f \in H$ (that can be possibly found in an iterative way via contraction mapping principle, as in the case of Volterra integral operators). Otherwise, call $v_1, ...v_k$ a basis of $\ker A^*$ (i.e. a maximal independent set of fixed points of T^*); then there are solutions of $Au = f$ provided f verifies the orthogonality conditions $< f, v_i >= 0$ for any $i = 1, ..., k$.

The Fredholm Alternative holds more generally for operators of the type $A = I - T$ with $T \in \mathcal{K}(E)$, E a Banach space.

Spectral theory for self-adjoint compact operators in Hilbert spaces: the eigenvalues are real and there exists a Hilbert basis made of eigenvectors, which "diagonalizes" the operator. In particular, for $T \in \mathcal{K}(H)$, $T^* = T$, and e_n a orthonormal basis of eigenvectors, i.e. $Te_n = \lambda_n e_n$ (with $\lambda_n \to 0$), we have the diagonal representation $Tv = T(\sum_n c_n e_n) = \sum_n \lambda_n c_n e_n$, i.e. the operator can be identified with $\tilde{T} \in \mathcal{K}(\ell^2)$ given by $\tilde{T}(c_n) = (\lambda_n c_n)$. Moreover, $\|T\|_{\mathcal{L}} = \max_n |\lambda_n|$.

12-dec-2013 (2 hrs). *Proof of the spectral theorem:* we consider a (iterated) contrained optimization problem on the unit closed ball $\overline{B} = \{\|v\| \leq 1\}$ of H for the quadratic form $Q(v) = \langle Tv, v\rangle$ associated to $T \in \mathcal{K}(H)$. Notice first that $Q(v)$ is weakly continuous, since $v_n \rightharpoonup v_0$ implies $Tv_n \to Tv_0$, and moreover $\|v_n\| \leq M$ (weakly convergent sequences are bounded), whence

$$|\langle Tv_n, v_n\rangle - \langle Tv_0, v_0\rangle| \leq |Tv_n - Tv_0| \cdot |v_n| + |\langle Tv_0, v_n - v_0\rangle| \to 0.$$

By Weierstrass Theorem, $|Q(v)|$ reaches its maximum on the unit closed ball $\overline{B}$, which is weakly compact. Let e_1 be a maximum point. We have necessarily $\|e_1\| = 1$ because $Q(\lambda v) = \lambda^2 Q(v)$ for $\lambda \in \mathbb{R}$. Moreover, for any $e \in H$ such that $\|e\| = 1$ and $\langle e, e_1\rangle = 0$, one has $\langle e, Te_1\rangle = 0$, since by the Lagrange multipliers theorem e_1 is a critical point of the function $Q(v) + \lambda\|v\|^2 = \psi(\alpha, \beta, \lambda)$, where $v = \alpha e_1 + \beta e_2$ belongs to the 2-dimensional space spanned by e_1 and e. In particular, one deduces $Te_1 = \langle Te_1, e_1\rangle \cdot e_1 = \lambda_1 e_1$, i.e. e_1 is an eigenvector of T and $|Q(e_1)| = |\langle Te_1, e_1\rangle| = |\lambda_1|$, i.e. the eigenvalue λ_1 has maximum modulus among the eigenvalues of T (actually we have $|\lambda_1| = \|T\|_{\mathcal{L}}$).

Iterating this procedure, one obtains, for $n \geq 1$, an eigenvector e_n of T, with $\|e_n\| = 1$, and such that $< e_n, e_m >= 0$ for any $m < n$, corresponding to the maximum point of $|Q(v)|$ on $(\text{span}\{e_1, ..., e_{n-1}\})^\perp \cap \overline{B}$, with $\lambda_n = Q(e_n)$ the corresponding eigenvalue. Moreover, it holds $|\lambda_{n-1}| \geq |\lambda_n|$.

If for some $n_0 \in \mathbb{N}$ one has $\lambda_{n_0} = Q(e_{n_0}) = 0$, then $(\text{span}\{e_1, ..., e_{n_0-1}\})^\perp = \ker T$. Indeed, $Q(w) = 0$ for any $w \in (\text{span}\{e_1, ..., e_{n_0-1}\})^\perp$, and if $\langle w, e_i\rangle = 0\ \forall\, i < n_0$, then $\langle Tw, e_i\rangle = \langle v, Te_i\rangle = 0$, i.e. also $Tw \in (\text{span}\{e_1, ..., e_{n_0-1}\})^\perp$. The polarization identity $4\langle Tv, u\rangle = Q(u + v) - Q(u - v)$ hence implies that $4\langle Tw, Tw\rangle = Q(w + Tw) - Q(w - Tw) = 0$ for any $w \in (\text{span}\{e_1, ..., e_{n_0-1}\})^\perp$, i.e. $Tw = 0$.

We deduce in this case that the set $\{e_1, ..., e_{n_0}\}$, completed with a (complete) orthonormal system of $\ker T$ yields a Hilbert basis of eigenvectors of T.

Otherwise, we are left with a orthonormal sequence $\{e_n\}_n$, so that in particular $e_n \rightharpoonup 0$ by Bessel inequality (for any $w \in H$, $\sum_n \langle e_n, w\rangle^2 \leq ||w||^2 \Rightarrow \langle e_n, w\rangle \to 0$ as $n \to +\infty$), and hence $|\lambda_n| = |Q(e_n)| \searrow 0$ by weak continuity of Q. Let $N = \overline{\text{span}\{e_1, ..., e_n, ...\}}^\perp$. For any $w \in N$ one necessarily has $|Q(w)| \leq |Q(e_n)|$ for any $n \in \mathbb{N}$, hence $Q(w) = 0$ and $N = \ker T$.

In this case, the set $\{e_n\}_{n\in\mathbb{N}}$, completed with a (complete) orthonormal system of $\ker T$ yields a Hilbert basis of eigenvectors of T. $\square$

The Lax-Milgram Lemma: given a bilinear form $a(u,v)$, continuous ($a(u,v) \leq M\|u\|\|v\|$) and coercive ($0 < \alpha\|u\|^2 \leq a(u,u)\ \forall\ u \neq 0$) on a Hilbert space H, for any bounded linear form $\phi \in H^*$ there exists a unique $u \in H$ such that $a(u,v) = \phi(v)$ for any $v \in H$. In particular, $\|u\| \leq \alpha^{-1}\|\phi\|_*$.

If moreover a is symmetric (i.e. $a(u,v) = a(v,u)$), we have the characterization $u = \arg\min\{\frac{1}{2}a(v,v) - \phi(v)\,,\ v \in H\}$.

Proof: by Riesz representation theorem, the equation to be solved can be rewritten as $\langle Au, v\rangle = \langle f, v\rangle$ for any $v \in H$, i.e. $Au = f$, where $A \in \mathcal{L}(\mathcal{H})$ verifies the estimates $0 < \alpha\|u\| \leq \|Au\| \leq M\|u\|\ \forall\ u \neq 0$.

From $\alpha\|u\| \leq \|Au\|$ (which is called an *a priori estimate*) it follows that $\ker A = 0$. Moreover, $\alpha\|u_n - u_m\| \leq \|Au_n - Au_m\|$ implies that if $y_n = Au_n \to y$ in H, i.e. Au_n is a Cauchy sequence in H, then also u_n is a Cauchy sequence, hence $u_n \to u$ in H by completeness, thus yielding $y = Au$. One concludes that A has a closed range $R(A)$ in H. Finally, if $v \perp R(A)$, then $\langle v, Au\rangle = 0\ \forall\ u \in H$. In particular, choosing $u = v$, we have $0 = \langle v, Av\rangle \geq \alpha\|v\|^2$, thus $v = 0$ and $R(A) = H$. We just proved that A is both injective and surjective, and the conclusion of the Lemma follows.

In case of a symmetric a, since $\alpha\|u\|^2 \leq a(u,u) \leq M\|u\|^2$, the scalar product $((u,v)) := a(u,v)$ is equivalent to $\langle\cdot,\cdot\rangle$, hence by Riesz representation theorem applied to H endowed with $((\cdot,\cdot))$, one has $\phi(v) = a(g,v)$ for a certain $g \in H$, whence u verifies $a(u-g,v) = 0\ \forall\ v \in H$, i.e. u is the orthogonal projection (with respect to the scalar product induced by a) of g on H, in other words u minimizes the (squared) distance (induced by a) $a(v-g, v-g)$, or, equivalently, the quadratic functional $F(v) = \frac{1}{2}a(v,v) - \phi(v)$, for $v \in H$, whose Euler-Lagrange equation $\partial_v F(u) \equiv \langle F'(u), v\rangle = 0$ for any direction $v \in H$ is precisely given by $a(u,v) = \phi(v)$ for any $v \in H$.

$\square$

A generalization of Lax-Milgram lemma is given by Stampacchia theorem.

16-dec-2013 (1 hr). The Galerkin approximation method: if $V_h \subset H$, $\dim V_h < +\infty$, one considers the solution u_h of the system $a(u,v) = \phi(v)\ \forall\ v \in V_h$. We have the uniform bound $\|u_h\| \leq \alpha^{-1}\|\phi\|_*$, which gives weak compactness of the sequence $\{u_h\}$. Moreover, the Lemma of Céa guarantees that $\|u - u_h\| \leq \frac{M}{\alpha}\mathrm{dist}\,(u, V_h)$ (in other words, u_h is comparable to the orthogonal projection of u on V_h): indeed, $a(u-u_h, u-u_h) = a(u-u_h, u-v)$ for any $v \in V_h$ since $a(u, v-u_h) = a(u_h, v-u_h) = \phi(v-u_h)$, whence $\alpha\|u-u_h\|^2 \leq M\|u-u_h\|\|u-v\|$ for any $v \in V_h$ and the conclusion follows.

Hence, considering a sequence of finite-dimensional spaces $V_h \subset V_{h+1}$ such that $H = \overline{\cup_h V_h}$, one has the convergence $u_h \to u$ in H as $h \to +\infty$.

Remark that the approximating finite-dimensional problem is a linear system with a positive definite coefficients matrix, called *stiffness matrix*, which is given by $[a(f_i, f_j)]$, with $\{f_i\}$ a basis for V_h.

The choice of the sequence V_h invading H and of a basis $\{f_i\}$ for V_h is aimed to efficiently solve the approximating linear system, and also to have the best possible convergence rate for the error estimate $||u_h - u||$. Here are some examples in case $H = L^2(\Omega)$, $\Omega \subset \mathbb{R}^n$:

1) if a is represented by a compact self-adjoint operator, then considering a Hilbert basis $\{e_n\}_{n\in\mathbb{N}}$ of $L^2(\Omega)$ made of eigenvectors, and setting V_h =span$< e_1, ..., e_h >$, the corresponding system is diagonal.

2) considering a basis $\{f_i\}$ of V_h made of *finite elements* (piecewise linear or polynomial function insisting on a fixed triangulation of the domain) yields a sparse stiffness matrix. Finite elements are used in numerical fluid dynamics, material science, elasticity,...

3) Haar basis, wavelets, radial basis functions: these Hilbert basis of $L^2(\Omega)$ are used in signal and image processing and statistical analysis, being not computationally expensive, and also since they are able to take into account localized oscillation phenomena at any scale in physical and in frequency space.

3) if the original problem admits a smooth solution (for example, $u \in C^\infty(\Omega)$ as for Laplace equation), it may be convenient to use *spectral methods* for its approximation, i.e. to consider a Hilbert basis of $L^2(\Omega)$ made of orthogonal polynomials (e.g. the trigonometric system, the Legendre polynomials, the Hermite polynomials): since the Lemma of Céa states that the error estimate $\|u-u_h\|$ is comparable to the distance of u to its orthogonal projection on V_h, hence the convergence rate will be better according to the regularity of u (for instance, the more regular u, the more rapidly its Fourier coefficients decay to 0).

18-dec-2013 (2 hrs). Weak / variational formulation of elliptic boundary value problems in dimension 1. Classical vs weak solutions. Homogeneous Dirichlet problem: weak formulation in H^1_0, existence, uniqueness, a priori estimates, variational characterization of the weak solution as the minimizer of the Dirichlet energy. Analysis of the (homogeneous) Sturm-Liouville problem, compactness and spectral decomposition of Sturm-Liouville operators.

9-jan-2014 (2 hrs). Basic facts from the theory of distributions. Motivation: existence theory for partial differential equations. For an open set $\Omega \subset \mathbb{R}^n$ define the space of smooth test functions $\mathcal{D}(\Omega) = C^\infty_c(\Omega)$. While the space of continuous (resp. C^k) test functions $C^0_c(\Omega)$ (resp. $C^k_c(\Omega)$) has a metric naturally induced by the norm $\|\phi\|_{C^0} \equiv \|\phi\|_\infty$ (resp. $\|\phi\|_{C^k} \equiv \sum_{0\le|\alpha|\le k} \|D^\alpha\phi\|_\infty$), the space of smooth test function has a metric built upon the family of semi-norms $\|D^\alpha\phi\|_\infty$, for any multi-index α (such a space is not a normed space, but it is called a Fréchét space), for instance consider

$$\text{dist}\,(\phi,\psi) = \sum_{k=0}^{\infty} 2^{-k} \frac{\|\phi-\psi\|_{C^k}}{1+\|\phi-\psi\|_{C^k}}.$$

Moreover, one says that $\phi_n \to \phi$ in $C_c^\infty(\Omega)$ if $\text{dist}\,(\phi_n, \phi) \to 0$ and $\text{spt}\,\phi_n \subset K$, with $K \subset \Omega$ a fixed compact set, independent of n. Remark that $\phi_n \to \phi$ in $C_c^\infty(\Omega)$ if and only if ϕ_n and all their derivatives converge uniformly on $K \subset \Omega$.

The space of distributions $\mathcal{D}'(\Omega) \equiv [C_c^\infty(\Omega)]'$ is defined as the space of linear and continuous functionals over the space of smooth test functions. In particular, for $T \in \mathcal{D}'(\Omega)$, stating $T(\phi_n) \to T(\phi)$ for any $\phi_n \to \phi$ in $C_c^\infty(\Omega)$ is equivalent to state that for any $K \subset \Omega$ compact, there exists $N \in \mathbb{N}$ and $C > 0$ (both possibly depending on K) such that

$$|T(\phi)| \le C \sup_{0 \le |\alpha| \le N} \|D^\alpha \phi\|_{L^\infty(K)}.$$

Convergence in the sense of distributions: for $T, T_j \in \mathcal{D}'(\Omega)$ we say that $T_n \rightharpoonup T$ in $\mathcal{D}'(\Omega)$ if $T_j(\phi) \to T(\phi)$ for any $\phi \in C_c^\infty(\Omega)$ (it is thus defined as a weak convergence).

Distribution associated to a locally integrable function $u \in L^1_{loc}(\Omega)$ (i.e. $u \in L^1(\Omega \cap B)$ for any ball B): set $T_u(\phi) = \int_\Omega u(x)\phi(x)\,dx$. Let $\text{spt}\,\phi = K$. We have $|T_u(\phi)| \le \|u\|_{L^1(K)} \cdot \|\phi\|_\infty$, so that $T_u \in \mathcal{D}'(\Omega)$.

Actually, if $u \in L^1(\Omega)$, we have $T_u \in [C_c^0(\Omega)]^*$, and $\|T_u\|_* = \sup\{T_u(\phi),\ \|\phi\|_\infty \le 1\} = \|u\|_1$.

Moreover, if $T_u = T_v$ for $u, v \in L^1_{loc}(\Omega)$ we have $0 = \int_\Omega (u(x) - v(x))\phi(x)\,dx$ for any $\phi \in C_c^\infty(\Omega)$, i.e. $u = v$ a.e. in Ω. The map $u \mapsto T_u$ gives thus an injection of $L^1_{loc}(\Omega)$ in $\mathcal{D}'(\Omega)$, (actually, in $[C_c^0(\Omega)]^* \subset \mathcal{D}'(\Omega)$).

Derivatives in the sense of distributions (distributional derivatives). For $T \in \mathcal{D}'(\Omega)$ we define $D^\alpha T(\phi) = (-1)^{|\alpha|} T(D^\alpha \phi)$, for $\phi \in \mathcal{D}(\Omega)$. If $u \in C^k(\Omega)$ and $|\alpha| \le k$ then we have as expected $D^\alpha(T_u) = T_{D^\alpha u}$ by iteratively applying the integration by part formula

$$\int_\Omega \partial_i u(x)\phi(x)\,dx = -\int_\Omega u(x)\partial_i \phi(x)\,dx\,.$$

Observe that if $T_j \rightharpoonup T$ in $\mathcal{D}'(\Omega)$ then $D^\alpha T_j \rightharpoonup D^\alpha T$ in $\mathcal{D}'(\Omega)$ for any multiindex α.

Examples: for $u(x) = |x|$, $x \in \mathbb{R}$, we have $(T_u)' = T_v$ where $v(x) = 1$ if $x > 0$ and $v(x) = -1$ if $x < 0$. Moreover, $(T_v)' = 2 \cdot T_{\delta_0}$, where $T_{\delta_0}(\phi) = \phi(0)$, for $\phi \in C_c^\infty(\Omega)$, is the distribution associated to the Dirac mass concentrated at the origin. In particular $T_{\delta_0} \in [C_c^0]^*$ and

$$\|T_{\delta_0}\|_* = \sup\{\phi(0),\ \phi \in C_c^0, \|\phi\|_\infty \le 1\} = 1.$$

On $\mathbb{R}$ we have $T_{\delta_0} = (T_H)'$, where $H(x) = 1$ if $x \ge 0$, $H(x) = 0$ if $x < 0$ is the Heaviside function. consider the following approximating sequence for H, given by $u_j(x) = 1$ if $x \ge j^{-1}$, $u_j(x) = j \cdot x$ for $0 \le x \le j^{-1}$, $u_j(x) = 0$ for $x < 0$. We have $(T_{u_j})' = T_{v_j}$ where $v_j(x) = j$ for $0 \le x \le j^{-1}$, and $v_j(x) = 0$ elsewhere in $\mathbb{R}$. Notice that $\|T_{v_j}\|_* = \int_\mathbb{R} v_j(x)\,dx = 1$ for any $j \in \mathbb{N}$. We have $T_{u_j} \rightharpoonup T_H$ in $\mathcal{D}'$ since $u_j \to H$ in L^1, so that $|T_{u_j}(\phi) - T_H(\phi)| \le \int_\mathbb{R} |u_j(x) - H(x)| \cdot |\phi(x)|\,dx$ vanishes as $j \to +\infty$. In particular we have $T_{v_j} = (T_{u_j})' \rightharpoonup (T_H)' = T_{\delta_0}$.

Observe that the Heaviside function has a classical derivative a.e. (actually everywhere except in the origin), which is identically zero, while the distributional derivative of H keeps the information on the unit jump of H at the origin.

13-jan-2014 (2 hrs). From the inclusions $C_c^\infty(\Omega) \subset C_c^{k+1}(\Omega) \subset C_c^k(\Omega)$ valid for any $k \geq 0$ we have the continuous injections

$$[C_c^0(\Omega)]^* \subset [C_c^1(\Omega)]^* \subset ... \subset [C_c^k(\Omega)]^* \subset ... \subset [C_c^\infty(\Omega)]' = \mathcal{D}'(\Omega).$$

Distributions in $[C_c^k(\Omega)]^*$ are called distributions of order k. In particular, distributions associated to locally integrable functions are of order zero.

The distribution T_μ associated to a (possibly σ-finite) Radon measure μ on Ω: $T_\mu(\phi) \equiv \langle T_\mu, \phi\rangle = \int_\Omega \phi(x) d\mu(x)$.

Example: the Dirac distribution T_{δ_0}: we have $\langle T_{\delta_0}, \phi\rangle = \int_\Omega \phi(x) d\delta_0(x) = \varphi(0)$.

For any $\mu \in \mathcal{M}(\Omega)$ (the space of finite Radon measures on Ω) it actually holds $T_\mu \in [C_c^0(\Omega)]^*$ and

$$\|T_\mu\|_* = \sup\{\langle T_\mu, \phi\rangle\,,\ \phi \in C_c^0(\Omega)\,,\ \|\phi\|_\infty \leq 1\} = |\mu|(\Omega),$$

where $|\mu|$ is the total variation measure of μ and $\|\mu\| = |\mu|(\Omega)$ is the *total variation* of the measure μ on Ω.

For $\mu, \nu \in \mathcal{M}(\Omega)$ two Radon measures on Ω, $T_\mu = T_\nu$ implies $\int_\Omega \phi(x)\, d\mu(x) = \int_\Omega \phi(x)\, d\nu(x)$, for any $\phi \in C_c^\infty(\Omega)$, so that by considering ϕ_j converging to the characteristic function of an open subset $A \subset \Omega$ se obtain, by the dominated convergence theorem, that $\mu(A) = \nu(A)$ for any $A \subset \Omega$ open, hence $\mu = \nu$ by the regularity of Radon measures. Hence $\mu \mapsto T_\mu$ gives an injection $\mathcal{M}(\Omega) \to [C_c^0(\Omega)]^* \subset \mathcal{D}'(\Omega)$. Hence (finite and σ-finite) Radon measures in Ω are distribution of order zero.

Riesz representation theorem: $\mu \in \mathcal{M}(\Omega) \mapsto T_\mu \in [C_c^0(\Omega)]^*$ is an isomorphism of Banach spaces, where the norm of a measure $\mu \in \mathcal{M}(\Omega)$ is given by its *total variation* $\|\mu\| = |\mu|(\Omega)$. In particular, it holds $|\mu|(\Omega) = \|\mu\| = \|T_\mu\|_*$. Weak* compactness and convergence in the sense of measures: by the Banach-Alaoglu theorem (bounded sets in the dual of a Banach space are relatively compact with respect to the weak* topology), a sequence of equibounded Radon measures μ_n on Ω (i.e. $|\mu_n|(\Omega) \leq C$) is weakly* compact, i.e. there exists a subsequence μ_{n_k} and a measure μ such that $\mu_{n_k} \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega)$, or in other words $\int_\Omega \phi(x)\, d\mu_{n_k}(x) \to \int_\Omega \phi(x)\, d\mu(x)$ for any $\phi \in C_c^0(\Omega)$.

The space of Radon measures is suited to solve optimization problems involving L^1 or total variation norms. For example, for $Y \subset \mathcal{M}(\Omega)$, Y closed convex and bounded, consider the minimization problem

$$\inf\{\|\mu\| = \mu(\Omega) = \|T_\mu\|_*,\ \mu \in Y\}\,.$$

Then, since Y is closed and convex it is weakly* closed by Hahn-Banach, and since it is also bounded it is weakly* compact by Banach-Alaogliu. The total variation norm is weakly* lowersemicontinuous being characterized as a supremum, as a dual norm. Hence the existence of a minimizer in Y is guaranteed by Weierstrass theorem (or equivalently by the direct method of the calculus of variations).

Product distribution $\psi \cdot T$, where $\psi \in C^\infty(\Omega)$: $\langle \psi \cdot T, \phi\rangle = \langle T, \psi \cdot \phi\rangle$. Convolution of a distribution with a test function: it is a smooth function $x \mapsto T * \phi(x)$ defined by

$T * \phi(x) = \langle T, \phi(x - \cdot)\rangle$. It holds $D^\alpha(T * \varphi) = (D^\alpha T) * \varphi = T * (D^\alpha \varphi)$. Support of a distribution T: it is the complementary of the largest open set O such that $T(\phi) = 0$ for any ϕ such that $\operatorname{spt}\phi \subset O$. In particular, if T has compact support, then $T * \phi \in C_c^\infty$. Convolution of distributions: for $T, S \in \mathcal{D}'(\Omega)$, with S with compact support, define $\langle T * S, \phi\rangle = \langle T, S * \phi\rangle$. The Dirac distribution δ_0 is the neutral element with respect of the convolution product.

Distributional gradient: it is a vector distribution in $[\mathcal{D}'(\Omega)]^n$ given by $\langle \nabla T, \vec{\varphi}\rangle = -\langle T, \operatorname{div}\vec{\varphi}\rangle$ for $\vec{\varphi} \in [\mathcal{D}(\Omega)]^n$. Remark that for distributions associated to functions of class C^1 previous formula coincides with Green (or integration by parts) formula.

Analogously , the distributional divergence of a vector distribution $\vec{T} = (T_1, ..., T_n)$ is defined as $\langle \operatorname{div}\vec{T}, \phi\rangle = -\langle \nabla T, \phi\rangle$. Distributional Laplacian $-\Delta T$: we have $\langle -\Delta T, \phi\rangle = \langle T, -\Delta\phi\rangle$.

Differential problems in the distributional sense. Fundamental solution of a linear and continuous operator L on $\mathcal{D}'(\mathbb{R}^n)$: it is a distribution G such that $L(G) = \delta_0$. For $F \in \mathcal{D}'(\mathbb{R}^n)$, the distribution $U = G * F$ is a solution of the equation $L(U) = F$ in $\mathcal{D}'(\mathbb{R}^n)$.

Example: the fundamental solution of the distributional Laplacian in $\mathbb{R}^n$, which satisfies $-\Delta G = \delta_0$ in $\mathcal{D}'(\mathbb{R}^n)$, is the function $G(x) = c_n|x|^{2-n}$ for $n > 2$, with $c_n > 0$ a suitable constant (e.g. $c_3 = \frac{1}{4\pi}$), and $G(x) = -\frac{1}{2\pi}\log|x|$ for $n = 2$.

15-jan-2014 (2 hrs). Distributional definition of the spaces $W^{1,p}(\Omega)$. Definition of $BV(\Omega)$ (functions of bounded variation): $u \in BV(\Omega)$ if $u \in L^1(\Omega)$ and the (distributional) gradient $Du = (D_1u, ..., D_nu)$ is a (vector) Radon measure, which satisfies the integration by part formula (Gauss-Green)

$$\int_\Omega u \operatorname{div}\vec{\phi} = -\int_\Omega \vec{\phi} \cdot dDu \qquad \text{for any } \vec{\phi} \in [C_c^0(\Omega)]^n .$$

Total variation of a vector Radon measure: for $\vec{\mu} = (\mu_1, ..., \mu_n)$ with $\mu_i \in \mathcal{M}(\Omega) = (C_c^0(\Omega))'$ we have the decomposition $\vec{\mu} = \vec{\nu}|\vec{\mu}|$, where $|\vec{\mu}|$ is a positive measure (called the total variation measure) and $|\vec{\nu}(x)| = 1$ for $|\vec{\mu}|$ a.e. $x \in \Omega$. The total variation of $\vec{\mu}$ is defined as

$$\|\vec{\mu}\| = \sup\left\{\int_\Omega \vec{\phi}\cdot d\vec{\mu} = \int_\Omega \vec{\phi}\cdot\vec{\nu}\, d|\vec{\mu}|\,, \quad \vec{\phi} \in [C_c^0(\Omega)]^n, \quad \|\vec{\phi}\|_\infty \le 1\right\} = |\vec{\mu}|(\Omega).$$

Example: the characteristic function $\mathbf{1}_E$ of an open bounded set $E \subset \mathbb{R}^n$ with $\partial E \cap \Omega$ of class C^1 belongs to $BV(\Omega)$, since by Gauss-Green formula

$$D\mathbf{1}_E(\vec{\phi}) = -\int_E \operatorname{div}\vec{\phi}\, dx = -\int_{\partial E} \vec{\phi}\cdot\vec{n}\, d\sigma\,,$$

where $\vec{n}$ is the unit outer normal to ∂E and $d\sigma$ is the surface measure on ∂E, so that $|D\mathbf{1}_E(\vec{\phi})| \le \|\vec{\phi}\|_\infty \cdot \text{Area}(\partial E \cap \Omega)$, i.e. $D\mathbf{1}_E$ is a vector Radon measure, and in particular $D\mathbf{1}_E = \vec{\nu}|D\mathbf{1}_E|$, where $\vec{\nu}(x) = -\vec{n}(x)$ is the inner unit normal to $\partial E \cap \Omega$ and $|D\mathbf{1}_E| = d\sigma$. By a suitable choice of the test function $\vec{\phi}$ in such a way that $|\vec{\phi}(x)| \le 1$ and $\vec{\phi} = -\vec{n}$ on $\partial E \cap \Omega$ one gets $|D\mathbf{1}_E|(\Omega) = \text{Area}(\partial E \cap \Omega)$.

Equivalent definitions of $W^{1,p}$ and BV norm. Some properties of $W^{1,p}(\Omega)$: completeness, reflexivity, separability, according to the exponent p. The Hilbert space $H^1(\Omega) = W^{1,2}(\Omega)$. The space $W^{1,\infty}(\Omega)$ coincides with the space of Lipschitz functions on Ω, when Ω is a regular domain (with boudary of class C^1 or lipschitz). Leibniz rule and chain rule n $W^{1,p}$.

Characterization of maps in $W^{1,p}(\Omega)$: weak derivatives as a bounded linear functionals on $L^{p'}$, uniformly bounded differential quotients w.r.t the L^p norm (this property implies that the injection $W^{1,p}(\Omega) \to L^p(\Omega)$ is compact, according to Fréchét-Kolmogorov theorem): in case $p = 1$ these properties characterize the space $BV(\Omega)$ of functions of bounded variation.

16-jan-2014 (2 hrs). (see [1], chapter 9) Density of smooth functions in $W^{1,p}(\Omega)$: extension of a function $u \in W^{1,p}(\Omega)$ to a function $\bar{u} \in W^{1,p}(\mathbb{R}^N)$ and regularization by convolution with a family of smoothing kernels $\rho_n \in C_c^\infty(\mathbb{R}^N)$ such that $\rho_n \rightharpoonup \delta_0$ in the sense of distributions. Proof of the compact injection of $W^{1,p}(\Omega)$ in $L^p(\Omega)$ for $\Omega \subset \mathbb{R}^N$ bounded: if $\mathcal{F}$ is a bounded family of $W^{1,p}(\Omega)$, and $\omega \subset\subset \Omega$, then $\rho_n * \mathcal{F}_{|\omega}$ is ϵ-close to $\mathcal{F}_{|\omega}$ for large n, and uniformly bounded in L^∞ and equi-uniformly continuous, hence relatively compact with respect to the $\|\cdot\|_\infty$ norm by Ascoli-Arzelà, so that in particular it is relatively compact in $L^p(\omega)$. An ϵ-net for $\rho_n * \mathcal{F}_{|\omega}$ in $L^p(\omega)$ is then a 2ϵ-net for $\mathcal{F}_{|\omega}$ and a 3ϵ-net for $\mathcal{F}$ in $L^p(\Omega)$ if ω is sufficiently close in measure to Ω. The space $W_0^{1,p}(\Omega)$, suited for homogeneous Dirichlet boundary value problems. Poincaré inequality. Sobolev inequalities in $\mathbb{R}^n$ and in domains Ω of class C^1. Rellich-Kondrachov compact embedding theorem.

20-jan-2014 (2 hrs). (see [1], chapter 9) Weak / variational formulation of elliptic boundary value problems in dimension N (see [1], chapter. Classical vs weak solutions. Homogeneous Dirichlet problem: weak formulation in H_0^1, existence, uniqueness, a priori estimates, H^2-regularity and higher regularity of the weak solution (analysis carried out only in the easier case $N = 1$). Variational characterization of the weak solution as the minimizer of the Dirichlet energy, and relationship with a discretization in time of the gradient flow of the Dirichlet energy (in other words, a discretization in time (Euler-type scheme) of the heat equation: denoting $t_n = n\Delta t$, $u(t_n, \cdot) = v_n(\cdot)$ one obtains v_{n+1} as the minimizer in of the Dirichlet energy functional $F(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{2\Delta t}\|v - v_n\|^2_{L^2(\Omega)}$). This gives also an example of Tychonoff regularization for signal (or image) processing.

Solvability of a general second-order elliptic boundary value problem in divergence form through the Fredholm Alternative (see [1], Theorem 9.23).

22-jan-2014 (2 hrs). Homogeneous Neumann problem. Maximum principle for elliptic equations, applications to uniqueness and stability of the Dirichlet problem. Spectral theory of the Laplacian: existence of a Hilbert basis of $L^2(\Omega)$ made by eigenfunctions of the Laplacian in $H_0^1(\Omega)$ guaranteed by compactness and self-adjointness of the solution operator. Integral representation of the solution of the homogeneous Dirichlet problem through the corresponding Green kernel. The linear heat equation and wave equation: some hints about methods of resolution through finite-dimensional Galerkin approximation of the Laplace operator, corresponding to the so-called technique of separation of variables.

An example of nonlinear partial differential equations arising from variational problems: the Rudin-Osher-Fatemi model $F(u) = |Du|(\Omega) + \frac{1}{2}\|u - f\|^2_{L^2(\Omega)}$, for $u \in BV(\Omega)$ is suitable in image processing, since it has the ability to regularize (denoise) a given image f (identified as its grey level function) by preserving at the same time edges and boundaries.

An example of geometric variational problem: the isoperimetric and the isovolumetric problem within the class of finite perimeter sets.

Definition of finite perimeter (or Caccioppoli) sets in $\Omega \subset \mathbb{R}^n$: they are Lebesgue measurable sets $E \subset \mathbb{R}^n$ such that

$$P_\Omega(E) \equiv |D\mathbf{1}_E|(\Omega) \equiv \sup\left\{\int_{E\cap\Omega} \mathrm{div}\vec{\phi}\,,\ \|\vec{\phi}\|_\infty \le 1,\quad \vec{\phi} \in [C_c^\infty(\Omega)]^n\right\} < +\infty,$$

in other words $\mathbf{1}_E \in BV(\Omega)$. Weak formulation of the isovolumetric problem in the class of finite perimeter sets in $\mathbb{R}^n$: fix $R > 1$ (sufficiently large) and set

$$\mathcal{P} = \left\{E \subset B_R(0),\, \mathcal{L}^n(E) = \int_{\mathbb{R}^n} \mathbf{1}_E\, d\mathcal{L}^n = 1,\ \mathbf{1}_E \in BV(B_{2R}(0))\right\},$$

i.e. $\mathcal{P}$ contains sets $E \subset B_R(0)$ having unit volume and finite perimeter $\|D\mathbf{1}_E\| \equiv |D\mathbf{1}_E|(B_{2R}(0))$ in $B_{2R}(0)$: observe that since $E \subset B_R(0)$, the perimeter of E in $B_{2R}(0)$ coincides with the whole perimeter of E in $\mathbb{R}^n$, i.e. with $|D\mathbf{1}_E|(\mathbb{R}^n)$. Consider the isovolumetric problem

$$\min_{E\in\mathcal{P}} \|D\mathbf{1}_E\|\,.$$

If $E_n \in \mathcal{P}$ is a minimizing sequence, i.e. $\|D\mathbf{1}_{E_n}\| \to \inf_{F\in\mathcal{P}} \|D\mathbf{1}_F\|$, we have

$$\|\mathbf{1}_{E_n}\|_{BV(B_{2R}(0))} = 1 + \|D\mathbf{1}_{E_n}\| \le C,$$

so that, up to a subsequence, $\mathbf{1}_{E_n} \to \mathbf{1}_E$ in $L^1(B_{2R}(0))$ by the compact embedding of $BV(B_{2R}(0))$ in $L^1(B_{2R}(0))$ (Rellich Theorem). We deduce $E \subset B_R(0)$ and $\mathcal{L}^n(E) = 1$. Moreover, we have $D\mathbf{1}_{E_n}(\vec{\phi}) \to D\mathbf{1}_E(\vec{\phi})$ for any $\vec{\phi} \in [C_c^\infty(\mathbb{R}^n)]^n$ (i.e. convergence in the sense of distributions) and

$$\|\nabla\mathbf{1}_E\| \le \liminf_{n\to+\infty} \|\nabla\mathbf{1}_{E_n}\| = \inf_{F\in\mathcal{P}} \|\nabla\mathbf{1}_F\|$$

by lower semicontinuity of the total variation norm. Hence E has minimum perimeter in the class $\mathcal{P}$.

The regularity theory (based for example on Steiner symmetrization) allows to conclude that the optimal set E is the unit volume round ball in $\mathbb{R}^n$.

References.

[1] Brézis, H.; *Functional Analysis, Sobolev spaces and Partial Differential Equations*, Springer (2010).
[2] Giusti, E.; *Minimal surfaces and functions of bounded variation*, Birkhäuser (1984).
[3] Kolmogorov, Fomin; *Elements of Theory of Functions and Functional Analysis*, Dover (1999).