Wavelets and filterbanks

Mallat 2009, Chapter 7

Outline

- Wavelets and Filterbanks
- Biorthogonal bases
- The dual perspective: from FB to wavelet bases
  - Biorthogonal FB
  - Perfect reconstruction conditions
- Separable bases (2D)
- Overcomplete bases
  - Wavelet frames (algorithme à trous, DDWF)
  - Curvelets

Wavelets and Filterbanks

Wavelet side

- Scaling function
  - Design (from multiresolution priors)
  - Signal approximation
  - Corresponding filtering operation
    - Condition on the filter h[n] → Conjugate Mirror Filter (CMF)
- Corresponding wavelet families

Filterbank side

- Perfect reconstruction conditions (PR)
  - Reversibility of the transform
- Equivalence with the conditions on the wavelet filters
  - Special case: CMFs → Orhogonal wavelets
  - General case → Biorthogonal wavelets

Wavelets and filterbanks

- The decomposition coefficients in a wavelet orthogonal basis are computed with a fast algorithm that cascades discrete convolutions with h and g, and subsample the output
- Fast orthogonal WT

$$f(t)=\sum_n a_0[n]\varphi(t-n)\in V_0$$

Since $\{\varphi(\text{t-n})\}_{n\in Z}$ is an orthonormal basis

$$a_0[n]=\langle f(t),\varphi(t-n)\rangle=\int_{-\infty}^{+\infty} f(t)\varphi^*(t-n)dt=\int_{-\infty}^{+\infty} f(t)\bar{\varphi}^*(n-t)dt=f*\bar{\varphi}(n)$$

$$\bar{\varphi}(t)=\varphi(-t)$$

Linking the domains

$$z = e^{j\omega}$$
$$\hat{f}(\omega) = \hat{f}(e^{j\omega}) \leftrightarrow f(z)$$
$$\hat{f}(\omega+\pi) = \hat{f}(e^{j(\omega+\pi)}) = \hat{f}(-e^{j\omega}) \leftrightarrow f(-z)$$
$$\hat{f}(-\omega) = \hat{f}(e^{-j\omega}) \leftrightarrow f(z^{-1})$$
$$\hat{f}^*(\omega) = \hat{f}(-\omega) \leftrightarrow f(z^{-1})$$

Switching between the Fourier and the z-domain

$$f[n] \leftrightarrow f(z) = \sum_{k=-\infty}^{+\infty} f[k] z^{-k}$$

$f[n-1] \leftrightarrow z^{-1} f(z)$ unit delay

$f[-n] \leftrightarrow f\left(z^{-1}\right)$ reverse the order of the coefficients

$(-1)^n f[n] \leftrightarrow f(-z)$ negate odd terms

Switching between the time and the z-domain

Fast orthogonal wavelet transform

- Fast FB algorithm that computes the orthogonal wavelet coefficients of a discrete signal $a_0[n]$. Let us define

$$f(t) = \sum_n a_0[n]\varphi(t-n) \in V_0$$

Since $\{\varphi(t-n)\}_{n\in Z}$ is orthonormal, then

$$a_0[n] = \langle f(t), \varphi(t-n)\rangle = f * \bar{\varphi}(n)$$

$$\Rightarrow \quad a_j[n] = \langle f, \varphi_{j,n}\rangle \text{ since } \varphi_{j,n} \text{ is an orthonormal basis for } V_j$$

$$d_j[n] = \langle f, \psi_{j,n}\rangle$$

- *A fast wavelet transform decomposes successively each approximation $PV_j f$ in the coarser approximation $PV_{j+1} f$ plus the wavelet coefficients carried by $PW_{j+1} f$.*
- *In the reconstruction, $PV_j f$ is recovered from $PV_{j+1} f$ and $PW_{j+1} f$ for decreasing values of j starting from J (decomposition depth)*

Fast wavelet transform

- Theorem 7.7
  - At the decomposition

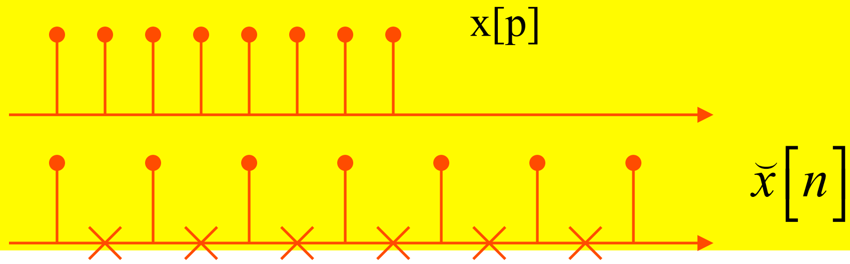
$$a_{j+1}[p] = \sum_{n=-\infty}^{+\infty} h[n-2p]a_j[n] = a_j * \bar{h}[2p] \quad (1)$$

$$d_{j+1}[p] = \sum_{n=-\infty}^{+\infty} g[n-2p]a_j[n] = a_j * \bar{g}[2p] \quad (2)$$

  - At the reconstruction

$$a_j[p] = \sum_{n=-\infty}^{+\infty} h[p-2n]a_{j+1}[n] + \sum_{n=-\infty}^{+\infty} g[p-2n]d_{j+1}[n] = \breve{a}_{j+1} * h[n] + \breve{d}_{j+1} * g[n] \quad (4)$$

$$\breve{x}[n] = \begin{cases} x[p] & n = 2p \\ 0 & n = 2p+1 \end{cases}$$



Proof: decomposition (1)

$$\varphi_{j+1}[p] \in V_{j+1} \subset V_j \rightarrow \varphi_{j+1}[p] = \sum_n \langle \varphi_{j+1}[p], \varphi_j[n] \rangle \varphi_j[n] \text{ (b)}$$

but

$$\langle \varphi_{j+1}[p], \varphi_j[n] \rangle = \int \frac{1}{\sqrt{2^{j+1}}} \varphi\left(\frac{t-2^{j+1}p}{2^{j+1}}\right) \frac{1}{\sqrt{2^j}} \varphi^*\left(\frac{t-2^j n}{2^j}\right) dt \text{ (a)}$$

let

$$t' = 2^{-j}t - 2p \rightarrow t = 2^j t' + 2^{j+1}p \rightarrow t - 2^{j+1}p = 2^j t' \rightarrow \frac{t-2^{j+1}p}{2^{j+1}} = \frac{t'}{2}$$

then

$$\varphi\left(\frac{t-2^{j+1}p}{2^{j+1}}\right) = \varphi\left(\frac{t'}{2}\right)$$

$$\varphi^*\left(\frac{t-2^j n}{2^j}\right) = \varphi^*\left(t' + 2p - n\right) \qquad \frac{t'}{2} = \frac{t}{2^{j+1}} - p \rightarrow \frac{t}{2^{j+1}} = \frac{t'}{2} + p \rightarrow \frac{t}{2^j} = t' + 2p$$

replacing into (a)

(3) $$\langle \varphi_{j+1}[p], \varphi_j[n] \rangle = \int \frac{1}{\sqrt{2}} \varphi\left(\frac{t'}{2}\right) \varphi^*\left(t' + 2p - n\right) dt' = \left\langle \frac{1}{\sqrt{2}} \varphi\left(\frac{t}{2}\right), \varphi\left(t + 2p - n\right) \right\rangle = h[n-2p]$$

thus (b) becomes

$$\boxed{\varphi_{j+1}[p] = \sum_n h[n-2p] \varphi_j[n]}$$

Proof: decomposition (2) qui

- Coming back to the projection coefficients

$$a_{j+1}[p] = \left\langle f, \varphi_{j+1,p} \right\rangle = \left\langle f, \sum_n h[n-2p]\varphi_{j,n} \right\rangle = \int_{-\infty}^{+\infty} f \sum_n h[n-2p]\varphi^*_{j,n} dt =$$

$$= \sum_n h[n-2p] \int_{-\infty}^{+\infty} f(t)\varphi^*_{j,n}(t)dt = \sum_n h[n-2p]\left\langle f, \varphi_{j,n} \right\rangle = \sum_n h[n-2p]a_j[n] \rightarrow$$

$$\boxed{a_{j+1}[p] = a_j * \bar{h}[2p]}$$

- Similarly, one can prove the relations for both the details and the reconstruction formula

Proof: decomposition (3)

- Details

$$\psi_{j+1,p} \in W_{j+1} \subset V_j \rightarrow \psi_{j+1,p} = \sum_n \langle \psi_{j+1,n}, \varphi_{j,n} \rangle \varphi_{j,n}$$

$$t^{'} = 2^{-j} t - 2p \rightarrow$$

(3bis) $$\langle \psi_{j+1,n}, \varphi_{j,n} \rangle = \left\langle \frac{1}{\sqrt{2}} \psi\left(\frac{t}{2}\right), \varphi(t - n + 2p) \right\rangle = g[n-2p] \rightarrow$$

$$\psi_{j+1,p} = \sum_n g[n-2p] \varphi_{j,n} \rightarrow$$

$$\langle f, \psi_{j+1,n} \rangle = \sum_n g[n-2p] \langle f, \varphi_{j,n} \rangle \rightarrow$$

$$d_{j+1}[p] = \sum_n g[n-2p] a_j[n]$$

Proof: Reconstruction

Since W_{j+1} is the orthonormal complement of V_{j+1} in V_j, the union of the two respective basis is a basis for V_j. Hence

$$V_j = V_{j+1} \oplus W_{j+1} \rightarrow \varphi_{j,p} = \sum_n \langle \varphi_{j,p}, \varphi_{j+1,n} \rangle \varphi_{j+1,n} + \sum_n \langle \varphi_{j,p}, \psi_{j+1,n} \rangle \psi_{j+1,n}$$

but
$$\langle \varphi_{j,p}, \varphi_{j+1,n} \rangle = h[p-2n]$$
$$\langle \varphi_{j,p}, \psi_{j+1,n} \rangle = g[p-2n]$$

(see (3) and (3bis), the analogous one for g)

thus

$$\varphi_{j,p} = \sum_n h[p-2n] \varphi_{j+1,n} + \sum_n g[p-2n] \psi_{j+1,n}$$

Taking the scalar product with f at both sides:

$$a_j[p] = \sum_{n=-\infty}^{+\infty} h[p-2n] a_{j+1}[n] + \sum_{n=-\infty}^{+\infty} g[p-2n] d_{j+1}[n] = \breve{a}_{j+1} * h[n] + \breve{d}_{j+1} * g[n]$$

$$\breve{x}[n] = \begin{cases} x[p] & n = 2p \\ 0 & n = 2p+1 \end{cases}$$

CVD

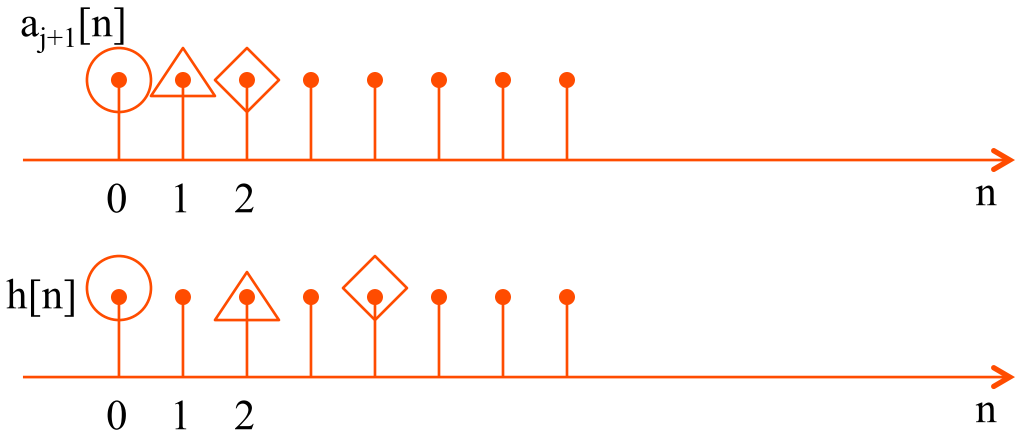
Graphically

$$a_j[p] = \sum_n h[p-2n]a_{j+1}[n] = \sum_n a_{j+1}[n]h[p-2n]$$

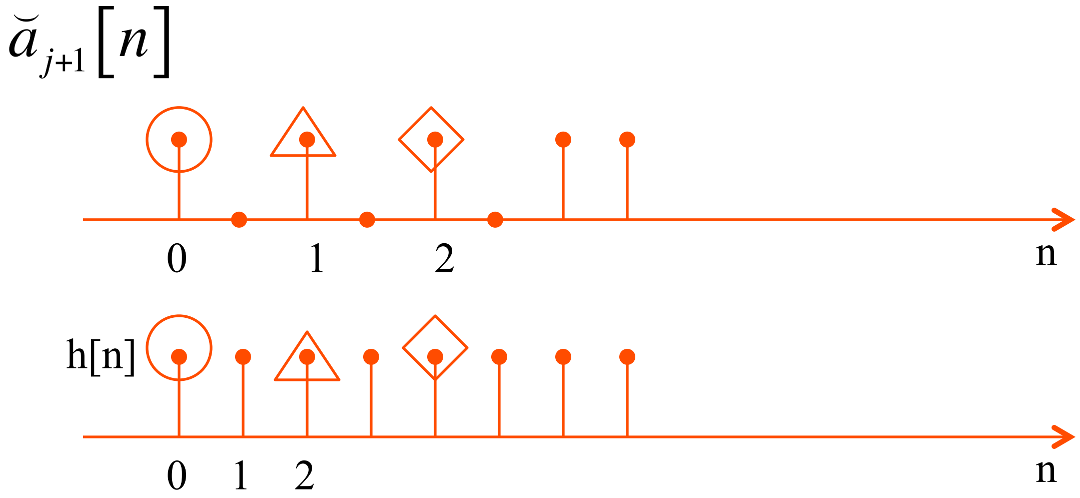
$$a_j[0] = \sum_n h[-2n]a_{j+1}[n] = \sum_n a_{j+1}[n]h[-2n]$$

Let's assume that h is symmetric

$$a_j[0] = \sum_n a_{j+1}[n]h[2n]$$



Graphically

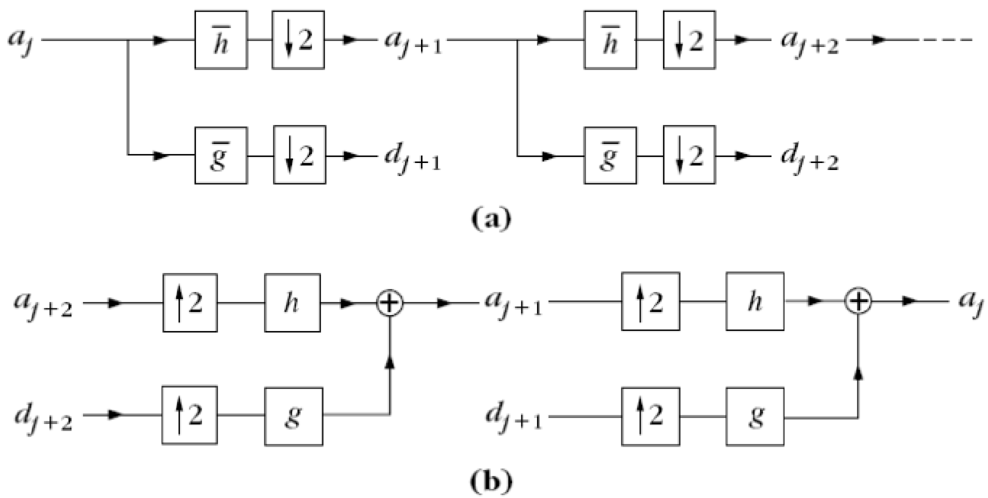


$$a_j[0] = \sum_n a_{j+1}[n]h[2n] = \sum_n \breve{a}_{j+1}[n]h[n]$$

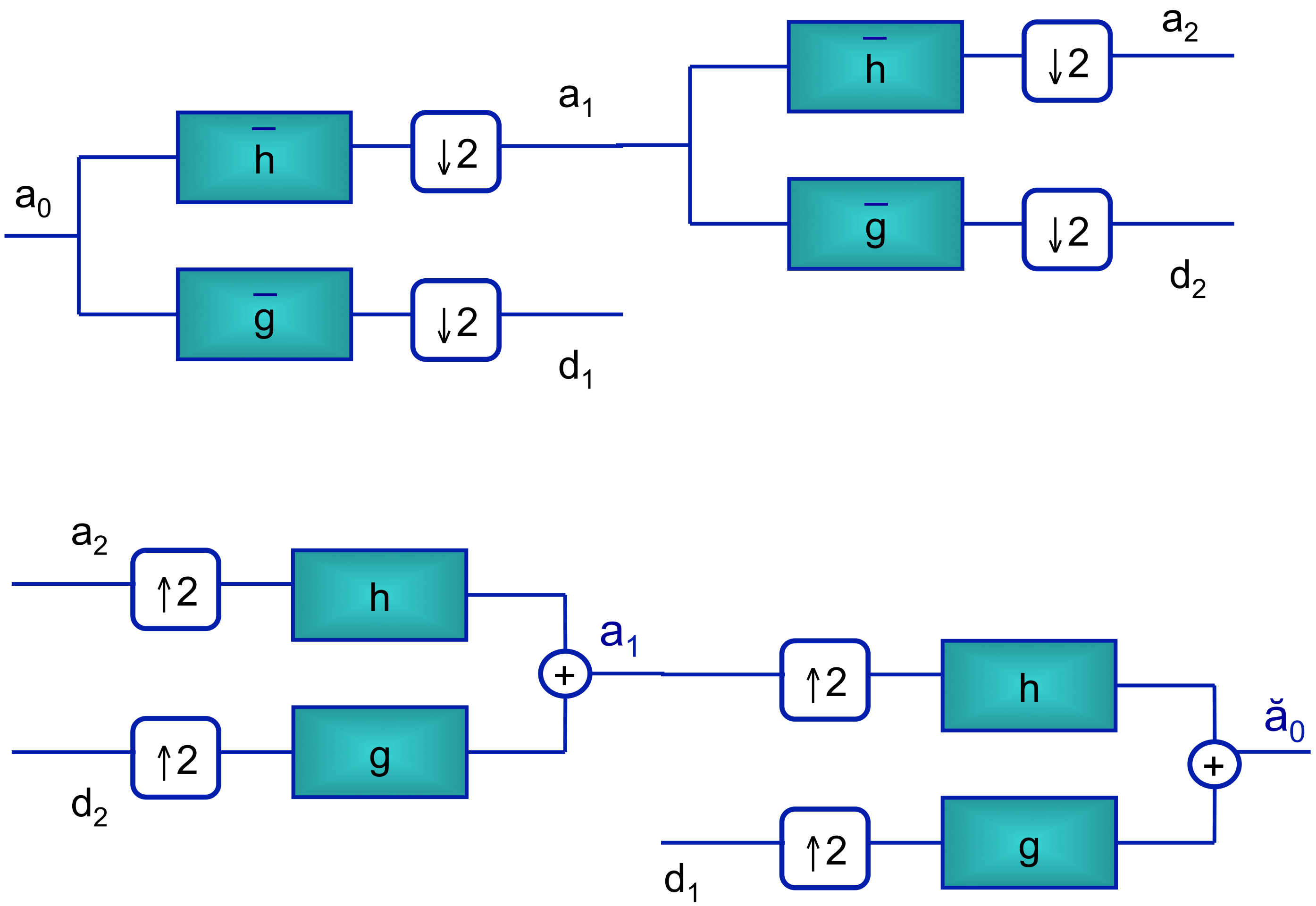
Summary

$$a_{j+1}[p] = a_j * \bar{h}[2p]$$

- The coefficients a_{j+1} and d_{j+1} are computed by taking every other sample of the convolution of a_j with $\bar{h}$ and $\bar{g}$ respectively.
- The filter $\bar{h}$ *removes* the higher frequencies of the inner product sequence a_j , whereas $\bar{g}$ is a high-pass filter that *collects* the remaining highest frequencies.
- The reconstruction is an interpolation that inserts zeroes to expand a_{j+1} and d_{j+1} and filters these signals, as shown in Figure.



Filterbank implementation



Fast DWT

- Theorem 7.10 proves that a_{j+1} and d_{j+1} are computed by taking every other sample of the convolution on a_j with $\bar{h}$ and $\bar{g}$ respectively
- The filter h removes the higher frequencies of the inner product and the filter g is a band-pass filter that collects such residual frequencies
- An orthonormal **wavelet representation** is composed of wavelet coefficients at scales

$$1 \le 2^j \le 2^J$$

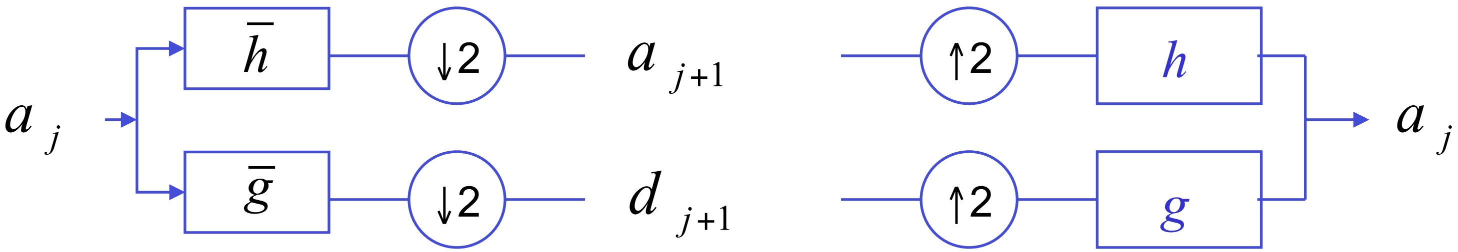
plus the remaining approximation at scale 2^J

$$\left[\left\{ d_j \right\}_{1 \le j \le J}, a_J \right]$$

Summary

Analysis or decomposition

Synthesis or reconstruction



Teorem 7.2 (Mallat&Meyer) and **Theorem 7.3** [Mallat&Meyer]

$$\forall \omega \in \mathbb{R}, \qquad \left|\hat{h}(\omega)\right|^2 + \left|\hat{h}(\omega+\pi)\right|^2 = 2$$

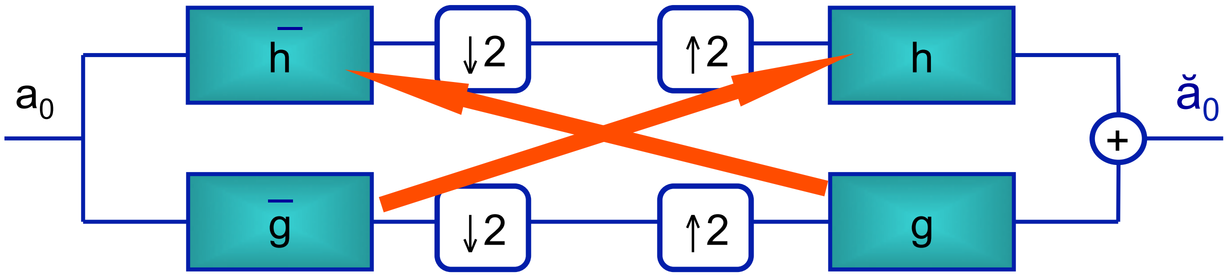
and

$$\hat{h}(0) = \sqrt{2}$$

$$\hat{g}(\omega) = e^{-j\omega}\hat{h}^*(\omega+\pi) \leftrightarrow g[n] = (-1)^{1-n} h[1-n]$$

The fast orthogonal WT is implemented by a filterbank that is completely specified by the filter *h*, which is a CMF
The filters are the same for every *j*

Filter bank perspective



Taking h[n] as reference (which amounts to choosing **the synthesis low-pass filter**) the following relations hold for an orthogonal filter bank:

$$\bar{h}[n] = h[-n]$$

$$g[n] = (-1)^{1-n} h[1-n] = (-1)^{1-n} \bar{h}[n-1]$$

$$\bar{g}[n] = g[-n] = (-1)^{-(1-n)} h[-(1-n)]$$

neglecting the unitary shift, as usually done in applications

$$g[n] = (-1)^{-n} h[-n] = (-1)^{-n} \bar{h}[n]$$

$$\bar{g}[n] = g[-n] = (-1)^{n} h[n]$$

Finite signals

- Issue: signal extension at borders
- Possible solutions:
  - Periodic extension
    - Works with any kind of wavelet
    - Generates large coefficients at the borders
  - Symmetryc/antisymmetric extension, depending on the wavelet symmetry
    - More difficult implementation
    - Haar filter is the only symmetric filter with compact support
  - Use different wavelets at boundary (boundary wavelets)
  - Implementation by *lifting steps*

Wavelet graphs

The graphs of ϕ and ψ are computed numerically with the inverse wavelet transform. If $f=\phi$, then $a_0[n]=\delta[n]$ and $d_j[n]=0$ for all $L<j\leqslant 0$. The inverse wavelet transform computes a_L and (7.111) shows that

$$N^{1/2}\, a_L[n] \approx \phi(N^{-1}n).$$

If ϕ is regular and N is large enough, we recover a precise approximation of the graph of ϕ from a_L.

Similarly, if $f=\psi$, then $a_0[n]=0$, $d_0[n]=\delta[n]$, and $d_j[n]=0$ for $L<j<0$. Then $a_L[n]$ is calculated with the inverse wavelet transform and $N^{1/2}\, a_L[n] \approx \psi(N^{-1}n)$. The Daubechies wavelets and scaling functions in Figure 7.10 are calculated with this procedure.

Orthogonal wavelet representation

- An *orthogonal wavelet representation* of $a_L = \langle f, \varphi_{L,n} \rangle$ is composed of wavelet coefficients of f at scales $2^L < 2^j <= 2^J$, plus the remaining approximation at the largest scale 2^J :

$$\left[\{d_j\}_{L<j\leqslant J},\ a_J\right].$$

- Initialization
  - Let *b[n]* be the discrete time input signal and let N^{-1} be the sampling period, such that the corresponding scale is $2^L = N^{-1}$
  - Then:

N^{-1}: discrete sample distance
$2^L = N^{-1}$ scale

$$f(t) = \sum_{n=-\infty}^{+\infty} b[n]\, \phi\left(\frac{t - 2^L n}{2^L}\right) \in \mathbf{V}_L.$$

original continuous time signal

discrete time signal

interpolation function

Initialization

$$f(t)=\sum_{n=-\infty}^{+\infty} b[n]\,\phi\left(\frac{t-2^L n}{2^L}\right)\in \mathbf{V}_L.$$

following the definition:

N^{-1}: discrete sample distance
$2^L = N^{-1}$ scale

$$\varphi_{L,n}=\frac{1}{\sqrt{2^L}}\varphi\left(\frac{t-2^L n}{2^L}\right)$$ Basis for V_L

$$2^L=\frac{1}{N}\rightarrow\frac{1}{\sqrt{2^L}}=N^{1/2}=\sqrt{N}\rightarrow\varphi_{L,n}=\sqrt{N}\varphi\left(\frac{t-N^{-1}n}{N^{-1}}\right)\rightarrow\varphi\left(\frac{t-N^{-1}n}{N^{-1}}\right)=\frac{1}{\sqrt{N}}\varphi_{L,n}$$

but

$$f(t)=\sum_{n=-\infty}^{+\infty} b[n]\varphi\left(\frac{t-N^{-1}n}{N^{-1}}\right)=\frac{1}{\sqrt{N}}\sum_{n=-\infty}^{+\infty} b[n]\varphi_{L,n}(t)$$

$$b[n]=\left\langle f,\varphi\left(\frac{t-N^{-1}n}{N^{-1}}\right)\right\rangle=\left\langle f,\frac{1}{\sqrt{N}}\varphi_{L,n}\right\rangle=\frac{1}{\sqrt{N}}a_L[n]$$ $$a_L[n]=\left\langle f,\varphi_{L,n}\right\rangle$$

since

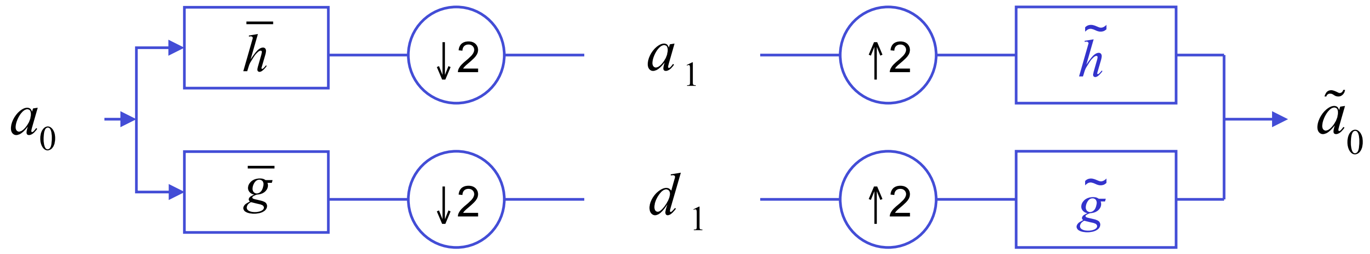
$$a_L[n]=\int_{-\infty}^{+\infty} f(t)\sqrt{N}\varphi\left(\frac{t-N^{-1}n}{N^{-1}}\right)dt$$ by definition, then

$$a_L[n]\approx\sqrt{N}f\left(N^{-1}n\right)$$ if f is regular, the sampled values can be considered as a local average in the neighborhood of $f(N^{-1}n)$

The filter bank perspective

Perfect reconstruction FB

- **Dual perspective**: given a filterbank consisting of 4 filters, we derive the *perfect reconstruction conditions*



- Goal: determine the conditions on the filters ensuring that

$$\tilde{a}_0 \equiv a_0$$

PR Filter banks

- The decomposition of a discrete signal in a multirate filter bank is interpreted as an expansion in $l^2(Z)$

since

$$a_1[l] = a_0 * \bar{h}[2l] = \sum_n a_0[n]\bar{h}[2l-n] = \sum_n a_0[n]h[n-2l]$$

then

$$a_1[l] = \sum_{n=-\infty}^{+\infty} a_0[n]\, h[n-2l] = \langle a_0[n], h[n-2l] \rangle,$$

$$d_1[l] = \sum_{n=-\infty}^{+\infty} a_0[n]\, g[n-2l] = \langle a_0[n], g[n-2l] \rangle.$$

and the signal is recovered by the reconstruction filter

$$a_0[n] = \sum_{l=-\infty}^{+\infty} a_1[l]\, \tilde{h}[n-2l] + \sum_{l=-\infty}^{+\infty} d_1[l]\, \tilde{g}[n-2l].$$

dual family of vectors

thus

$$a_0[n] = \sum_{l=-\infty}^{+\infty} \langle f[k], h[k-2l] \rangle\, \tilde{h}[n-2l] + \sum_{l=-\infty}^{+\infty} \langle f[k], g[k-2l] \rangle\, \tilde{g}[n-2l].$$

points to biorthogonal wavelets

The two families are biorthogonal

Theorem 7.13. If h, g, $\tilde{h}$, and $\tilde{g}$ are perfect reconstruction filters, and their Fourier transforms are bounded, then $\{\tilde{h}[n-2l], \tilde{g}[n-2l]\}_{l\in\mathbb{Z}}$ and $\{h[n-2l], g[n-2l]\}_{l\in\mathbb{Z}}$ are biorthogonal Riesz bases of $\ell^2(\mathbb{Z})$.

Thus, a PR FB projects a discrete time signals over a biorthogonal basis of $l^2(Z)$.
If the dual basis is the same as the original basis than the projection is orthonormal.

Discrete Wavelet basis

- Question: why bother with the construction of wavelet basis if a PR FB can do the same easily?
- Answer: because conjugate mirror filters are most often used in filter banks that cascade several levels of filterings and subsamplings. Thus, it is necessary to understand the behavior of such a cascade

N^{-1}: discrete sample distance

$2^L = N^{-1}$ scale

$$a_L[n] = \langle f, \varphi_{L,n} \rangle$$ discrete signal at scale 2^L

$$\varphi\left(\frac{t - N^{-1}n}{N^{-1}}\right) = \frac{1}{\sqrt{N}}\varphi_{L,n}$$

for depth j>L

$$a_j[l] = a_L \star \bar{\phi}_j[2^{j-L}l] = \langle a_L[n], \phi_j[n - 2^{j-L}l] \rangle$$

$$d_j[l] = a_L \star \bar{\psi}_j[2^{j-L}l] = \langle a_L[n], \psi_j[n - 2^{j-L}l] \rangle.$$

$$\hat{\phi}_j(\omega) = \prod_{p=0}^{j-L-1} \hat{h}(2^p\omega)$$

$$\hat{\psi}_j(\omega) = \hat{g}(2^{j-L-1}\omega) \prod_{p=0}^{j-L-2} \hat{h}(2^p\omega).$$

Discrete wavelet basis

For conjugate mirror filters, one can verify that this family is an orthonormal basis of $\ell^2(\mathbb{Z})$. These discrete vectors are close to a uniform sampling of the continuous time-scaling functions $\phi_j(t) = 2^{-j/2}\phi(2^{-j}t)$ and wavelets $\psi_j(t) = 2^{-j/2}\phi(2^{-j}t)$. When the number $L-j$ of successive convolutions increases, one can verify that $\phi_j[n]$ and $\psi_j[n]$ converge, respectively, to $N^{-1/2}\,\phi_j(N^{-1}n)$ and $N^{-1/2}\,\psi_j(N^{-1}n)$.

The factor $N^{-1/2}$ normalizes the $\ell^2(\mathbb{Z})$ norm of these sampled functions. If $L-j = 4$, then $\phi_j[n]$ and $\psi_j[n]$ are already very close to these limit values. Thus, the impulse responses $\phi_j[n]$ and $\psi_j[n]$ of the filter bank are much closer to continuous time-scaling functions and wavelets than they are to the original conjugate mirror filters h and g. This explains why wavelets provide appropriate models for understanding the applications of these filter banks. Chapter 8 relates more general filter banks to wavelet packet bases.

Perfect reconstruction FB

- **Theorem 7.7 (Vetterli)** The FB performs an exact reconstruction for any input signal iif

$$\hat{h}^*(\omega)\hat{\tilde{h}}(\omega)+\hat{g}^*(\omega)\hat{\tilde{g}}(\omega)=2$$

$$\hat{h}^*(\omega+\pi)\hat{\tilde{h}}(\omega)+\hat{g}^*(\omega+\pi)\hat{\tilde{g}}(\omega)=0$$

(*alias free*)

Matrix notations

$$\begin{pmatrix}\hat{\tilde{h}}^*(\omega)\\ \hat{\tilde{g}}^*(\omega)\end{pmatrix}=\frac{2}{\Delta(\omega)}\begin{pmatrix}\hat{g}(\omega+\pi)\\ -\hat{h}(\omega+\pi)\end{pmatrix}$$

$$\Delta(\omega)=\hat{h}(\omega)\hat{g}(\omega+\pi)-\hat{h}(\omega+\pi)\hat{g}(\omega)$$

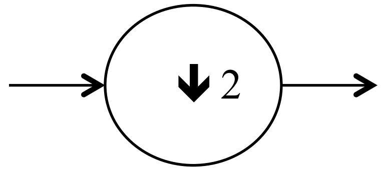
When all the filters are FIR, the determinant can be evaluated, which yields simpler relations between the decomposition and the reconstruction filters.

Changing the sampling rate

- Downsampling

$$\hat{y}(2\omega) = \frac{1}{2}\left(\hat{x}(\omega) + \hat{x}(\omega+\pi)\right) = \sum_{n=-\infty}^{+\infty} x[2n]e^{-jn\omega}$$

$\hat{x}(\omega)$



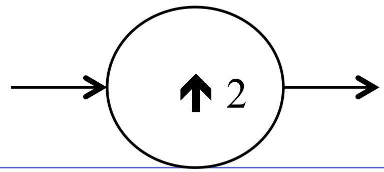
$\hat{x}_{down}(\omega) = \hat{y}(\omega)$

$$\hat{y}(\omega) = \frac{1}{2}\left(\hat{x}\left(\frac{\omega}{2}\right) + \hat{x}\left(\frac{\omega}{2}+\pi\right)\right)$$

- Upsampling

$$\hat{y}(\omega) = \hat{x}(2\omega) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j2n\omega}$$

$\hat{x}(\omega)$



$\hat{x}_{up}(\omega) = \hat{y}(\omega)$

Subsampling: proof

$$\hat{y}(\omega) = \ldots y[0] + y[1]e^{-j\omega} + y[2]e^{-j2\omega} + \ldots =$$

$$= \ldots x[0] + x[2]e^{-j\omega} + x[4]e^{-j2\omega} + \ldots \rightarrow$$

thus

$$\hat{y}(2\omega) = \ldots x[0] + x[2]e^{-j2\omega} + x[4]e^{-j4\omega} + \ldots$$

but

$$x[1]e^{-j\omega} + x[1]e^{-j(\omega+\pi)} = 0 \rightarrow \frac{1}{2}\left(x[1]e^{-j\omega} + x[1]e^{-j(\omega+\pi)}\right) = 0$$

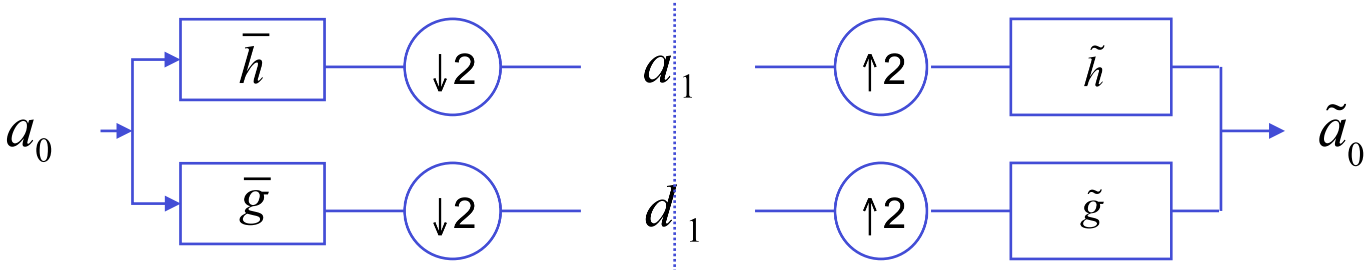
$$x[2]e^{-j2\omega} = \frac{1}{2}\left(x[2]e^{-j2\omega} + x[2]e^{-j2(\omega+\pi)}\right)$$

thus

$$\hat{y}(2\omega) = \ldots x[0] + \frac{1}{2}\left(x[1]e^{-j\omega} + x[1]e^{-j(\omega+\pi)}\right) + \frac{1}{2}\left(x[2]e^{-j2\omega} + x[2]e^{-j2(\omega+\pi)}\right) + \ldots =$$

$$\hat{y}(2\omega) = \frac{1}{2}\left(\hat{x}(\omega) + \hat{x}(\omega+\pi)\right)$$

Perfect Reconstruction conditions



$$a_1(2\omega) = \frac{1}{2}\left(a_0(\omega)\hat{\bar{h}}(\omega) + a_0(\omega+\pi)\hat{\bar{h}}(\omega+\pi)\right)$$

since h and g are real

$$h[n] \rightarrow h(\omega)$$

$$h[-n] = \bar{h}[n] \rightarrow \hat{\bar{h}}(\omega) = \hat{h}(-\omega) = h^*(\omega)$$

thus, replacing in the first equation

$$a_1(2\omega) = \frac{1}{2}\left(a_0(\omega)\hat{h}^*(\omega) + a_0(\omega+\pi)\hat{h}^*(\omega+\pi)\right)$$

Similarly, for the high-pass branch

$$d_1(2\omega) = \frac{1}{2}\left(a_0(\omega)\hat{g}^*(\omega) + a_0(\omega+\pi)\hat{g}^*(\omega+\pi)\right)$$

$$\hat{\tilde{a}}_0(\omega) = \hat{a}_1(2\omega)\hat{\tilde{h}}(\omega) + \hat{d}_1(2\omega)\hat{\tilde{g}}(\omega)$$

Perfect Reconstruction conditions

- Putting all together

$$\hat{\tilde{a}}_0(\omega) = \hat{a}_1(2\omega)\hat{\tilde{h}}(\omega) + \hat{d}_1(2\omega)\hat{\tilde{g}}(\omega) =$$

$$= \frac{1}{2}\left(a_0\left(\omega\right)\hat{h}^*\left(\omega\right) + a_0\left(\omega+\pi\right)\hat{h}^*\left(\omega+\pi\right)\right)\hat{\tilde{h}}(\omega)$$

$$+ \frac{1}{2}\left(a_0\left(\omega\right)\hat{g}^*\left(\omega\right) + a_0\left(\omega+\pi\right)\hat{g}^*\left(\omega+\pi\right)\right)\hat{\tilde{g}}(\omega)$$

$$\hat{\tilde{a}}_0(\omega) = \frac{1}{2}\left(\hat{h}^*\left(\omega\right)\hat{\tilde{h}}(\omega) + \hat{g}^*\left(\omega\right)\hat{\tilde{g}}(\omega)\right)a_0\left(\omega\right) + \frac{1}{2}\left(\hat{h}^*\left(\omega+\pi\right)\hat{\tilde{h}}(\omega) + \hat{g}^*\left(\omega+\pi\right)\hat{\tilde{g}}(\omega)\right)a_0\left(\omega+\pi\right)$$

=1 =0 (alias-free)

$$\hat{h}^*\left(\omega\right)\hat{\tilde{h}}(\omega) + \hat{g}^*\left(\omega\right)\hat{\tilde{g}}(\omega) = 2$$

$$\hat{h}^*\left(\omega+\pi\right)\hat{\tilde{h}}(\omega) + \hat{g}^*\left(\omega+\pi\right)\hat{\tilde{g}}(\omega) = 0$$

(alias free)

Matrix notations

$$\begin{pmatrix} \hat{\tilde{h}}^*(\omega) \\ \hat{\tilde{g}}^*(\omega) \end{pmatrix} = \frac{2}{\Delta(\omega)}\begin{pmatrix} \hat{g}(\omega+\pi) \\ -\hat{h}(\omega+\pi) \end{pmatrix}$$

$$\Delta(\omega) = \hat{h}(\omega)\hat{g}(\omega+\pi) - \hat{h}(\omega+\pi)\hat{g}(\omega)$$

Perfect reconstruction biorhogonal filters

- Theorem 7.8. Perfect reconstruction filters also satisfy

$$\hat{h}^*(\omega)\hat{\tilde{h}}(\omega)+\hat{h}^*(\omega+\pi)\hat{\tilde{h}}(\omega+\pi)=2$$

Furthermore, if the filters have a finite impulse response there exists *a in R* and *l in Z* such that

$$\hat{g}(\omega)=ae^{-i(2l+1)\omega}\hat{\tilde{h}}^*(\omega+\pi)$$
$$\hat{\tilde{g}}(\omega)=\frac{1}{a}e^{-i(2l+1)\omega}\hat{h}^*(\omega+\pi)$$

⇒ *a=1, l=0* ⇒

$$\hat{g}(\omega)=e^{-j\omega}\hat{\tilde{h}}^*(\omega+\pi)$$
$$\hat{\tilde{g}}(\omega)=e^{-j\omega}h^*(\omega+\pi)$$

Correspondingly

$$g[n]=(-1)^{1-n}\tilde{h}[1-n]$$
$$\tilde{g}[n]=(-1)^{1-n}h[1-n]$$

- Conjugate Mirror Filters: $\tilde{h}=h\rightarrow\left|\hat{h}(\omega)\right|^2+\left|\hat{h}(\omega+\pi)\right|^2=2$

Perfect reconstruction biorthogonal filters

Given h and $\tilde{h}$ and setting *a=1* and *l=0* in (2) the remaining filters are given by the following relations

(3)

$$\hat{g}(\omega) = e^{-i\omega}\hat{\tilde{h}}^*(\omega+\pi)$$

$$\hat{\tilde{g}}(\omega) = e^{-i\omega}\hat{h}^*(\omega+\pi)$$

- The filters h and $\tilde{h}$ are related to the scaling functions φ and ~φ via the corresponding two-scale relations, as was the case for the orthogonal filters (see eq. 1).

Switching to the z-domain

$$g(z) = z^{-1}\tilde{h}(-z^{-1})$$
$$\tilde{g}(z) = z^{-1}h(-z^{-1})$$

Signal domain

$$g[n] = (-1)^{1-n}\tilde{h}[1-n]$$
$$\tilde{g}[n] = (-1)^{1-n}h[1-n]$$

Biorthogonal filter banks

- A 2-channel **multirate** filter bank convolves a signal a_0 with

  a low pass filter $\bar{h}[n] = h[-n]$

  and a high pass filter $\bar{g}[n] = g[-n]$

  and sub-samples the output by 2

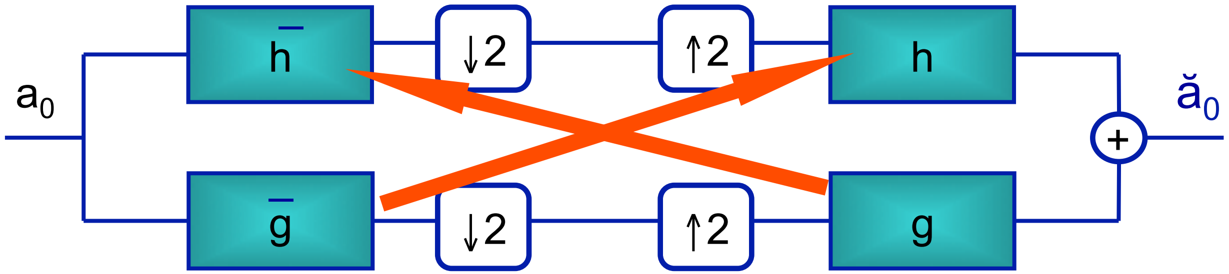
$$a_1[n] = a_0 * \bar{h}[2n]$$
$$d_1[n] = a_0 * \bar{g}[2n]$$

A reconstructed signal $\tilde{a}_0$ is obtained by filtering the zero-expanded signals with a *dual low-pass* $\tilde{h}[n]$ *and high pass filter* $\tilde{g}[n]$

$$\tilde{a}_0[n] = \breve{a}_1 * \tilde{h}[n] + \breve{d}_1 * \tilde{g}[n]$$
$$y[n] = \breve{x}[n] = \begin{cases} x[p] & n = 2p \\ 0 & n = 2p+1 \end{cases}$$

Imposing the PR condition (output signal=input signal) one gets the relations that the different filters must satisfy (Theorem 7.7)

Revisiting the orthogonal case (CMF)



Taking $\bar{h}[n] = h[-n]$ as reference (which amounts to choosing the analysis low-pass filter) the following relations hold for an orthogonal filter bank:

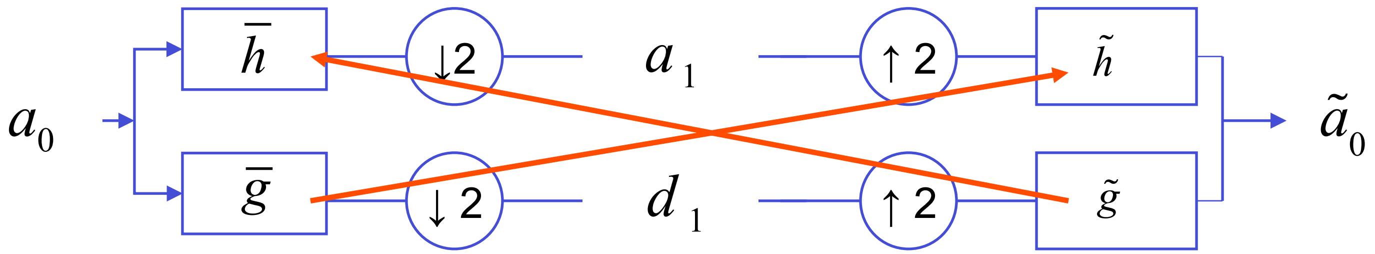
$$\bar{h}[n] = h[-n] \leftrightarrow h[n] = \bar{h}[-n]$$

synthesis low-pass (interpolation) filter: reverse the order of the coefficients

$$g[n] = (-1)^{1-n} h[1-n]$$

negate every other sample

Orthogonal vs biorthogonal PRFB



$\tilde{h} \neq h$ *Biorthogonal* PRFB

$$\hat{h}^*(\omega)\hat{\tilde{h}}(\omega) + \hat{h}^*(\omega+\pi)\hat{\tilde{h}}(\omega+\pi) = 2$$

$$\hat{g}(\omega) = e^{-j\omega}\hat{\tilde{h}}^*(\omega+\pi)$$

$$\hat{\tilde{g}}(\omega) = e^{-j\omega}h^*(\omega+\pi)$$

In the signal domain

$$g[n] = (-1)^{1-n}\tilde{h}[1-n]$$

$$\tilde{g}[n] = (-1)^{1-n}h[1-n]$$

$\tilde{h} = h$ *Orthogonal* PRFB

$$\left|\hat{h}(\omega)\right|^2 + \left|\hat{h}(\omega+\pi)\right|^2 = 2$$

$$\tilde{g} = g$$

Fast BWT

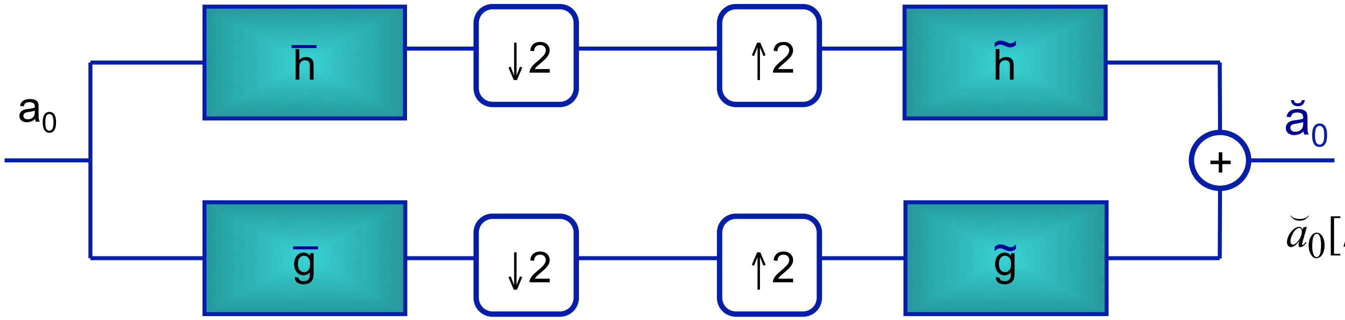
- Two different sets of basis functions are used for analysis and synthesis

$$a_{j+1}[n] = a_j * \bar{h}[2n]$$
$$d_{j+1}[n] = a_j * \bar{g}[2n]$$

$$a_j[n] = \breve{a}_{j+1} * \tilde{h}[n] + \breve{d}_{j+1} * \tilde{g}[n]$$

- PR filterbank

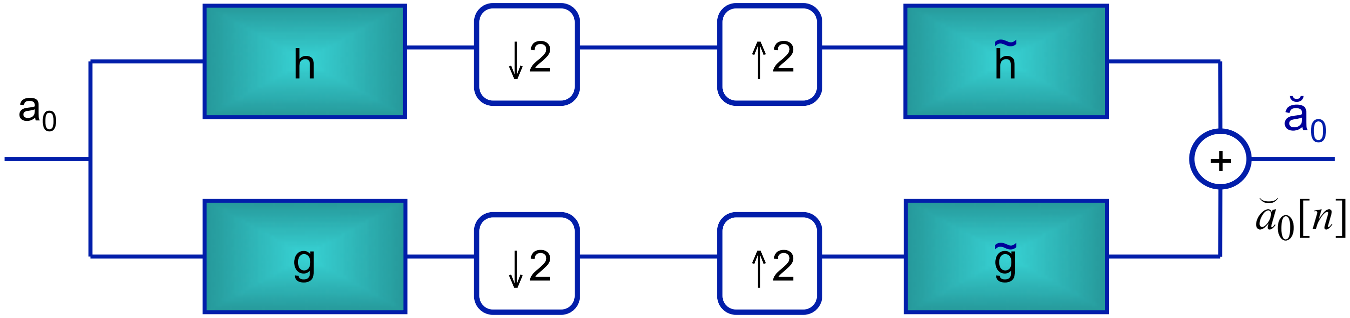
$$g[n] = (-1)^{1-n}\tilde{h}[1-n]$$
$$\tilde{g}[n] = (-1)^{1-n}h[1-n]$$



$$\breve{a}_0[n] = a_0[n]$$

Be careful with notations!

- In the simplified notation where
  - *h[n]* is the analysis low pass filter and *g[n]* is the analysis band pass filter, as it is the case in most of the literature;
  - the delay factor is not made explicit;
- The relations among the filters modify as follows



$\breve{a}_0[n] = a_0[n]$

$$g[n] = (-1)^{-n}\tilde{h}[n]$$

$$\tilde{g}[n] = (-1)^{-n}h[n]$$

Slightly different formulation: the high pass filters are obtained by the low pass filters by negating the odd terms

Biorthogonal bases

Orthonormal basis

$\{e_n\}_{n\in N}$: basis of Hilbert space

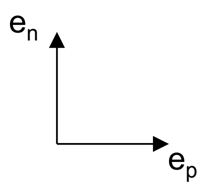
Ortogonality condition $< e_n, e_p >=0 \quad \forall n\neq p$

$\forall y \in H$,

There exists a sequence

$$\lambda[n] = \langle y, e_n \rangle : \quad y = \sum_n \lambda[n] e_n$$

$|e_n|^2=1$ ortho-*normal* basis



Bi-orthogonal basis

$\{e_n\}_{n\in N}$: linearly independent

$\forall y \in H, \quad \exists A>0$ and $B>0$:

$$\lambda[n] = \langle y, e_n \rangle : \quad y = \sum_n \lambda[n] \tilde{e}_n$$

$$\frac{|y|^2}{B} \leq \sum_n |\lambda[n]|^2 \leq \frac{|y|^2}{A}$$

Biorthogonality condition:

$$\langle e_n, \tilde{e}_p \rangle = \delta[n-p]$$

$$y = \sum_n \langle f, \tilde{e}_n \rangle e_n = \sum_n \langle f, e_n \rangle \tilde{e}_n$$

$A=B=1 \Rightarrow$ orthogonal basis

Biorthogonal bases

If h and $\tilde{h}$ are FIR

$$\hat{\tilde{\Phi}}(\omega) = \prod_{p=1}^{+\infty} \frac{\hat{\tilde{h}}\left(2^{-p}\omega\right)}{\sqrt{2}} \hat{\tilde{\Phi}}(0), \qquad \hat{\Phi}(\omega) = \prod_{p=1}^{+\infty} \frac{\hat{h}\left(2^{-p}\omega\right)}{\sqrt{2}} \hat{\Phi}(0)$$

Though, some other conditions must be imposed to guarantee that φ^ and φ^tilde are FT of finite energy functions. The theorem from Cohen, Daubechies and Feaveau provides *sufficient* conditions (Theorem 7.10 in M1999 and Theorem 7.13 in M2009)

The functions $\hat{\phi}$ and $\hat{\tilde{\phi}}$ satisfy the biorthogonality relation

$$\left\langle \varphi(t), \tilde{\varphi}(t-n) \right\rangle = \delta[n]$$

The two wavelet families $\left\{\psi_{j,n}\right\}_{(j,n)\in Z^2}$ and $\left\{\tilde{\psi}_{j,n}\right\}_{(j,n)\in Z^2}$ are Riesz bases of $L^2(R)$

$$\left\langle \psi_{j,n}, \tilde{\psi}_{j',n'} \right\rangle = \delta[n-n']\delta[j-j']$$

Any $f \in L^2(R)$ has two possible decompositions in these bases

$$f = \sum_{n,j} \left\langle f, \psi_{j,n} \right\rangle \tilde{\psi}_{j,n} = \sum_{n,j} \left\langle f, \tilde{\psi}_{j,n} \right\rangle \psi_{j,n}$$

Reminder

Theorem 7.13. If h, g, $\tilde{h}$, and $\tilde{g}$ are perfect reconstruction filters, and their Fourier transforms are bounded, then $\{\tilde{h}[n-2l], \tilde{g}[n-2l]\}_{l\in\mathbb{Z}}$ and $\{h[n-2l], g[n-2l]\}_{l\in\mathbb{Z}}$ are biorthogonal Riesz bases of $\ell^2(\mathbb{Z})$.

Summary of Biorthogonality relations

- An infinite cascade of PR filter banks $(h,g),(\tilde{h},\tilde{g})$ yields two scaling functions and two wavelets whose Fourier transform satisfy

$$\hat{\Phi}(2\omega)=\frac{1}{\sqrt{2}}\hat{h}(\omega)\hat{\Phi}(\omega) \quad \leftrightarrow \quad \varphi\left(\frac{t}{2}\right)=\sum_{n=-\infty}^{+\infty} h[n]\varphi(t-n) \qquad (i)$$

$$\hat{\tilde{\Phi}}(2\omega)=\frac{1}{\sqrt{2}}\hat{\tilde{h}}(\omega)\hat{\tilde{\Phi}}(\omega) \quad \leftrightarrow \quad \tilde{\varphi}\left(\frac{t}{2}\right)=\sum_{n=-\infty}^{+\infty} \tilde{h}[n]\tilde{\varphi}(t-n) \qquad (ii)$$

$$\hat{\Psi}(2\omega)=\frac{1}{\sqrt{2}}\hat{g}(\omega)\hat{\Phi}(\omega) \quad \leftrightarrow \quad \psi\left(\frac{t}{2}\right)=\sum_{n=-\infty}^{+\infty} g[n]\varphi(t-n) \qquad (iii)$$

$$\hat{\tilde{\Psi}}(2\omega)=\frac{1}{\sqrt{2}}\hat{\tilde{g}}(\omega)\hat{\tilde{\Phi}}(\omega) \quad \leftrightarrow \quad \tilde{\psi}\left(\frac{t}{2}\right)=\sum_{n=-\infty}^{+\infty} \tilde{g}[n]\tilde{\varphi}(t-n) \qquad (iv)$$

Properties of biorthogonal filters

Imposing the zero average condition to ψ in equations *(iii)* and *(iv)*

$$\hat{\Psi}(0) = \hat{\tilde{\Psi}}(0) = 0 \quad \rightarrow \quad \boxed{\hat{g}(0) = \hat{\tilde{g}}(0) = 0}$$

replacing into the relations (3) (also shown below)

$$\hat{g}(\omega) = e^{-i\omega}\hat{\tilde{h}}^*(\omega+\pi) \qquad \hat{\tilde{g}}(\omega) = e^{-i\omega}\hat{h}^*(\omega+\pi) \rightarrow \boxed{\hat{h}^*(\pi) = \hat{\tilde{h}}(\pi) = 0}$$

Furthermore, replacing such values in the PR condition (1)

$$\hat{h}^*(\omega)\hat{\tilde{h}}(\omega) + \hat{g}^*(\omega)\hat{\tilde{g}}(\omega) = 2 \rightarrow \boxed{\hat{h}^*(0)\hat{\tilde{h}}(0) = 2}$$

It is common choice to set

$$\boxed{\hat{h}^*(0) = \hat{\tilde{h}}(0) = \sqrt{2}}$$

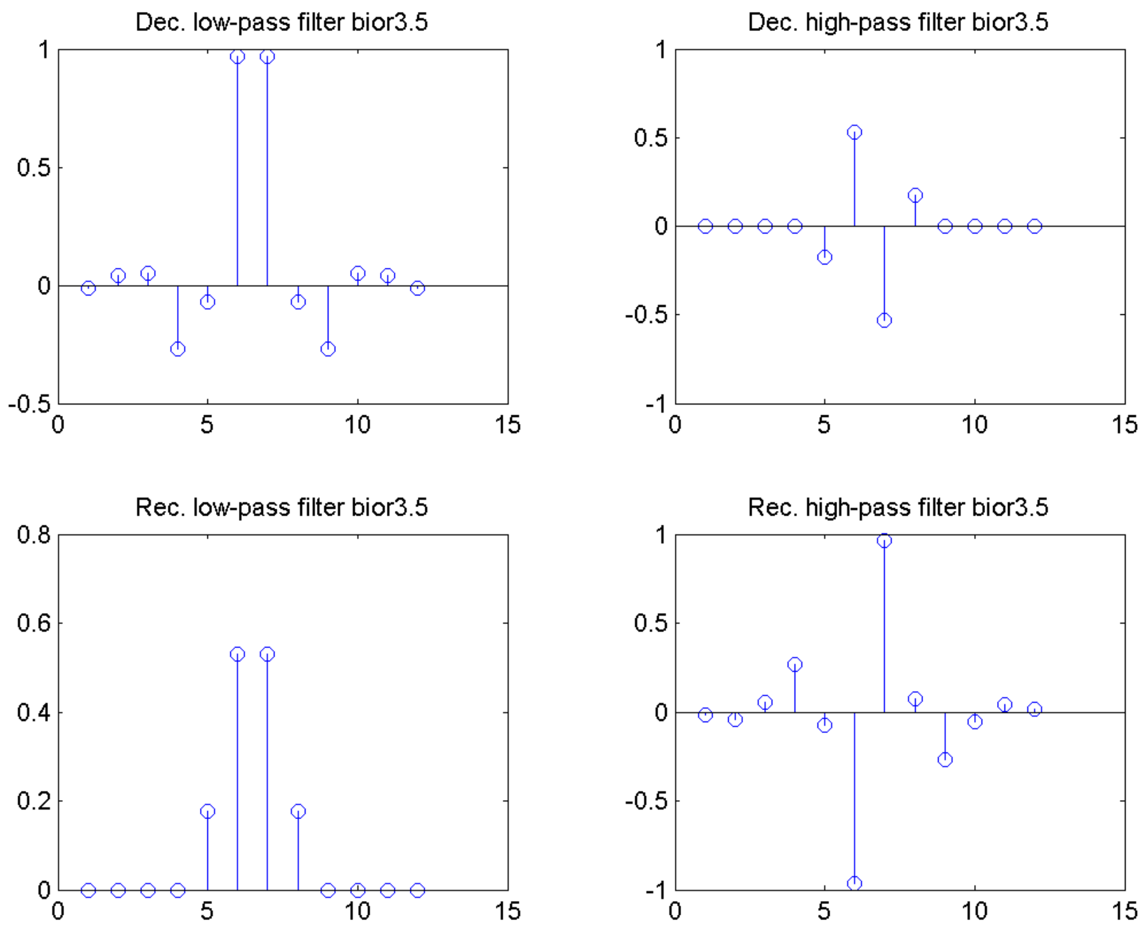
Biorthogonal bases

- If the decomposition and reconstruction filters are different, the resulting bases is non-orthogonal
- The cascade of J levels is equivalent to a signal decomposition over a non-orthogonal basis

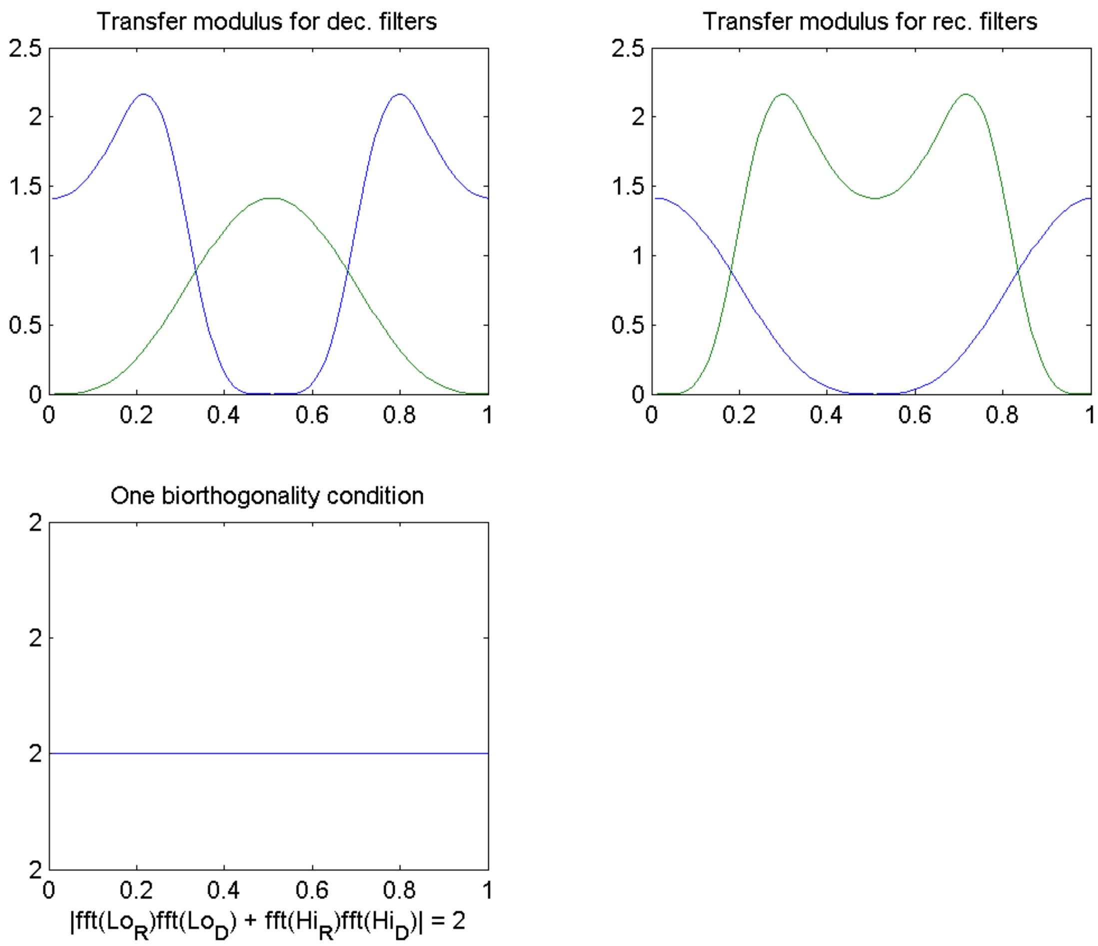
$$\left[\left\{ \varphi_J \left[k - 2^J n \right] \right\}_{n \in Z}, \left\{ \psi_j \left[k - 2^j n \right] \right\}_{1 \le j \le J, n \in Z} \right]$$

- The dual bases is needed for reconstruction

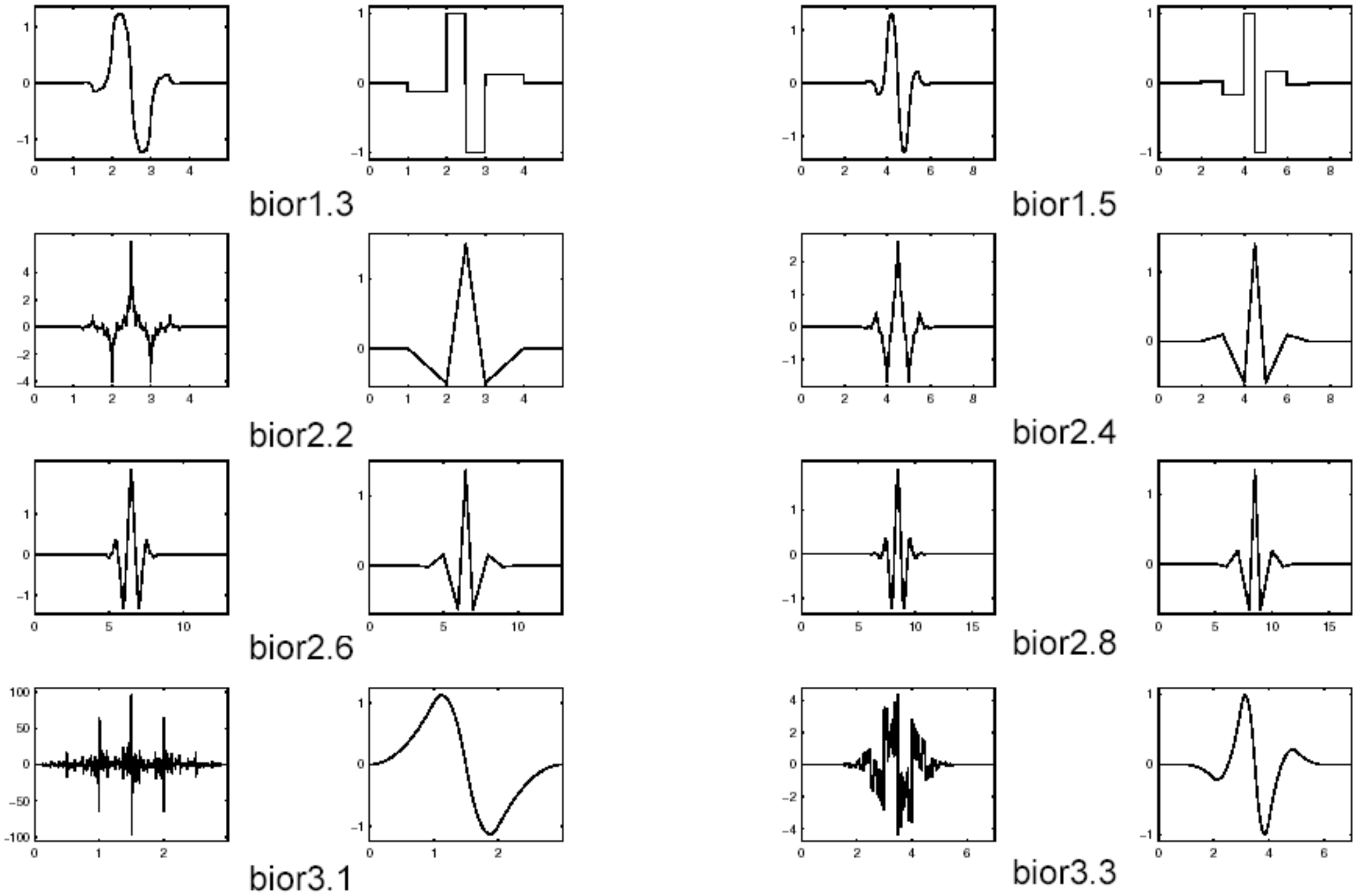
Example: bior3.5



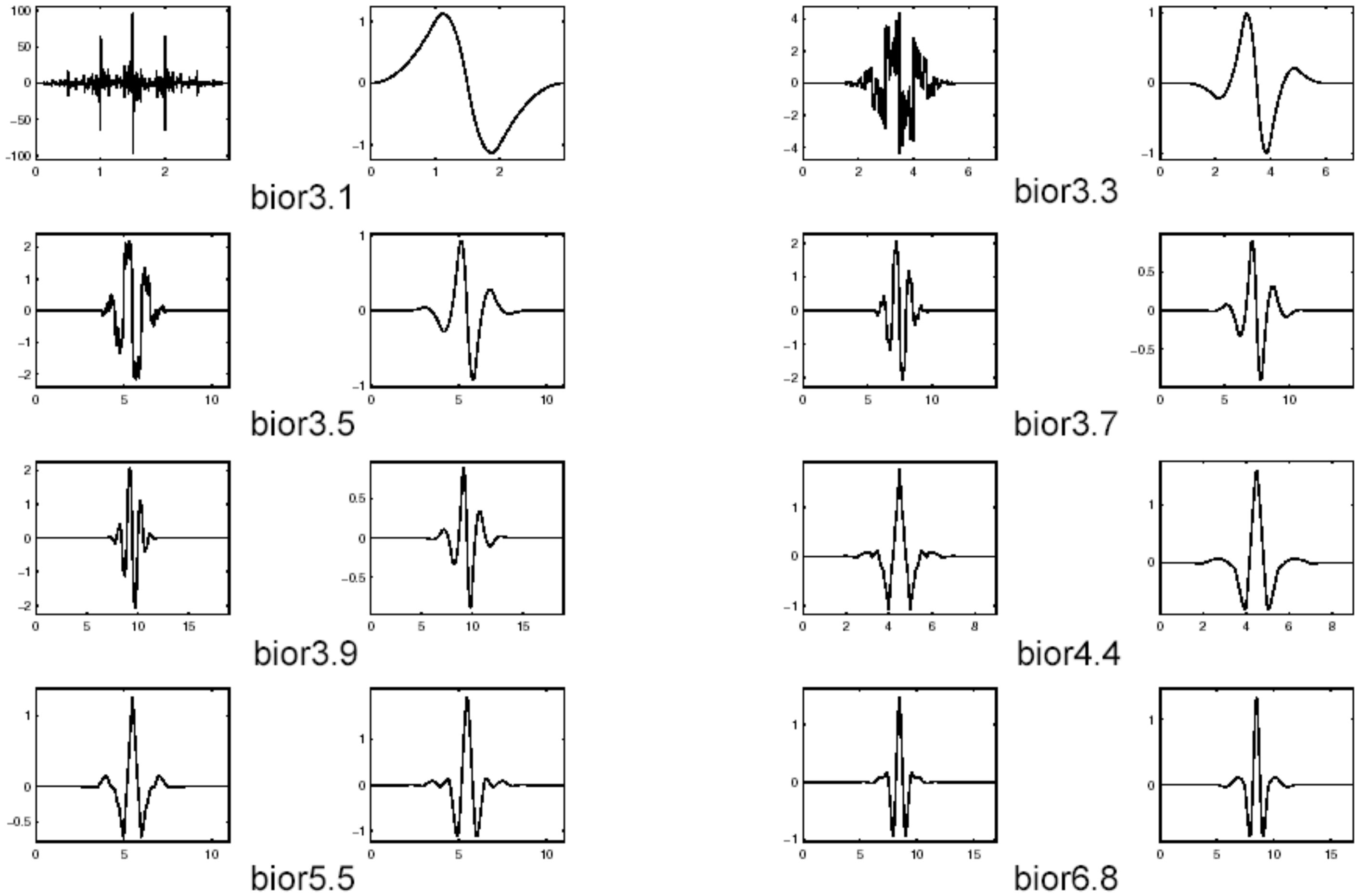
Example: bior3.5



Biorthogonal bases



Biorthogonal bases



CMF : orhtogonal filters

- PR filter banks decompose the signals in a basis of $l^2(Z)$. This basis is *orthogonal* for *Conjugate Mirror Filters* (CMF).
- [Smith&Barnwell,1984]: Necessary and sufficient condition for PR *orthogonal FIR* filter banks, called CMFs
  - Imposing that *the decomposition filter h is equal to the reconstruction filter h~,* eq. (1) becomes

$$\hat{h}^*(\omega)\hat{\tilde{h}}(\omega)+\hat{h}^*(\omega+\pi)\hat{\tilde{h}}(\omega+\pi)=2 \quad (1) \rightarrow$$

$$\hat{h}^*(\omega)\hat{h}(\omega)+\hat{h}^*(\omega+\pi)\hat{h}(\omega+\pi)=2 \rightarrow$$

$$|\hat{h}(\omega)|^2+|\hat{h}(\omega+\pi)|^2=2$$

  - Correspondingly

$$\tilde{h}[n]=h[n]$$
$$\tilde{g}[n]=g[n]=(-1)^{1-n}h[1-n]$$

Summary

- PR filter banks decompose the signals in a basis of $l^2(Z)$. This basis is *orthogonal* for *Conjugate Mirror Filters* (CMF).
- [Smith&Barnwell,1984]: Necessary and sufficient condition for PR orthogonal FIR filter banks, called CMFs
  - Imposing that the decomposition filter h is equal to the reconstruction filter $h^{\sim}$, eq. (1) becomes

$$\hat{h}^*(\omega)\hat{\tilde{h}}(\omega) + \hat{h}^*(\omega+\pi)\hat{\tilde{h}}(\omega+\pi) = 2 \rightarrow$$

$$\hat{h}^*(\omega)\hat{h}(\omega) + \hat{h}^*(\omega+\pi)\hat{h}(\omega+\pi) = 2 \rightarrow$$

$$|\hat{h}(\omega)|^2 + |\hat{h}(\omega+\pi)|^2 = 2$$

  - Correspondingly

$$\tilde{h}[n] = h[n]$$
$$\tilde{g}[n] = g[n] = (-1)^{1-n} h[1-n]$$

Properties

- Support
  - h, $\tilde{h}$ are FIR → scaling functions and wavelets have compact support
- Vanishing moments
  - The number of vanishing moments of Ψ is equal to the order $\tilde{p}$ of zeros of $\tilde{h}$ in π. Similarly, the number of vanishing moments of $\tilde{\psi}$ is equal to the order p of zeros of h in π.
- Regularity
  - One can show that the regularity of Ψ and φ increases with the number of vanishing moments of $\tilde{\Psi}$, thus with the order p of zeros of h in π.
  - Viceversa, the regularity of $\tilde{\psi}$ and $\tilde{\varphi}$ increases with the number of vanishing moments of Ψ, thus with the order $\tilde{p}$ of zeros of $\tilde{h}$ in π.
- Symmetry
  - It is possible to construct both symmetric and anti-symmetric bases using linear phase filters
    - In the orthogonal case only the Haar filter is possible as FIR solution.