— Lectures on

DIFFERENTIAL GEOMETRY AND TOPOLOGY

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Lecture V

* k-forms (continued)

If $\omega_1, \omega_2 \ldots \omega_k \in \Lambda^1(V^*)$, one finds

again $\dim V = n$ …

$$(\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_k)(v_1, v_2 \ldots, v_k) =$$

$\leftarrow v \text{ fixed} \rightarrow$

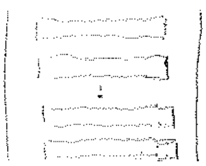
determinant

$$\begin{vmatrix} \omega_1(v_1) & \cdots & \omega_k(v_1) \\ \vdots & & \\ \omega_1(v_k) & \cdots & \omega_k(v_k) \end{vmatrix} \quad \updownarrow \ \omega \text{ fixed}$$

which represents the volume of a suitable hyperparallelopode in $\mathbb{R}^k$

(set $\mathbb{R}^n \ni \xi \longmapsto (\omega_1(\xi) \ldots \omega_k(\xi)) \in \mathbb{R}^k$

(row notation))



* Further examples

* In $\mathbb{R}^3$ let

$$\omega^1_a = \sum_{i=1}^{3} a_i e_i^* \qquad \underline{a} = (a_1, a_2, a_3)$$

$$\omega^1_b = \sum_{i=1}^{3} b_i e_i^* \qquad \underline{b} = (b_1, b_2, b_3)$$

Compute $\omega^1_a \wedge \omega^1_b$. One gets

$$\omega^1_a \wedge \omega^1_b = \sum_{i,j} a_i b_j \, e_i^* \wedge e_j^* = \dots =$$

$$= \begin{aligned} &{}^{z}\ (a_1 b_2 - a_2 b_1)\, e_1^* \wedge e_2^* \ + \\ &{}^{x}\ (a_2 b_3 - a_3 b_2)\, e_2^* \wedge e_3^* \ + \\ &{}^{y}\ (a_3 b_1 - a_1 b_3)\, e_3^* \wedge e_1^* \end{aligned}$$

components of $\underline{a} \times \underline{b}$

flux 2-form (different notation)

Now set $\omega^2_c(v_1, v_2) := \langle c, v_1 \times v_2 \rangle = \det(c, v_1, v_2)$

One easily checks that

$$\omega^2_c = c_1 e_2^* \wedge e_3^* + c_2 e_3^* \wedge e_1^* + c_3 e_1^* \wedge e_2^*$$

Indeed r.h.s. $(e_1, e_2) = \dots \ c_3$

and $\omega^2_c(e_1, e_2) = \det(c, e_1, e_2) = \begin{vmatrix} c_1 & 1 & 0 \\ c_2 & 0 & 1 \\ c_3 & 0 & 0 \end{vmatrix} \underset{\text{Laplace}}{=} c_3$

etc.

Therefore $\omega^1_a \wedge \omega^1_b = \omega^2_c$, $\underline{c} = \underline{a} \times \underline{b}$

$\Rightarrow$ $\wedge$, with a grain of salt (a metric is involved!) can be viewed as a generalization of $\times$

* A linear-algebraic intermezzo

* **Direct sums** Given vector spaces U and W over the same field K, their direct sum $U \oplus W$ is defined as follows:

$$U \oplus W = \{ (u, w) \,/\, u \in U,\ w \in W \}$$

with operations

$$(u_1, w_1) \underset{\text{to be defined}}{+} (u_2, w_2) := (\underset{\text{in } U}{u_1 + u_2},\ \underset{\text{in } W}{w_1 + w_2})$$

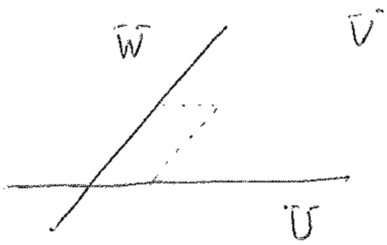
$$\alpha \underset{\text{to be defined}}{\cdot} (u, w) := (\underset{\text{in } U}{\alpha u},\ \underset{\text{in } W}{\alpha \cdot w})$$

notice that
$$U \cong \{(u, 0)\}_{u \in U} \leq U \oplus W$$
$$W \cong \{(0, w)\}_{w \in W} \leq U \oplus W$$

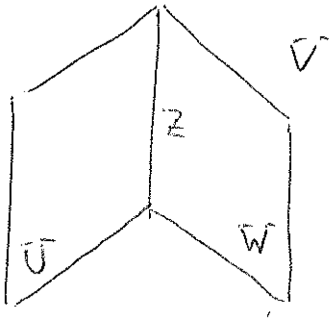
(i.e. U and W are naturally realized as vector subspaces of $U \oplus W$)

Now, given V and $U \leq V$, $W \leq V$, V is the sum of U and W, notation $V = U + W$, if any $v \in V$ can be written as $v = u + w$ for some $u \in U$, $w \in W$

If u and v are unique (and this is equivalent to $U \cap W = \{0\}$), then U and W are said to be in direct sum, and one writes $V = U \oplus W$



here $\bar V = U \oplus \bar W$



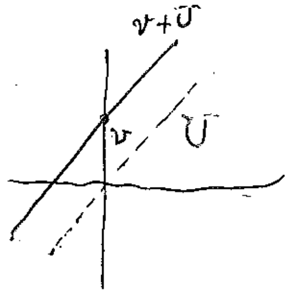
$Z = U \cap \bar W$ $(\le \bar V)$

one has $\bar V = U + \bar W$
but the sum is not direct

★ Quotient spaces (fundamental for the sequel: homology & cohomology theories)

Let $\bar U \le \bar V$. Define the quotient vector space

$$\bar V / \bar U = \{ [v] := v + \bar U \}$$



quotient of abelian groups. ($\bar U$ is normal in $\bar V$)

linear varieties, i.e. affine subspaces of $\bar V$ of "direction", or "position $\bar U$

structured as follows: $[v_1] + [v_2] := [v_1 + v_2]$

abelian group structure with respect to $+$

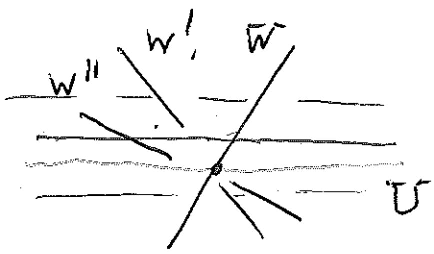
to be defined $\quad \alpha \cdot [v] = [\alpha \cdot v]$

in $\bar V$

one checks that $+$ and $\cdot$ are well-defined, i.e. independent of the choice of a vector in its class. One also has

$$\bar V / \bar U \cong \bar W, \quad \bar W \text{ any direct complement of } \bar U \text{ in } \bar V$$

(i.e. $\bar W$ such that $U \oplus W = V$)



Consider, in particular, the k-forms

$$e^*_{i_1} \wedge e^*_{i_2} \wedge \dots \wedge e^*_{i_k} \qquad i_1 < i_2 < \dots < i_k$$

They yield a basis of $\Lambda^k(V^*)$

recall $e^*_i(e_j) = \delta_{ij}$

One easily checks that

$$(e^*_{i_1} \wedge \dots e^*_{i_k})(e_{j_1}, e_{j_2} \dots e_{j_k}) =$$

$$= \begin{cases} \pm 1 & \text{if } \{i_1 \dots i_k\} = \{j_1 \dots j_k\} \\ & \text{(sign according to parity of the corresponding permutation} \\ 0 & \text{otherwise} \end{cases}$$

and this implies

$$e^*_I = e^*_{i_1} \wedge \dots e^*_{i_k}$$

defined previously

* <u>Exterior</u> (or Grassmann) <u>algebra</u>

$\dim V = n$

Let

$$\underset{\underset{\Lambda}{|||}}{\Lambda(V^*)} = \underbrace{\bigoplus_{k=0}^{n}}_{\text{direct sum}} \underbrace{\Lambda^k(V^*)}_{k\text{-forms}} \quad \Lambda^k(V^*) \equiv \Lambda^k$$

(it is a vector space)

Extend $\wedge$ by distributivity, one has a quadruple $(\Lambda, \underbrace{+, \cdot}_{\text{vector space operations}}, \underbrace{\wedge}_{\text{exterior product}})$ [Λ is structured via $+, \cdot, \wedge$]

called exterior, or Grassmann algebra (over V^*); its elements are called <u>forms</u> on V, elements in Λ^k are k-forms (forms of degree k)

One easily finds that

$$\dim \Lambda(V^*) = 2^n$$

($\sum_{k=0}^{n} \binom{n}{k} = 2^n$, this immediately following from $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$ upon setting $a = b = 1$).