Predicate Logic
(first order logic)

formula	intuitive meanings
$\exists x P(x)$	there is an x with property P
$\forall y P(y)$	?
$\forall x \exists y (x = 2y)$	?
$\forall \varepsilon (\varepsilon > 0 \rightarrow \exists n (1 < \varepsilon))$	?
$x < y \rightarrow \exists z (x < z \wedge z < y)$	?
$\forall x \exists y (x.y = 1)$	?

formula	intuitive meanings
$\exists x P(x)$	there is an x with property P
$\forall y P(y)$	for all y P holds (all y have the property P)
$\forall x \exists y (x = 2y)$	for all x there is a y such that x is two times y
$\forall \varepsilon (\varepsilon > 0 \rightarrow \exists n (1 < \varepsilon))$	for all positive ε there is an n such that 1< ε
$x < y \rightarrow \exists z (x < z \wedge z < y)$	if x < y, then there is a z such that x < z and z < y
$\forall x \exists y (x.y = 1)$	for each x there exists an inverse y

The semantics of predicate logics

Structure

$$\langle A\ ,\ R_1,\ldots,R_n\ ,\ F_1,\ldots,F_m\ ,\ \{c_i | i \in I\}\rangle$$

Structure

$$\mathfrak{A} = \langle A, R_1,\dots,R_n, F_1,\dots,F_m, \{c_i \mid i \in I\}\rangle$$

- A – A non-empty set
- $R_1,\dots,R_n$ – relations on A
- $F_1,\dots,F_m$ – functions on A
- $\{c_i \mid i \in I\}$ – elements of A

notation $|\mathfrak{A}| = A$

$\langle R, +, \cdot, ^{-1}, 0, 1\rangle$ – the field of real numbers,

$\langle N, <\rangle$ – the ordered set of natural numbers.

Definition 2.2.2 *The similarity type of a structure* $\mathfrak{A} = \langle A, R_1, \ldots, R_n, F_1, \ldots, F_m, \{c_i | i \in I\}\rangle$ *is a sequence,* $\langle r_1, \ldots, r_n; a_1, \ldots, a_m; \kappa\rangle$*, where* $R_i \subseteq A^{r_i}$*,* $F_j : A^{a_j} \to A$*,* $\kappa = |\{c_i \ | i \in I\}|$ *(cardinality of* I*).*

what is A^0 ?

what is $f: A^0 \to A$?

what is A^0 ?

what is $f: A^0 \to A$?

what is A^n ?

what is $f: \varnothing \to A$?

Write down the similarity type for the following structures:

(i) $\langle \mathbb{Q}, <, 0 \rangle$

(ii) $\langle \mathbb{N}, +, \cdot, S, 0, 1, 2, 3, 4, \ldots, n, \ldots \rangle$,where$S(x) = x + 1$,

(iii) $\langle \mathcal{P}(\mathbb{N}), \subseteq, \cup, \cap, ^c, \emptyset \rangle$,

(iv) $\langle \mathbb{Z}/(5), +, \cdot, -, ^{-1}, 0, 1, 2, 3, 4 \rangle$,

(v) $\langle \{0, 1\}, \wedge, \vee, \rightarrow, \neg, 0, 1 \rangle$, where $\wedge, \vee, \rightarrow, \neg$ operate according to the ordinary truth tables,

(vi) $\langle \mathbb{R}, 1 \rangle$,

(vii) $\langle \mathbb{R} \rangle$,

Give structures with type $\langle 1, 1; -; 3 \rangle$, $\langle 4; -; 0 \rangle$.

alphabet

$\langle r_1,\ldots,r_n ; a_1,\ldots,a_m;\kappa\rangle$, with $r_i \geq 0, a_j > 0$.

1. Predicate symbols: sequence $P_1, \ldots, P_n$, plus $\doteq$
2. Function symbols: sequence $f_1,\ldots,f_m$
3. Constant symbols $\mathbf{c}_i$ for $i \in I$ with $|I|=\kappa$
4. Variables: $x_0,x_1,x_2,\ldots$(countably many)
5. Connectives: $\vee,\wedge,\rightarrow,\neg,\leftrightarrow,\bot\ \forall,\exists$
6. auxiliary symbols: (,),

we write also $\langle <P_1, \ldots, P_n; f_1,\ldots,f_m,\{\mathbf{c}_i\}_{i\in I}\rangle$ to relate with

$$\langle r_1,\ldots,r_n ;a_1,\ldots,a_m;\kappa\rangle$$

$\langle r_1,\ldots,r_n ; a_1,\ldots,a_m;\kappa\rangle$, with $r_i \geq 0, a_j > 0$.

Definition *TERM is the smallest set X with the properties*

(i) $\overline{c}_i \in X\,(i \in I)$ *and* $x_i \in X\,(i \in N)$,

(ii) $t_1, \ldots, t_{a_i} \in X \Rightarrow f_i(t_1, \ldots, t_{a_i}) \in X$, *for* $1 \leq i \leq m$

TERM is our *set of terms.*

Definition *FORM is the smallest set X with the properties:*

(i) $\perp \in X; P_i \in X$ *if* $r_i = 0$; $t_1, \ldots, t_{r_i} \in TERM \Rightarrow$
$P_i(t_1, \ldots, t_{r_i}) \in X; t_1, t_2 \in TERM \Rightarrow t_1 = t_2 \in X$,

(ii) $\varphi, \psi \in X \Rightarrow (\varphi \square \psi) \in X$, *where* $\square \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$,

(iii) $\varphi \in X \Rightarrow (\neg\varphi) \in X$,

(iv) $\varphi \in X \Rightarrow ((\forall x_i)\varphi), ((\exists x_i)\varphi) \in X$.

proof by induction

Lemma *Let* $A(t)$ *be a property of terms. If* $A(t)$ *holds for* t *a variable or a constant, and if* $A(t_1), A(t_2), \ldots, A(t_n) \Rightarrow A(f(t_1, \ldots, t_n))$, *for all function symbols* f, *then* $A(t)$ *holds for all* $t \in$ *TERM.*

Lemma *Let* $A(\varphi)$ *be a property of formulas. If*

(i) $A(\varphi)$ *for atomic* φ,

(ii) $A(\varphi), A(\psi) \Rightarrow A(\varphi \Box \psi)$,

(iii) $A(\varphi) \Rightarrow A(\neg\varphi)$,

(iv) $A(\varphi) \Rightarrow A((\forall x_i)\varphi), A((\exists x_i)\varphi)$ *for all* i, *then* $A(\varphi)$ *holds for all* $\varphi \in$ *FORM.*

Example of a language of type $\langle 2;2,1;1 \rangle$.

predicate symbols: L, =
function symbols: p, i
constant symbol: e

Definition by Recursion on TERM: Let $H_0 : Var \cup Const \to A$ (i.e.H_0 is defined on variables and constants), $H_i : A^{a_i} \to A$, then there is a unique mapping $H : TERM \to A$ such that

$$\begin{cases} H(t) = H_0(t) \text{ for } t \text{ a variable or a constant,} \\ H(f_i(t_1, \ldots, t_{a_i})) = H_i(H(t_1), \ldots, H(t_{a_i})). \end{cases}$$

Definition by Recursion on FORM:

$$\begin{array}{ll} \text{Let } H_{at} : At \to A & \text{(i.e.} H_{at} \text{ is defined on atoms),} \\ \quad H_{\Box} : A^2 \to A, & (\Box \in \{\vee, \wedge, \to, \leftrightarrow\}) \\ \quad H_{\neg} : A \to A, & \\ \quad H_{\forall} : A \times N \to A, & \\ \quad H_{\exists} : A \times N \to A. & \end{array}$$

then there is a unique mapping $H : FORM \to A$ such that

$$\begin{cases} H(\varphi) & = H_{at}(\varphi) \text{ for atomice } \varphi, \\ H(\varphi \Box \psi) & = H_{\Box}(H(\varphi), H(\psi)), \\ H(\neg\varphi) & = H_{\neg}(H(\varphi)), \\ H(\forall x_i \varphi) & = H_{\forall}(H(\varphi), i), \\ H(\exists x_i (\varphi) & = H_{\exists}(H(\varphi), i). \end{cases}$$

free variables

Definition *The set $FV(t)$ of free variables of t is defined by*

$$
\begin{array}{lll}
(i) & FV(x_i) & := \{x_i\}, \\
 & FV(\overline{c}_i) & := \emptyset \\
(ii) & FV(f(t_1, \ldots, t_n)) & := FV(t_1) \cup \ldots \cup FV(t_n).
\end{array}
$$

Definition *The set $FV(\varphi)$ of free variables of φ is defined by*

$$
\begin{array}{lll}
(i) & FV(P(t_1, \ldots, t_p)) & := FV(t_1) \cup \ldots \cup FV(t_p), \\
 & FV(t_1 = t_2) & := FV(t_1) \cup FV(t_2), \\
 & FV(\bot) = FV(P) & := \emptyset \text{ for } P \text{ a proposition symbol}, \\
(ii) & FV(\varphi \Box \psi) & := FV(\varphi) \cup FV(\psi), \\
 & FV(\neg \varphi) & := FV(\varphi), \\
(iii) & FV(\forall x_i \varphi) := FV(\exists x_i \varphi) & := FV(\varphi) - \{x_i\}.
\end{array}
$$

- t or φ is called closed if $FV(t) = \varnothing$, resp. $FV(\varphi) = \varnothing$.
- a closed formula is also called a sentence.
- a formula without quantifiers is called open.
- $TERM_c$ denotes the set of closed terms;
- SENT denotes the set of sentences.

Exercise:

- define the set $BV(\varphi)$ of bound variables of φ
- $FV(\varphi) \cap BV(\varphi) = \varnothing$?

The notion of SUBFORMULa

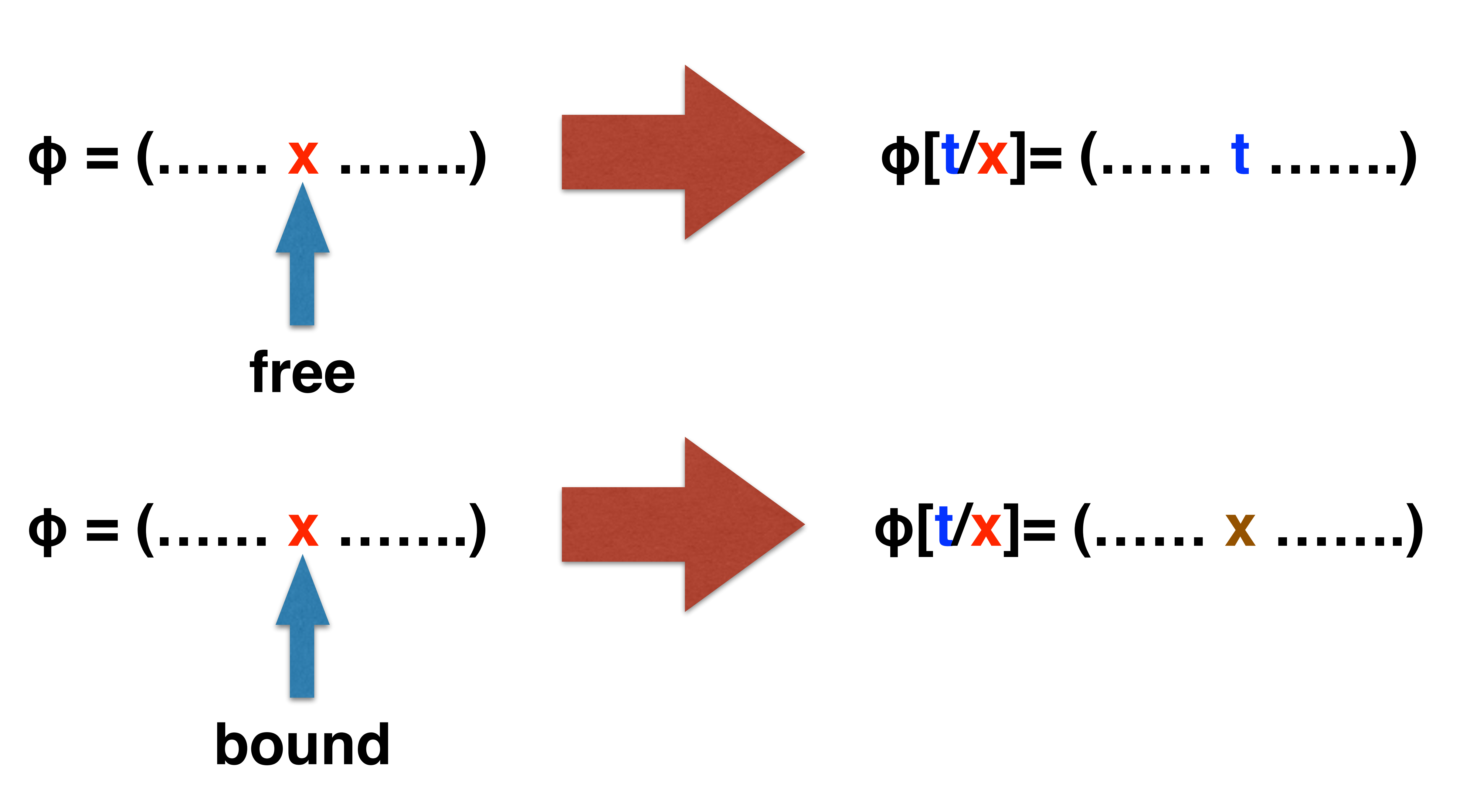
- $Sub(\varphi) = \{\varphi\}$ for atomic φ
- $Sub(\varphi_1 \square \varphi_2) = Sub(\varphi_1) \cup Sub(\varphi_2) \cup \{\varphi_1 \square \varphi_2\}$ for $\square \in \{\wedge, \vee, \rightarrow\}$
- $Sub_+(\neg\varphi) = Sub(\varphi) \cup \{\neg\varphi\}$
- $Sub_+(Qx.\varphi) = Sub(\varphi) \cup \{Qx.\varphi\}$ for $Q \in \{\forall, \exists\}$

Free and Bound occurrences of variables

- an occurrence of a variable x in δ is BOUND, if x occurs in $\varphi \in SUB\{\delta\}$ and $\varphi \equiv \{Qx.\theta\}$ for $Q \in \{\forall, \exists\}$
- an occurrence of a variable x in δ is FREE, if x does not occur in any $\varphi \in SUB\{\delta\}$ with $\varphi \equiv \{Qx.\theta\}$ for $Q \in \{\forall, \exists\}$

SUBSTITUTION

$\varphi[t/x]$?



SUBSTITUTION

Definition *Let s and t be terms, then* $s[t/x]$ *is defined by:*

$$(i)\quad y[t/x] := \begin{cases} y \text{ if } y \not\equiv x \\ t \text{ if } y \equiv x \end{cases}$$

$$c[t/x] := c$$

$$(ii)\quad f(t_1, \ldots, t_p)[t/x] := f(t_1[t/x], \ldots, t_p[t/x]).$$

Definition $\varphi[t/x]$ *is defined by:*

$$(i)\quad \bot\, [t/x] := \bot,$$

$$P[t/x] := P \text{ for propositions } P,$$

$$P(t_1, \ldots, t_p)[t/x] := P(t_1[t/x], \ldots, t_p[t/x]),$$

$$(t_1 = t_2)[t/x] := t_1[t/x] = t_2[t/x],$$

$$(ii)\quad (\varphi \Box \psi)[t/x] := \varphi[t/x] \Box \psi[t/x],$$

$$(\neg\varphi)[t/x] := \neg\varphi[t/x]$$

$$(iii)\quad (\forall y \varphi)[t/x] := \begin{cases} \forall y \varphi[t/x] \text{ if } x \not\equiv y \\ \forall y \varphi \text{ if } x \equiv y \end{cases}$$

$$(\exists y \varphi)[t/x] := \begin{cases} \exists y \varphi[t/x] \text{ if } x \not\equiv y \\ \exists y \varphi \text{ if } x \equiv y \end{cases}$$

Define symultaneus substitution $\delta[t_1,\ldots,t_n/x_1,\ldots,x_n]$

$$\exists x (y < x)[x/y] = \exists x\, (x < x)$$

?

$\exists x(y < x)[x/y] = \exists x\,(x < x)$

We must forbid dangerous substitutions

$t = (\ldots y \ldots)$ $\quad \phi = (\ldots(\exists y \ldots x \ldots)\ldots)$

$\phi[t/x] = (\ldots(\exists y \ldots(\ldots y \ldots)\ldots)\ldots)$

y is now bound!

Definition

t is free for x in ϕ if

(i) ϕ is atomic,

(ii) $\phi := \phi_1 \square \phi_2$ (or $\phi := \neg\phi_1$) and t is free for x in ϕ_1 and ϕ_2 (resp. ϕ_1),

(iii) $\phi := \exists y \psi$ (or $\phi := \forall y \psi$) and if $x \in FV(\phi)$, then $y \notin FV(t)$ and t is free for x in ψ.

$$t = (\ldots y \ldots) \qquad \phi = (\ldots(\exists y \ldots x \ldots)\ldots)$$

$$\phi[t/x] = (\ldots(\exists y \ldots(\ldots y \ldots)\ldots)\ldots)$$

t is NOT free for x in ϕ

proposition

t is free for x in $\varphi \Leftrightarrow$ the variables of t in $\varphi[t/x]$ are not bound by a quantifier.

proof by induction (exercise!)

Check which terms are free in the following cases, and carry out the substitution:

(a) x for x in $x = x$,

(b) y for x in $x = x$,

(c) $x + y$ for y in $z = \overline{0}$,

(d) $\overline{0} + y$ for y in $\exists x(y = x)$,

(e) $x + y$ for z in $\exists w(w + x = \overline{0})$,

(f) $x + w$ for z in $\forall w(x + z = \overline{0})$,

(g) $x + y$ for z in $\forall w(x + z = \overline{0}) \wedge \exists y(z = x)$,

(h) $x + y$ for z in $\forall u(u = v) \rightarrow \forall z(z = y)$.

NATURAL DEDUCTION

Notation

in the same context **φ(x)** and **φ(t)** denote respectively **φ** and **φ[t/x]**

hp$\mathcal{D}$

$$\forall I\ \frac{\begin{matrix}\mathcal{D}\\ \varphi(x)\end{matrix}}{\forall x \varphi(x)}$$

x∉FV(hp$\mathcal{D}$)

$$\forall E\ \frac{\begin{matrix}\mathcal{D}\\ \forall x \varphi(x)\end{matrix}}{\varphi(t)}$$

t free for x in φ

$$\dfrac{\dfrac{\dfrac{\dfrac{[x = 0]}{\forall x(x = 0)}}{x = 0 \to \forall x(x = 0)}}{\forall x(x = 0 \to \forall x(x = 0))}}{0 = 0 \to \forall x(x = 0)}$$

$$
\cfrac{\cfrac{\cfrac{\cfrac{[x = 0]}{\forall x(x = 0)}}{x = 0 \rightarrow \forall x(x = 0)}}{\forall x(x = 0 \rightarrow \forall x(x = 0))}}{0 = 0 \rightarrow \forall x(x = 0)}
$$

NO!

$$\dfrac{\dfrac{\dfrac{\dfrac{[x = 0]}{\forall x(x = 0)}}{x = 0 \to \forall x(x = 0)}}{\forall x(x = 0 \to \forall x(x = 0))}}{0 = 0 \to \forall x(x = 0)}$$

$$\dfrac{\dfrac{[\forall x \neg \forall y (x = y)]}{\neg \forall y (y = y)}}{\forall x \neg \forall y (x = y) \to \neg \forall y (y = y)}$$

$$\dfrac{\dfrac{[\forall x \neg \forall y (x = y)]}{\neg \forall y (y = y)} \textbf{ NO!}}{\forall x \neg \forall y (x = y) \to \neg \forall y (y = y)}$$

$$\cfrac{\cfrac{[\forall x \neg \forall y (x = y)]}{\neg \forall y (y = y)}}{\forall x \neg \forall y (x = y) \rightarrow \neg \forall y (y = y)}$$

$$\forall x \forall y \varphi(x, y) \rightarrow \forall y \forall x \varphi(x, y)$$

$$\frac{[\forall x \forall y \varphi(x,y)]}{\forall y \varphi(x,y)}\,\forall E$$

$$\dfrac{\dfrac{[\forall x \forall y \varphi(x,y)]}{\forall y \varphi(x,y)}\,\forall E}{\varphi(x,y)}\,\forall E$$

$$\dfrac{\dfrac{\dfrac{[\forall x \forall y \varphi(x,y)]}{\forall y \varphi(x,y)}\,\forall E}{\varphi(x,y)}\,\forall E}{\forall x \varphi(x,y)}\,\forall I$$

$$
\begin{array}{cl}
[\forall x \forall y \varphi(x, y)] & \\
\hline
\forall y \varphi(x, y) & \forall E \\
\hline
\varphi(x, y) & \forall E \\
\hline
\forall x \varphi(x, y) & \forall I \\
\hline
\forall y \forall x (\varphi(x, y) & \forall I
\end{array}
$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall x \forall y \varphi(x, y)]}{\forall y \varphi(x, y)}\,\forall E}{\varphi(x, y)}\,\forall E}{\forall x \varphi(x, y)}\,\forall I}{\forall y \forall x (\varphi(x, y)}\,\forall I}{\forall x \forall y \varphi(x, y) \rightarrow \forall y \forall x \varphi(x, y)}\,\rightarrow I$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall x(\varphi(x) \wedge \psi(x))]}{\varphi(x) \wedge \psi(x)}}{\varphi(x)}}{\forall x \varphi(x)} \qquad \dfrac{\dfrac{\dfrac{[\forall x(\varphi(x) \wedge \psi(x))]}{\varphi(x) \wedge \psi(x)}}{\psi(x)}}{\forall x \psi(x)}}{\forall x \varphi(x) \wedge \forall x \psi(x)}}{\forall x(\varphi \wedge \psi) \to \forall x \varphi \wedge \forall x \psi}
$$

Let $x \notin FV(\varphi)$

$$\forall x(\varphi \longrightarrow \psi(x)) \longrightarrow (\varphi \longrightarrow \forall x(\psi(x))$$

Let $x \notin FV(\varphi)$

$$\cfrac{\cfrac{\cfrac{\cfrac{[\forall x(\varphi \to \psi(x))]}{\varphi \to \psi(x)}\,\forall E \qquad [\varphi]}{\psi(x)}\,\to E}{\forall x \psi(x)}\,\forall I}{\varphi \to \forall x \psi(x)}\,\to I$$

$$\overline{\forall x(\varphi \to \psi(x)) \to (\varphi \to \forall x(\psi(x))}$$

$$\mathit{\Gamma} \vdash \varphi(x) \Rightarrow \mathit{\Gamma} \vdash \forall x \varphi(x) \text{ if } x \notin FV(\psi) \text{ for all } \psi \in \mathit{\Gamma}$$
$$\mathit{\Gamma} \vdash \forall x \varphi(x) \Rightarrow \mathit{\Gamma} \vdash \varphi(t) \text{ if } t \text{ is free for } x \text{ in } \varphi.$$

1. Show: (i)$\vdash \forall x(\varphi(x) \to \psi(x)) \to (\forall x\varphi(x) \to \forall x\psi(x))$,

(ii)$\vdash \forall x\varphi(x) \to \neg\forall x\neg\varphi(x)$,

(iii)$\vdash \forall x\varphi(x) \to \forall z\varphi(z)$ if z does not occur in $\varphi(x)$,

SEMANTICS

In describing a structure it is a great help to be able to refer to all elements of $|\mathfrak{A}|$ individually, i.e. to have names for all the elements of $|\mathfrak{A}|$

extended language

The extended language, $L(\mathfrak{A})$, of $\mathfrak{A}$ is obtained from the language L, of the type of $\mathfrak{A}$, by adding constant symbols for all elements of $\mathfrak{A}$. We denote each constant symbol, corresponding to $a \in |\mathfrak{A}|$, by $\boldsymbol{a}$.

an interpretation of the closed terms of $L(\mathfrak{U})$ in $\mathfrak{U}$, is a mapping

$[\![-]\!]_{\mathfrak{U}} : TERM_C \to |\mathfrak{U}|$ satisfying:

(i) $[\![\mathbf{c}]\!]_{\mathfrak{U}} = c$

(ii) $[\![\mathbf{a}]\!]_{\mathfrak{U}} = a$,

(iii) $[\![\mathbf{F_i}(t_1,\dots,t_k)]\!]_{\mathfrak{U}} = F_i([\![t_1]\!]_{\mathfrak{U}},\dots,[\![t_k]\!]_{\mathfrak{U}})$, with $k=a_i$

an interpretation of the sentences φ of $L(\mathfrak{U})$ in $\mathfrak{U}$, is a mapping

$[\![-]\!]_{\mathfrak{U}} : SENT \to \{0,1\}$, satisfying:

i) $[\![\bot]\!]_{\mathfrak{U}} = 0$

ii) $[\![\mathbf{P_i}(t_1,\dots,t_k)]\!]_{\mathfrak{U}} = 1 \iff ([\![t_1]\!]_{\mathfrak{U}},\dots,[\![t_k]\!]_{\mathfrak{U}}) \in P_i$ (with $k = r_i$)

iii) $[\![t_1 = t_e]\!]_{\mathfrak{U}} = 1 \iff$

iv) $[\![\varphi \wedge \delta]\!]_{\mathfrak{U}} = 1 \iff [\![\varphi]\!]_{\mathfrak{U}} = 1$ **and** $[\![\delta]\!]_{\mathfrak{U}} = 1$

v) $[\![\varphi \vee \delta]\!]_{\mathfrak{U}} = 1 \iff [\![\varphi]\!]_{\mathfrak{U}} = 1$ **or** $[\![\delta]\!]_{\mathfrak{U}} = 1$

vi) $[\![\varphi \to \delta]\!]_{\mathfrak{U}} = 1 \iff [\![\varphi]\!]_{\mathfrak{U}} = 0$ **or** $[\![\delta]\!]_{\mathfrak{U}} = 1$

vii) $[\![\forall x.\varphi]\!]_{\mathfrak{U}} = 1 \iff$ **for all** $a \in |\mathfrak{U}|$ $[\![\varphi[x/\mathbf{a}]]\!]_{\mathfrak{U}} = 1$

viii) $[\![\exists x.\varphi]\!]_{\mathfrak{U}} = 1 \iff$ **there exists** $a \in |\mathfrak{U}|$ s.t. $[\![\varphi[x/\mathbf{a}]]\!]_{\mathfrak{U}} = 1$

when there is no ambiguity we write $[\![-]\!]$ instead of $[\![-]\!]_{\mathfrak{U}}$

Given a fixed similarity type and a sentence φ:

$\mathfrak{A} \vDash \varphi$ stands for $[\![\varphi]\!]^{\mathfrak{A}} = 1$

$\vDash \varphi$ stands for **for each $\mathfrak{A}$. $\mathfrak{A} \vDash \varphi$**

Let $FV(\varphi) = \{z_1 \ldots, z_k\}$, then $Cl(\varphi) := \forall z_1 \ldots z_k \varphi$ is the universal closure of φ

(we assume the order of variables z_i to be fixed in some way).

- $\mathfrak{A} \vDash \varphi$ ($\mathfrak{A}$ is a model of φ) $\Leftrightarrow$ $\mathfrak{A} \vDash Cl(\varphi)$,
- $\vDash \varphi$ (φ is valid/true) $\Leftrightarrow$ $\mathfrak{A} \vDash \varphi$ for all $\mathfrak{A}$ (of the appropriate type),
- $\mathfrak{A} \vDash \Gamma$ ($\mathfrak{A}$ is a model of Γ) $\Leftrightarrow$ $\mathfrak{A} \vDash \psi$ for all $\psi \in \Gamma$,
- $\Gamma \vDash \varphi$ (φ is consequence of Γ) $\Leftrightarrow$

  for each $\mathfrak{A}(\mathfrak{A} \vDash \Gamma \Rightarrow \mathfrak{A} \vDash \varphi)$, where $\Gamma \cup \{\varphi\}$ consists of sentences.

For the rest of the course let us suppose to fix an enumeration (without repetitions) of variables $\{x_i\}_{i \in N^*}$ ($\mathbb{N}^* = \mathbb{N} - \{0\}$)

When we write that $FV(\varphi) = \{z_1,\dots,z_k\}$ we means that

for each $j \in [1,k]$ $z_j = x_{i_j}$ and for each $m,n \in [1,k]$ $m < n \Rightarrow i_m < i_n$

If φ is a formula with $FV(\varphi) = \{z_1,\dots,z_k\}$, then we say that

- φ is satisfied by $a_1,\dots,a_k \in |\mathfrak{U}|$ if $\mathfrak{U} \models \varphi[\mathbf{a_1},\dots,\mathbf{a_k}/z_1,\dots,z_k]$
- φ is called satisfiable in $\mathfrak{U}$ if there are $a_1,\dots,a_k \in |\mathfrak{U}|$ such that φ is satisfied by $a_1,\dots,a_k \in |\mathfrak{U}|$
- φ is called satisfiable if it is satisfiable in some $\mathfrak{U}$.

Note that φ is satisfiable in $\mathfrak{U}$ iff $\mathfrak{U} \models \exists z_1,\dots,z_k.\varphi$.

If we restrict ourselves to sentences, we have

(i) $\mathfrak{A} \models \varphi \wedge \psi \Leftrightarrow \mathfrak{A} \models \varphi$ $\mathfrak{A}$ and $\mathfrak{A} \models \psi$,

(ii) $\mathfrak{A} \models \varphi \vee \psi \Leftrightarrow \mathfrak{A} \models \varphi$ or $\mathfrak{A} \models \psi$

(iii) $\mathfrak{A} \models \neg\varphi \Leftrightarrow \mathfrak{A} \not\models \varphi$,

(iv) $\mathfrak{A} \models \varphi \rightarrow \psi \Leftrightarrow (\mathfrak{A} \models \varphi \Rightarrow \mathfrak{A} \models \psi)$,

(vi) $\mathfrak{A} \models \forall x\varphi \Leftrightarrow \mathfrak{A} \models \varphi[\mathbf{a}/x]$, for each $a \in |\mathfrak{A}|$.

(vii) $\mathfrak{A} \models \exists x\varphi \Leftrightarrow \mathfrak{A} \models \varphi[\mathbf{a}/x]$, for some $a \in |\mathfrak{A}|$.

$\mathfrak{A} \models \exists x \varphi \Leftrightarrow \mathfrak{A} \models \varphi[\mathbf{a}/x]$, for some $a \in |\mathfrak{A}|$.

$\mathfrak{U} \models \exists x \varphi \Leftrightarrow \mathfrak{U} \models \varphi[\mathbf{a}/x]$, for some $a \in |\mathfrak{U}|$.

i) $[\![\exists x.\varphi]\!]_{\mathfrak{U}} = 1 \iff$ **there exists** $a \in |\mathfrak{U}|$ s.t. $[\![\varphi[x/\mathbf{a}]]\!]_{\mathfrak{U}} = 1$

exercises

$$(i) \quad \models \neg\forall x\varphi \leftrightarrow \exists x\neg\varphi$$
$$(ii) \quad \models \neg\exists x\varphi \leftrightarrow \forall x\neg\varphi$$
$$(iii) \quad \models \forall x\varphi \leftrightarrow \neg\exists x\neg\varphi$$
$$(iv) \quad \models \exists x\varphi \leftrightarrow \neg\forall x\neg\varphi$$

exercises

$$(i) \models \neg\forall x\varphi \leftrightarrow \exists x\neg\varphi$$
$$(ii) \models \neg\exists x\varphi \leftrightarrow \forall x\neg\varphi$$
$$(iii) \models \forall x\varphi \leftrightarrow \neg\exists x\neg\varphi$$
$$(iv) \models \exists x\varphi \leftrightarrow \neg\forall x\neg\varphi$$

(i) Let $FV(\forall x\varphi) = \{z_1, \ldots, z_k\}$, then we must show
$\mathfrak{A} \models \forall z_1 \ldots z_k(\neg\forall x\varphi(x, z_1, \ldots, z_k) \leftrightarrow \exists x\neg\varphi(x, z_1, \ldots, z_k))$, for all $\mathfrak{A}$.
So we have to show $\mathfrak{A} \models \neg\forall x\varphi(x, \overline{a}_1, \ldots, \overline{a}_k) \leftrightarrow \exists x\neg\varphi(x, \overline{a}_1, \ldots, \overline{a}_k)$ for arbitrary $a_1, \ldots, a_k \in |\mathfrak{A}|$. We apply the properties of $\models$ as listed in Lemma 2.4.5:
$\mathfrak{A} \models \neg\forall x\varphi(x, \overline{a}_1, \ldots, \overline{a}_k) \Leftrightarrow \mathfrak{A} \not\models \forall x\varphi(x, \overline{a}_1, \ldots, \overline{a}_k) \Leftrightarrow$ not for all $b \in |\mathfrak{A}|$ $\mathfrak{A} \models \varphi(\overline{b}, \overline{a}_1, \ldots, \overline{a}_k) \Leftrightarrow$ there is a $b \in |\mathfrak{A}|$ such that
$\mathfrak{A} \models \neg\varphi(\overline{b}, \overline{a}_1, \ldots, \overline{a}_k) \Leftrightarrow \mathfrak{A} \models \exists x\neg\varphi(x, \overline{a}_1, \ldots, \overline{a}_n)$.

$$
\begin{array}{ll}
(i) & \models \forall x \forall y \varphi \leftrightarrow \forall y \forall x \varphi, \\
(ii) & \models \exists x \exists y \varphi \leftrightarrow \exists y \exists x \varphi, \\
(iii) & \models \forall x \varphi \leftrightarrow \varphi \ \ if\ x \notin FV(\varphi), \\
(iv) & \models \exists x \varphi \leftrightarrow \varphi \ \ if\ x \notin FV(\varphi).
\end{array}
$$

$$
\begin{array}{ll}
(i) & \models \forall x(\varphi \wedge \psi) \leftrightarrow \forall x \varphi \wedge \forall x \psi, \\
(ii) & \models \exists x(\varphi \vee \psi) \leftrightarrow \exists x \varphi \vee \exists x \psi, \\
(iii) & \models \forall x(\varphi(x) \vee \psi) \leftrightarrow \forall x \varphi(x) \vee \psi \ \ if\ x \notin FV(\psi), \\
(iv) & \models \exists x(\varphi(x) \wedge \psi) \leftrightarrow \exists x \varphi(x) \wedge \psi \ \ if\ x \notin FV(\psi).
\end{array}
$$

$\forall x(\varphi(x) \vee \psi(x)) \to \forall x\varphi(x) \vee \forall x\psi(x)$, and
$\exists x\varphi(x) \wedge \exists x\psi(x) \to \exists x(\varphi(x) \wedge \psi(x))$ are *not* true.

$$\forall x(\phi(x) \vee \psi(x)) \rightarrow \forall x\phi(x) \vee \forall x\psi(x),$$

similarity type: **<1,1;;0>**

alphabet: <p,q;;>

structure: $\mathfrak{A}$=<{a,b}, P,Q>

P={a}, Q={b}

$$\forall x(p(x) \vee q(x)) \rightarrow \forall x p(x) \vee \forall x q(x),$$

$$\mathfrak{A} \nvDash \forall x(p(x) \vee q(x)) \rightarrow \forall x p(x) \vee \forall x q(x)$$

Let x and y be distinct variables such that $x \notin FV(r)$,

then $(t[s/x])[r/y] = (t[r/y])[s[r/y]/x]$

(classroom exercise)

let x and y be distinct variables such that $x \notin FV(s)$ and let t and s be free for x and y in φ, then $(\varphi[t/x])[s/y] = (\varphi[s/y])[t[s/y]/x]$, Let x and y be distinct variables such that $x \notin FV(r)$,

then $(t[s/x])[r/y] = (t[r/y])[s[r/y]/x]$

(classroom exercise)

let x and y be distinct variables such that $x \notin FV(s)$ and let t and s be free for x and y in φ, then $(\varphi[t/x])[s/y] = (\varphi[s/y])[t[s/y]/x]$,

- By induction on the length of t
- $t = c$, trivial.
- $t = x$. Then $t[s/x] = s$ and $(t[s/x])[r/y] = s[r/y]$; $(t[r/y])[s[r/y]/x] = x[s[r/y]/x] = s[r/y]$.
- $t = y$. Then $(t[s/x])[r/y] = y[r/y]=r$ and $(t[r/y])[s[r/y]/x]=r(s[r/y]/x]=r$, since $x \notin FV(r)$.
- $t = z$, where $z \neq x,y$, trivial.
- $t = f(t_1,\ldots,t_n)$. Then $(t[s/x])[r/y] = (f(t_1[s/x],\ldots)[r/y]$ = (by IH) $=f((t_1[s/x])[r/y], \ldots) = f((t_1[r/y])[s[r/y]/x], \ldots) = f(t_1[r/y],\ldots)[s[r/y]/x]$ $=(t[r/y])[s[r/y]/x]$. [1]

(i) If $z \notin FV(t)$, then $t[\mathbf{a}/x] = (t[z/x])[a/z]$,

(ii)If $z \notin FV(\varphi)$and z free for x in φ, then

$\varphi[\mathbf{a}/z]=(\varphi[z/x])[\mathbf{a}/z]$

Change of Bound Variables

If x, y are free for z in φ and $x,y \notin FV(\varphi)$),

(or simply: if x and y does not occur in φ) then

$\models \exists x(\varphi[x/z]) \leftrightarrow \exists y(\varphi[y/z])$,

$\models \forall x(\varphi[x/z]) \leftrightarrow \forall y(\varphi[y/z])$.

Every formula is equivalent to one in which no variable occurs both free and bound.

Substitution Theorem

(i) $\models t' = t'' \rightarrow s[t'/x] = s[t''/x]$

(ii) $\models t' = t'' \rightarrow \varphi[t'/x] \leftrightarrow \varphi[t''/x])$

[t] is the constant corresponding to ⟦t⟧

$$[\![s[t/x]]\!] = [\![s[\mathbf{[t]}/x]]\!]$$
$$[\![\phi[t/x]]\!] = [\![\phi[\mathbf{[t]}/x]]\!]$$

IDENTITY

1. $\forall x(x = x)$,
2. $\forall xy(x=y \rightarrow y=x)$,
3. $\forall xyz(x=y \wedge y=z \rightarrow x=z)$,
4. $\forall x_1 \ldots x_n y_1 \ldots y_n(\bigwedge_{i=1,n} x_i = y_i \rightarrow t(x_1,\ldots,x_n) = t(y_1,\ldots,y_n))$
5. $\forall x_1 \ldots x_n y_1 \ldots y_n(\bigwedge_{i=1,n} x_i = y_i \rightarrow (\phi(x_1,\ldots,x_n) \rightarrow \phi(y_1,\ldots,y_n)))$

exercise:

$\models \forall x \exists y(x = y)$

Γ a set of formulas

let $X = \{x1, x2, . . .\}$ be the injective (and surjective) enumeration of all the variables)

$\rho=(a_1, a_2, . . .)$a denumerable sequence of elements in $|\mathfrak{A}|$ i.e

$\rho:\mathbb{N}^*\rightarrow|\mathfrak{A}|$ **(we do not require injectivity)**

Γ(ρ) is obtained by replacing simultaneously in all formulas of Γ all the free occurrences of the x_j-s by the corresponding $\mathbf{a}_j$-s (for each $j \geq 1$)

$\Gamma(\rho)=\{\psi(\rho): \psi(\rho)\in\Gamma\}=\{\psi[\mathbf{a_1}, \mathbf{a_2},.../\mathbf{x_1}, \mathbf{x_2},...]: \psi\in\Gamma\}$

(i) $\mathfrak{A}\models\Gamma(\rho)$ if $\mathfrak{A}\models\psi$ for all $\psi\in\Gamma(\rho)$

(ii)$\Gamma \models\sigma$ if for all $\mathfrak{A},\rho$. $\mathfrak{A}\models\Gamma(\rho)\Rightarrow\mathfrak{A}\models\sigma(\rho)$

If $\Gamma=\varnothing$, we write $\models \sigma$

$\rho[i\mapsto a]$ is the sequence obtained by replacing in ρ the i-th element with a

Soundness

$$\Gamma \vdash \sigma \Rightarrow \Gamma \models \sigma$$

$\mathbf{hp}\mathcal{D} \subseteq \Gamma$ $x \notin FV(\mathbf{hp}\mathcal{D})$ and $x \equiv x_k$ in the enumeration

$$\frac{\begin{array}{c}\mathcal{D}\\ \varphi(x)\end{array}}{\forall x \varphi(x)}$$

by Induction hypothesis

$\Gamma \models \varphi$ i.e. for each $\mathfrak{U}$ and for each ρ,. $\mathfrak{U} \models \mathbf{hp}\mathcal{D}(\rho) \Rightarrow \mathfrak{U} \models \phi(\rho)$

$\forall \rho, a\ (\mathfrak{U} \models \mathbf{hp}\mathcal{D}(\rho[k \mapsto a]) \Rightarrow \mathfrak{U} \models \varphi(\rho[k \mapsto a])) \Rightarrow$

$\Rightarrow \forall \rho,\ (\forall a\ \mathfrak{U} \models \mathbf{hp}\mathcal{D}(\rho[k \mapsto a])) \Rightarrow (\forall a\ \mathfrak{U} \models \varphi(\rho[k \mapsto a])) \Rightarrow$

(because $\mathfrak{U} \models \mathrm{hp}\mathcal{D}(\rho[k \mapsto a]) \Leftrightarrow \mathfrak{U} \models \mathrm{hp}\mathcal{D}(\rho)$)

$\forall \rho\ (\mathfrak{U} \models \mathbf{hp}\mathcal{D}(\rho)) \Rightarrow \mathfrak{U} \models \forall x.\varphi(\rho) \Rightarrow$

$\mathfrak{U} \models \mathbf{hp}\mathcal{D}(\rho) \Rightarrow \mathfrak{U} \models \forall x.\varphi(\rho) \Rightarrow$

$\mathfrak{U} \models \Gamma(\rho) \Rightarrow \mathfrak{U} \models \forall x.\varphi(\rho) \Rightarrow$

$\Gamma \models \forall x.\varphi$

$$(\forall E) \quad \frac{\begin{array}{c}\mathcal{D}\\ \forall x \varphi(x)\end{array}}{\varphi(t)}$$

[t] is the constant corresponding to ⟦t⟧

⟦s[t/x]⟧=⟦s[**[t]**/x]⟧
⟦ϕ[t/x]⟧=⟦ϕ[**[t]**/x]⟧

let $x \equiv x_k$ in the enumeration
by IH: Γ ⊨∀x.φ
i.e. for each 𝖀 and for each ρ,. 𝖀 ⊨ Γ(ρ) ⇒ 𝖀 ⊨ ∀x.φ(ρ)

𝖀 ⊨ ∀x.φ(ρ) ⇒ ∀a 𝖀⊨φ(ρ[k↦a])⇔ ∀a 𝖀⊨(φ[**a**/x])(ρ) ⇒

∀t 𝖀⊨(φ[**[t**(ρ)**]**/x])(ρ) ⇔ ∀t 𝖀⊨(φ[t/x])(ρ)

and therefore ∀t(𝖀 ⊨ ∀x.φ(ρ) ⇒ 𝖀⊨(φ[t/x])(ρ))

Adding the Existential Quantifier

$$\frac{\varphi(t)}{\exists x\ \varphi(x)}\ \exists I$$

t free for x in φ

$$\frac{\exists x \varphi(x) \qquad \begin{array}{c}[\varphi] \\ \mathcal{D} \\ \psi\end{array}}{\psi}\ \exists E$$

$x \notin FV(hp\mathcal{D} - \{\varphi\}) \cup FV(\psi)$

$$
\dfrac{\dfrac{\dfrac{[\exists x\varphi(x)]^2 \qquad \dfrac{\dfrac{[\forall x(\varphi(x) \to \psi)]^3}{\varphi(x) \to \psi}\,\forall E \qquad [\varphi(x)]^1}{\psi}\to E}{\psi}\,\exists E_1}{\exists x\varphi(x) \to \psi}\to I_2}{\forall x(\varphi(x) \to \psi) \to (\exists x\varphi(x) \to \psi)}\to I_3
$$

$$\exists x(\varphi(x) \vee \psi(x)) \to \exists x\varphi(x) \vee \exists x\psi(x)$$

$$
\dfrac{[\exists x(\varphi(x) \vee \psi(x))]^3 \qquad \dfrac{[\varphi(x) \vee \psi(x)]^2 \qquad \dfrac{\dfrac{[\varphi(x)]^1}{\exists x \varphi(x)}}{\exists x \varphi(x) \vee \exists x \psi(x)} \qquad \dfrac{\dfrac{[\psi(x)]^1}{\exists x \psi(x)}}{\exists x \varphi(x) \vee \exists x \psi(x)}}{\exists x \varphi(x) \vee \exists x \psi(x)} \vee E_1}{\dfrac{\exists x \varphi(x) \vee \exists x \psi(x)}{\exists x(\varphi(x) \vee \psi(x)) \rightarrow \exists x \varphi(x) \vee \exists x \psi(x)} \rightarrow I_3} \exists E_2
$$

$$\vdash \exists x \varphi(x) \leftrightarrow \neg \forall x \neg \varphi(x).$$

$$\forall I \; \frac{\varphi}{\forall x \varphi} \qquad \forall E \; \frac{\forall x \varphi}{\varphi[t/x]}$$

$$\exists I \; \frac{\varphi[t/x]}{\exists x \varphi} \qquad \exists E \; \frac{\exists x \varphi \qquad \begin{array}{c} [\varphi] \\ \vdots \\ \psi \end{array}}{\psi}$$

$$\cfrac{\cfrac{\forall x(x = x)}{x = x}\;\forall E}{\exists y(x = y))}\;\exists I$$

?

$$\cfrac{\cfrac{\forall x(x = x)}{x = x}\;\forall E}{\exists y(x = y))}\;\exists I$$

yes!

$$\frac{\forall x.\phi}{\phi[t/x]} \equiv \psi[u/y]$$

$$\frac{\forall x(x = x)}{(x{=}x)[x/x]} \forall E$$

$$\frac{\forall x.\phi}{\phi[t/x]} \equiv \psi[u/y]$$

$$\frac{\forall x(x = x)}{(x=x)[x/x]} \equiv (x=y)[x/y]$$

$$\frac{\forall x.\phi}{\psi[u/y]}$$
$$\frac{\psi[u/y]}{\exists y.\psi}$$

$$\frac{\dfrac{\forall x(x = x)}{(x=y)[x/y]}\,\forall E}{\exists y(x = y))}\,\exists I$$

1. $\vdash \exists x(\varphi(x) \wedge \psi) \leftrightarrow \exists x \varphi(x) \wedge \psi$ if $x \notin FV(\psi)$,
2. $\vdash \forall x(\varphi(x) \vee \psi) \leftrightarrow \forall x \varphi(x) \vee \psi$ if $x \notin FV(\psi)$,
3. $\vdash \forall x \varphi(x) \leftrightarrow \neg \exists x \neg \varphi(x)$,
4. $\vdash \neg \forall x \varphi(x) \leftrightarrow \exists x \neg \varphi(x)$,
5. $\vdash \neg \exists x \varphi(x) \leftrightarrow \forall x \neg \varphi(x)$,
6. $\vdash \exists x(\varphi(x) \rightarrow \psi) \leftrightarrow (\forall x \varphi(x) \rightarrow \psi)$ if $x \notin FV(\psi)$,
7. $\vdash \exists x(\varphi \rightarrow \psi(x)) \leftrightarrow (\varphi \rightarrow \exists x \psi(x))$ if $x \notin FV(\varphi)$,
8. $\vdash \exists x \exists y \varphi \leftrightarrow \exists y \exists x \varphi$,
9. $\vdash \exists x \varphi \leftrightarrow \varphi$ if $x \notin FV(\varphi)$.

Natural Deduction and Identity

$$\frac{x = y}{y = x}\ \mathrm{RI}_2$$

$$\frac{x = y \quad y = z}{x = z}\ \mathrm{RI}_3$$

$$\frac{x_1 = y_1, \ldots, x_n = y_n}{t(x_1, \ldots, x_n) = t(y_1, \ldots, y_n)}\ \mathrm{RI}_4$$

$$\frac{x_1 = y_1, \ldots, x_n = y_n \quad \varphi(x_1, \ldots, x_n)}{\varphi(y_1, \ldots, y_n)}\ \mathrm{RI}_4$$

$$\frac{x_1 = y_1, \ldots, x_n = y_n}{t[x_1, \ldots, x_n / z_1, \ldots, z_n] = t[y_1, \ldots, y_n / z_1, \ldots, z_n]}$$

$$\frac{x_1 = y_1, \ldots, x_n = y_n \quad \varphi[x_1, \ldots, x_n / z_1, \ldots, z_n]}{\varphi[y_1, \ldots, y_n / z_1, \ldots, z_n]}$$

$$\frac{x = y \quad x^2 + y^2 > 12x}{2y^2 > 12x} \qquad \frac{x = y \quad x^2 + y^2 > 12x}{x^2 + y^2 > 12y} \qquad \frac{x = y \quad x^2 + y^2 > 12x}{2y^2 > 12y}$$

Lemma 2.10.2 *Let L be of type $\langle r_1, \ldots, r_n; a_1, \ldots, a_m; k\rangle$. If the rules*

$$\frac{x_1 = y_1, \ldots, x_{r_i} = y_{r_i} \quad \boldsymbol{P}_i(x_1, \ldots, x_{r_i})}{\boldsymbol{P}_i\,(y_1, \ldots, y_{r_i})} \text{ for all } i \leq n$$

and

$$\frac{x_1 = y_1, \ldots, x_{a_j} = y_{a_j}}{f_j(x_1, \ldots, x_{a_j}) = f_j(y_1, \ldots, y_{a_j})} \text{ for all } j \leq m$$

are given, then the rules RI_4 are derivable.

Completeness

(Model Existence Lemma)
If Γ is a consistent set of sentences, then Γ has a model.

Let L be a language of cardinality κ. If Γ is a consistent set of sentences in L, then Γ has a model of cardinality $\leq \kappa$

Definition

(i) A theory T is a collection of sentences s.t. for each sentence φ, $T \vdash \varphi \Rightarrow \varphi \in T$ (a theory is closed under derivability).

(ii) Given a theory T, a set Γ (of sentences) such that $T = \{\varphi : \Gamma \vdash \varphi$ and φ is a sentence$\}$ is called an axiom set of the theory T. The elements of Γ are called axioms.

(iii) T is called a Henkin theory if **for each sentence $\exists x\varphi(x)$ there is a constant c** such that $\exists x\varphi(x) \rightarrow \varphi(c) \in T$ (such a c is called a ***witness*** for $\exists x\varphi(x)$).

Definition

Let T and T′ be theories in the languages L and L′.
(i) T′ is an extension of T if $T \subseteq T'$,
(ii) T′ is a conservative extension of T if $T' \cap L = T$ (i.e. all theorems of T′ in the language L are already theorems of T).

Definition

Let T be a theory with language L.

The language L* is obtained from L by adding a constant c_φ for each sentence of the form $\exists x\varphi(x)$.

T* is the theory with axiom set
$T \cup \{\exists x\varphi(x) \to \varphi(c_\varphi) \mid \exists x\varphi(x)$ closed, with witness $c_\varphi\}$

Theorem[const-var]

Let x be a variable not occurring in Γ or φ. (i) $\Gamma \vdash \varphi \Rightarrow \Gamma[x/c] \vdash \varphi[x/c]$.

(ii) If c does not occur in Γ, then $\Gamma \vdash \varphi(c) \Rightarrow \Gamma \vdash \forall x\varphi(x)$.

Lemma T* is conservative over T

(a) Let $\exists x\varphi(x) \rightarrow \varphi(c)$ be one of the new axioms.

Suppose $\Gamma, \exists x\varphi(x) \rightarrow \varphi(c) \vdash \psi$, where ***ψ does not contain c*** and where **Γ is a set of sentences, none of which contains the constant c**. We show $\Gamma \vdash \psi$ in a number of steps.

1. $\Gamma \vdash (\exists x\varphi(x) \rightarrow \varphi(c)) \rightarrow \psi$,
2. $\Gamma \vdash (\exists x\varphi(x) \rightarrow \varphi(y)) \rightarrow \psi$, where y is a variable that does not occur in the associated derivation. *2 follows from 1 by Theorem [const-var].*
3. $\Gamma \vdash \forall y[(\exists x\varphi(x) \rightarrow \varphi(y)) \rightarrow \psi]$. This application of ($\forall$I) is correct, since c did not occur in Γ.
4. $\Gamma \vdash \exists y(\exists x\varphi(x) \rightarrow \varphi(y)) \rightarrow \psi$
5. $\Gamma \vdash (\exists x\varphi(x) \rightarrow \exists y\varphi(y)) \rightarrow \psi$,
6. $\vdash \exists x\varphi(x) \rightarrow \exists y\varphi(y)$.
7. $\Gamma \vdash \psi$, (from 5,6).

(b) Let $T^* \vdash \psi$ for a $\psi \in L$. By the definition of derivability $T \cup \{\sigma_1, \ldots, \sigma_n\} \vdash \psi$, where the σ_i are new axioms of the form $\exists x\varphi(x) \rightarrow \varphi(c)$. We show $T \vdash \psi$ by induction on n.

For n=0 we are done.

Let us suppose that $T \cup \{\sigma_1, \ldots, \sigma_n\} \vdash \psi \Rightarrow T \vdash \psi$ we prove that $T \cup \{\sigma_1, \ldots, \sigma_{n+1}\} \vdash \psi \Rightarrow T \vdash \psi$

Let $T \cup \{\sigma_1, \ldots, \sigma_{n+1}\} \vdash \psi$. Put $\Gamma' = T \cup \{\sigma 1, \ldots, \sigma n\}$, then $T', \sigma n+1 \vdash \psi$ and we may apply (a). Hence $T \cup \{\sigma 1, \ldots, \sigma n\} \vdash \psi$. By ***induction hypothesis*** $T \vdash \psi$.

Lemma

Let $T_0 := T$; $T_{n+1} := (T_n)^*$; $T\omega := \bigcup\{T_n : n \geq 0\}$.

Then $T\omega$ is a Henkin theory and it is conservative over T .

Proof.

Call the language of T_n (resp. $T\omega$) L_n (resp. $L\omega$).

(i) **T_n is conservative over T** . Induction on n.

(ii)**$T\omega$ is a theory**.

Suppose $T\omega \vdash \sigma$, then $\varphi_0,\ldots,\varphi_n \vdash \sigma$ for certain $\varphi_0,\ldots,\varphi_n \in T\omega$.

$\forall i \leq n \ \exists\, m_i \ \ \varphi_i \in T_{m_i}$. $m = \max\{m_i : i \leq n\}$.

Since $\forall k\ T_k \subseteq T_{k+1}$, we have $T_{m_i} \subseteq T_m$ $(i \leq n)$.

Therefore $T_m \vdash \sigma$. T_m is (by definition) a theory, so $\sigma \in T_m \subseteq T\omega$.

(iii) **$T\omega$ is a Henkin theory**.

Let $\exists x\varphi(x) \in L\omega$, then $\exists x\varphi(x) \in L_n$ for some n.

By definition $\exists x\varphi(x) \rightarrow \varphi(c) \in T_{n+1}$ for a certain c.

So $\exists x\varphi(x) \rightarrow \varphi(c) \in T\omega$.

(iv)**$T\omega$ is conservative over T** .

Observe that $T\omega \vdash \sigma$ if $T_n \vdash \sigma$ for some n and apply (i).

corollary

if T is consistent then $T\omega$ is consistent.

proof:

For suppose $T\omega$ inconsistent, then $T\omega \vdash \bot$. As $T\omega$ is conservative over T (and $\bot \in L$) $T \vdash \bot$. Contradiction.

XORN'S LEMMA

If *<P,≤> is PO set, and each* chain C (C ⊆ P and C totally ordered by ≤) has an upper bound in *P*. Then the set *P* contains at least one maximal element.

Lindenbaum Lemma

Each consistent theory is contained in a maximally consistent theory.

Proof. We give a straightforward application of Zorn's Lemma. Let T be consistent.

Consider the PO <**A**,⊆> with **A**={T′: T constant extension of T}

Claim: A has a maximal element.

1. Let $\{Ti\}_{i \in I}$ be a chain. Then $T' = \bigcup_{i \in I} Ti$ is a consistent extension of T containing all Ti's (Exercise!). So T' is an upper bound.
2. by means of (1) and Zorn's lemma **A** has a maximal element Tm.
3. Tm is a maximally consistent extension of T (i.e. if Tm ⊆ T′ and T′∈A, then Tm =T′)

Lemma An extension of a Henkin theory with the same language is again a Henkin theory.

Proof

For, the language remains fixed, so if for an existential statement $\exists x\varphi(x)$ there is a witness c such that $\exists x\varphi(x) \to \varphi(c) \in T$, then trivially, $\exists x\varphi(x) \to \varphi(c) \in Tm$

Model Existence Lemma

If Γ is consistent, then Γ has a model.

Model Existence Lemma

If Γ is consistent set of sentences, then Γ has a model.

Proof of Model Existence Lemma

Let $T = \{\sigma : \Gamma \vdash \sigma\}$ be the theory given by Γ.

Any model of T is, of course, a model of Γ .

Let **Tm** be a maximally consistent Henkin extension of T (which exists by the preceding lemmas), with language Lm.

We will construct a model of **Tm** using **Tm** itself.

1. $A = \{t \in Lm : t \text{ is closed}\}$.
2. For each k-ary function symbol f we define a function $\mathbf{f}^*: A^k \to A$ by

$$\mathbf{f}^*(t1,\ldots,tk) \equiv f(t1,\ldots,tk).$$

3. For each p-ary predicate symbol P we define a relation $\mathbf{P}^* \subseteq A^p$ by

$$\langle t_1,\ldots,t_p\rangle \in \mathbf{P}^* \Leftrightarrow T_m \vdash P(t_1,\ldots,t_p) .$$

4. For each constant symbol c we define a constant

$$\mathbf{c}^* \equiv c.$$

I_1 $\forall x(x = x)$,
I_2 $\forall xy(x = y \to y = x)$,
I_3 $\forall xyz(x = y \wedge y = z \to x = z)$,
I_4 $\forall x_1 \dots x_n y_1 \dots y_n (\bigwedge_{i \le n} x_i = y_i \to t(x_1, \dots, x_n) = t(y_1, \dots, y_n))$,
$\forall x_1 \dots x_n y_1 \dots y_n (\bigwedge_{i \le n} x_i = y_i \to (\varphi(x_1, \dots, x_n) \to \varphi(y_1, \dots, y_n)))$.

$\vdash I_1, \vdash I_2, \vdash I_3, \vdash I_4,$

REMARK '=' is not interpreted as the real equality.

We can only assert that:

(a)The relation **~** ⊆TERMxTERM defined by

$$t \sim s \Leftrightarrow \mathbf{Tm} \vdash t=s \text{ for } t,s \in A$$

is an ***equivalence relation***.

Tm ⊢ ∀x(x = x), and hence Tm ⊢ t=t, namely t**~**t.

Symmetry and transitivity follow in the same way (use I_2 and I_3)

(b) $t_i \sim s_i$ ($i \le p$) and $\langle t_1,\dots,t_p\rangle \in \mathbf{P}^*$ ⇒ $\langle s_1,\dots,s_p\rangle \in \mathbf{P}^*$

$t_i \sim s_i$($i \le k$) ⇒ $\mathbf{f}^*(t1,\dots,tk) \sim \mathbf{f}^*(s1,\dots,sk)$ for all symbols P and f. (use I_4)

[t] is the equivalence class of t under $\sim$

Define

$\mathfrak{U}=\langle A/\sim, \mathbf{P_1}^{\sim},\ldots,\mathbf{P_n}^{\sim},\mathbf{f_1}^{\sim},\ldots,\mathbf{f_m}^{\sim}, \{\mathbf{c_i}^{\sim}: i\in I\}\rangle$,

where

$\mathbf{P_i}^{\sim} := \{\langle [t_1],\ldots,[t_{r_i}]\rangle \mid \langle t_1,\ldots,t_{r_i}\rangle \in \mathbf{P_1}^*\}$

$\mathbf{f_j}^{\sim}([t_1],\ldots,[t_{r_j}]) = [\mathbf{f_1}^*(t_1,\ldots,t_{a_j})]$

$\mathbf{c_i}^{\sim} := [\mathbf{c_i}^*]$

$\mathbf{f}^*(t1,\ldots,tk) \equiv f(t1,\ldots,tk)$.

$\langle t_1,\ldots,t_p\rangle \in \mathbf{P}^* \Leftrightarrow T_m \vdash P(t_1,\ldots,t_p)$.

$\mathbf{c}^* \equiv c$.

$t^{\mathfrak{U}} \equiv [\![t]\!]_{\mathfrak{U}}$

claim: $\mathbf{t}^{\mathfrak{U}}=[t]$

base: $t=c$, then $t^{\mathfrak{U}}=\mathbf{c}^{\sim}=[\mathbf{c}^*]=[c]=[t]$,

$t=f(t_1,\ldots,t_k)$, then

$$\mathbf{t}^{\mathfrak{U}}=\mathbf{f}^{\sim}(\mathbf{t_1}^{\mathfrak{U}},\ldots,\mathbf{t_k}^{\mathfrak{U}}) =^{IH} \mathbf{f}^{\sim}([t_1],\ldots,[t_k])= [\mathbf{f_1}^*(t_1,\ldots,t_k)]=[f(t1,\ldots,tk)]$$

claim: $\mathfrak{U} \models \varphi(t) \Leftrightarrow \mathfrak{U} \models \varphi([\mathbf{t}])$ (exercise)

Claim. $\mathfrak{A} \models \varphi(t) \Leftrightarrow Tm \vdash \varphi(t)$ for all sentences in the language Lm of Tm

(which, by the way, is also L($\mathfrak{A}$), since each element of $\mathbf{A}/_{\sim}$ has a name in Lm)

by induction on φ:

(i) φ is atomic. $\mathfrak{A} \models P(t_1^{\mathfrak{A}}, \ldots, t_p^{\mathfrak{A}}) \leftrightarrow \langle t_1^{\mathfrak{A}}, \ldots, t_p^{\mathfrak{A}} \rangle \in \mathbf{P}^{\sim} \Leftrightarrow \langle [t1], \ldots, [tp] \rangle \in \mathbf{P}^{\sim} \Leftrightarrow$

$\langle t1, \ldots, tp \rangle \in \mathbf{P}^{*} \Leftrightarrow Tm \vdash P(t1, \ldots, tp)$.

Let Γ be maximally consistent;

a) $\forall \phi$ either $\phi \in \Gamma$, or $\neg \phi \in \Gamma$,

b) $\forall \phi, \psi.\ \phi \rightarrow \psi \in \Gamma \Leftrightarrow (\phi \in \Gamma \Rightarrow \psi \in \Gamma)$.

The case $\varphi = \perp$ is trivial.

(ii) $\varphi = \sigma \rightarrow \tau$.

$Tm \vdash \sigma \rightarrow \tau \Leftrightarrow (Tm \vdash \sigma \Rightarrow Tm \vdash \tau)$ (by maximally consistence of Tm).

$\mathfrak{A} \models \phi \rightarrow \tau \Leftrightarrow (\mathfrak{A} \models \sigma \Rightarrow \mathfrak{A} \models \tau) \Leftrightarrow$ (by IH) $(Tm \vdash \sigma \Rightarrow Tm \vdash \tau) \Leftrightarrow Tm \vdash \sigma \rightarrow \tau$.

(iii) $\varphi = \forall x\psi(x)$. $\mathfrak{U} \models \forall x\psi(x) \Leftrightarrow \mathfrak{U} \not\models \exists x\neg\psi(x) \Leftrightarrow$

$\mathfrak{U} \not\models \neg\psi(\mathbf{a})$ for all $a \in |\mathfrak{U}| \Leftrightarrow$ for all $a \in |\mathfrak{U}|$. $\mathfrak{U}\models\psi(\mathbf{a})$

Assuming $\mathfrak{U}\models \forall x\psi(x)$, we get in particular $\mathfrak{U}\models \psi(c)$ for the witness c belonging to $\exists x\neg\psi(x)$.

By induction hypothesis: $Tm \vdash \psi(c)$. $Tm \vdash \exists x\neg\psi(x) \rightarrow \neg\psi(c)$, so

$Tm \vdash \psi(c) \rightarrow \neg\exists x\neg\psi(x)$.

Hence $Tm \vdash \forall x\varphi(x)$.

Conversely: $Tm \vdash \forall x\psi(x) \Rightarrow Tm \vdash \psi(t)$, so $Tm \vdash \psi(t)$ for all closed t, and therefore by induction hypothesis, $\mathfrak{U} \models \psi(t)$ for all closed t. Hence $\mathfrak{U} \models \forall x\psi(x)$.

Now we see that $\mathfrak{U}$ is a model of Γ,as $\Gamma \subseteq Tm$.

Model Existence Lemma

If Γ is consistent set of sentences, then Γ has a model.

Corollary

$\Gamma \nvdash \phi \Rightarrow \Gamma \nvDash \phi$.

proof

$\Gamma \nvdash \phi \Rightarrow \Gamma \cup \{\neg\phi\} \nvdash \bot \Rightarrow \exists \mathfrak{U}$ such that $\mathfrak{U} \vDash \Gamma \cup \{\neg\phi\} \Rightarrow \Gamma \nvDash \phi$

Theorem Let x be a variable not occurring in Γ or φ. (i) $\Gamma \vdash \phi \Rightarrow \Gamma\,[x/c] \vdash \phi[x/c]$.

$\Gamma \vDash \sigma$ if for all $\mathfrak{U}, \rho$. $\mathfrak{U} \vDash \Gamma(\rho) \Rightarrow \mathfrak{U} \vDash \sigma(\rho) \Rightarrow$

for all ρ. $\Gamma(\rho) \vDash \sigma(\rho) \Rightarrow$ for all ρ $\Gamma(\rho) \vdash \sigma(\rho) \Rightarrow \Gamma \vdash \sigma$

$$\Gamma \vdash \phi \Leftrightarrow \Gamma \vDash \phi$$

From the Model Existence Lemma we get the following:

Theorem (Compactness Theorem)

Γ has a model $\Leftrightarrow$ each finite subset Δ of Γ has a model.

An equivalent formulation is:

Γ has no model $\Leftrightarrow$ some finite $\Delta \subseteq \Gamma$ has no model.

Proof. We consider the second version.

$\Leftarrow$ Trivial.

$\Rightarrow$ Suppose Γ has no model, then by the Model Existence Lemma Γ is inconsistent, i.e. $\Gamma \vdash \perp$. Therefore there are $\sigma_1,\ldots,\sigma_n \in \Gamma$ such that $\sigma_1,\ldots,\sigma_n \vdash \perp$. This shows that $\Delta = \{\sigma_1,\ldots,\sigma_n\}$ has no model.

Lemma If Γ has arbitrarily large finite models, then Γ has an infinite model.

Proof. Put $\Gamma^* = \Gamma \cup \{\lambda_n | n > 1\}$, where λ_n expresses the sentence "there are at least n distinct elements" (**exercise**) Apply the Compactness Theorem. Let $\Delta \subseteq \Gamma^*$, be finite, and let λ_m be the sentence λ_n in Δ with the largest index n. Verify that $Mod(\Delta) \supseteq Mod(\Gamma \cup \{\lambda_m\})$.

Now Γ has arbitrarily large finite models, so Γ has a model $\mathfrak{A}$ with at least m elements, i.e. $\mathfrak{A} \in Mod(\Gamma \cup \{\lambda_m\})$. So $Mod(\Delta) \neq \varnothing$.

By compactness $Mod(\Gamma^*) \neq \varnothing$, but in virtue of the axioms λ_m, a model of Γ' is infinite. Hence Γ^*, and therefore Γ, has an infinite model.

$Mod(\Gamma) = \{\mathfrak{U} \mid \mathfrak{U} \models \sigma \text{ for all } \sigma \in \Gamma\}$.

For convenience we will often write $\mathfrak{U} \models \sigma$ for $\mathfrak{U} \in Mod(\Gamma)$.

We write $Mod(\phi_1,\ldots,\phi_2)$ instead of $Mod(\{\phi_1,\ldots,\phi_n\})$.

In general $Mod(\Gamma)$ is not a set (in the technical sense of set theory: $Mod(\Gamma)$ is most of the time a proper class).

Conversely, let $\mathscr{K}$ be a class of structures (we have fixed the similarity type), then $Th(\mathscr{K}) = \{\sigma \mid \mathfrak{U} \models \sigma \text{ for all } \mathfrak{U} \in \mathscr{K}\}$.

$Mod(\forall xy(x \leq y \wedge \leq y \leq x \leftrightarrow x = y), \forall xyz(x \leq y \wedge y \leq z \rightarrow x \leq z))$

is the class of posets.

Lemma

Γ is consistent ↔ Γ has a model of cardinality at most the cardinality of the language.

- If L has finitely many constants, then L is countable. –
- If L has $\kappa \geq \aleph_0$ constants, then $|L| = \kappa$.

Theorem (Downward Skolem-Löwenheim Theorem) Let Γ be a set of sentences in a language of cardinality κ, and let $\kappa < \lambda$. If Γ has a model of cardinality λ, then Γ has a model of cardinality κ', with $\kappa \leq \kappa' < \lambda$.

Examples.

1. The theory of real numbers, Th(R), in the language of fields, has a countable model.

Upward Skolem-Löwenheim Theorem

Let Γ have a language L of cardinality κ, and $\mathfrak{U} \in \mathrm{Mod}(\Gamma)$ with cardinality $\lambda \geq \kappa$. For each $\mu > \lambda$ Γ has a model of cardinality μ.

THEORIES

Let **Φ** a Recursively Enumerable (RE) set of formulas (not necessarily sentences)
the ***natural deduction system with axioms*** **Φ** is obtained by adding to the standard natural deduction system, for each $\sigma \in$ **Φ,** a 0-ary rule (namely a rule without premises)

$$\frac{}{\sigma}$$

such a new rules are called **axioms**
Derivability in the natural deduction system with axioms **Φ** is denoted with

$$\vdash_{\boldsymbol{\Phi}}$$

Let **Φ** a Recursively Enumerable (RE) set of formulas (not necessarily sentences)

$$Cl(\boldsymbol{\Phi}), \Gamma \vdash \alpha \Leftrightarrow \Gamma \vdash_{\boldsymbol{\Phi}} \alpha$$

In the following we will use both the concepts

$$Cl(\mathbf{\Phi}),\ \Gamma \vdash \alpha \ \text{ and } \ \Gamma \vdash_{\mathbf{\Phi}} \alpha$$

A theory T is called **axiomatizable** if:

there is a Recursively Enumerable (RE) set of formulas, called postulates or axioms, $\mathbf{\Phi}$ s.t.
$T=\{\alpha \mid \ \vdash_{\mathbf{\Phi}} \alpha$ and α is a sentence $\}$

or equivalently

there is a Recursively Enumerable (RE) set of sentences $\mathbf{\Sigma}$, called postulates or axioms, s.t.
$T=\{\alpha \mid \mathbf{\Sigma} \vdash \alpha$ and α is a sentence $\}$

If the set of postulates for a theory T is actually given, we say that T is **axiomatic**