# FIRST-ORDER LOGIC (CONTINUED)

## The first-order predicate calculus

By a *generalization* of $\boldsymbol{\alpha}$ we mean any formula of the form $\forall \mathbf{x}_1 \ldots \forall \mathbf{x}_k \boldsymbol{\alpha}$, where $k \geqslant 1$ and $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are any variables, not necessarily distinct.

what is an axiomatization?

# the propositional case

only 0-ary predicate letters (infinite), called propositional symbols

modus ponens

$$\frac{\alpha \quad \alpha \to \beta}{\beta}$$

As *propositional axioms* of $\mathscr{L}$ we take all $\mathscr{L}$-formulas of the following forms:

(Ax. I) $\alpha \to \beta \to \alpha$,

(Ax. II) $(\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma$,

(Ax. III) $(\neg\alpha \to \beta) \to (\neg\alpha \to \neg\beta) \to \alpha$,

Notice that we have got here not three *single* axioms but three *axiom schemes,* each representing infinitely many axioms obtained by all possible choices of formulas $\alpha$, $\beta$, $\gamma$.

Let $\mathbf{\Phi}$ be a set of $\mathscr{L}$-formulas. By a *propositional deduction from* $\mathbf{\Phi}$ *in* $\mathscr{L}$ we mean a finite non-empty sequence of $\mathscr{L}$-formulas $\varphi_1,\ldots,\varphi_n$ such that, for each $k$ $(1 \leqq k \leqq n)$, $\varphi_k$ is a propositional axiom of $\mathscr{L}$, or $\varphi_k \in \mathbf{\Phi}$ or $\varphi_k$ is obtained by *modus ponens* from earlier formulas in the same sequence (i.e., there are $i,j<k$ such that $\varphi_j = \varphi_i \rightarrow \varphi_k$).

In this connection $\mathbf{\Phi}$ is called a set of *hypotheses.*

A propositional deduction in $\mathscr{L}$ from the empty set of hypotheses is called a *propositional proof in* $\mathscr{L}$.

We write "$\mathbf{\Phi} \vdash_0 \alpha$" to assert that $\alpha$ is *deducible* from $\mathbf{\Phi}$ (i.e., that there is a deduction of $\alpha$ from $\mathbf{\Phi}$). If $\mathbf{\Phi}$ is empty, so that $\alpha$ is *provable* (i.e., there

is a proof of $\alpha$), we write simply "$\vdash_0 \alpha$". Also, we write, e.g., "$\varphi \vdash_0 \alpha$" instead of "$\{\varphi\} \vdash_0 \alpha$".

10.3 LEMMA. *For any* $\alpha$, $\vdash_0 \alpha \to \alpha$.

PROOF. Here is a propositional proof of $\alpha \to \alpha$:[1]

| | |
|---|---|
| $(\alpha \to (\alpha \to \alpha) \to \alpha) \to (\alpha \to \alpha \to \alpha) \to \alpha \to \alpha,$ | (Ax. II) |
| $\alpha \to (\alpha \to \alpha) \to \alpha,$ | (Ax. I) |
| $(\alpha \to \alpha \to \alpha) \to \alpha \to \alpha,$ | (m.p.) |
| $\alpha \to \alpha \to \alpha,$ | (Ax. I) |
| $\alpha \to \alpha.$ | (m.p.) |

In the sequel we shall make implicit use of the following simple facts about deductions:

If $\Phi \subseteq \Psi$, then any deduction from $\Phi$ is also a deduction from $\Psi$.

If $\varphi_1, \ldots, \varphi_n$ is a deduction from $\Phi$ and $1 \leqslant k \leqslant n$, then $\varphi_1, \ldots, \varphi_k$ is also a deduction from $\Phi$.

If $\varphi_1, \ldots, \varphi_m$ is a deduction from $\Phi$ and $\psi_1, \ldots, \psi_n$ is a deduction from $\Psi$, then the concatenation of the two sequences (i.e., $\varphi_1, \ldots, \varphi_m, \psi_1, \ldots, \psi_n$) is a deduction from $\Phi \cup \Psi$.

**10.5. THEOREM.** *For any* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$,

(a) $\neg\neg\alpha \vdash_0 \alpha$,

(b) $\alpha \vdash_0 \neg\neg\alpha$,

(c) $\beta, \neg\beta \vdash_0 \alpha$.

$\vdots$

| | |
|---|---|
| $\neg\alpha \to \neg\alpha$, | |
| $(\neg\alpha \to \neg\alpha) \to (\neg\alpha \to \neg\neg\alpha) \to \alpha$, | (Ax. III) |
| $(\neg\alpha \to \neg\neg\alpha) \to \alpha$, | (m.p.) |
| $\neg\neg\alpha$, | (hyp.) |
| $\neg\neg\alpha \to \neg\alpha \to \neg\neg\alpha$, | (Ax. I) |
| $\neg\alpha \to \neg\neg\alpha$, | (m.p.) |
| $\alpha$. | (m.p.) |

$\vdots$

$\neg\neg\neg\alpha \to \neg\alpha,$

$(\neg\neg\neg\alpha \to \alpha) \to (\neg\neg\neg\alpha \to \neg\alpha) \to \neg\neg\alpha,$ (Ax. III)

$\alpha,$ (hyp.)

$\alpha \to \neg\neg\neg\alpha \to \alpha,$ (Ax. I)

$\neg\neg\neg\alpha \to \alpha,$ (m.p.)

$(\neg\neg\neg\alpha \to \neg\alpha) \to \neg\neg\alpha,$ (m.p.)

$\neg\neg\alpha.$ (m.p.)

| | |
|---|---|
| $(\neg\alpha \to \beta) \to (\neg\alpha \to \neg\beta) \to \alpha$, | (Ax. III) |
| $\beta$, | (hyp.) |
| $\beta \to \neg\alpha \to \beta$, | (Ax. I) |
| $\neg\alpha \to \beta$, | (m.p.) |
| $(\neg\alpha \to \neg\beta) \to \alpha$, | (m.p.) |
| $\neg\beta$, | (hyp.) |
| $\neg\beta \to \neg\alpha \to \neg\beta$, | (Ax. I) |
| $\neg\alpha \to \neg\beta$, | (m.p.) |
| $\alpha$. | (m.p.) |

# The first-order predicate calculus

As *first-order axioms* of $\mathscr{L}$ we take all $\mathscr{L}$-formulas of the following eight groups:

(Ax.1) All propositional axioms of $\mathscr{L}$.

(Ax.2) $\forall \mathbf{x}(\boldsymbol{\alpha} \to \boldsymbol{\beta}) \to \forall \mathbf{x}\boldsymbol{\alpha} \to \forall \mathbf{x}\boldsymbol{\beta}$,
where $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ are any $\mathscr{L}$-formulas and $\mathbf{x}$ is any variable.

(Ax.3) $\boldsymbol{\alpha} \to \forall \mathbf{x}\boldsymbol{\alpha}$,
where $\boldsymbol{\alpha}$ is any $\mathscr{L}$-formula and the variable $\mathbf{x}$ is not free in $\boldsymbol{\alpha}$.

(Ax.4) $\forall \mathbf{x}\boldsymbol{\alpha} \to \boldsymbol{\alpha}(\mathbf{x}/\mathbf{t})$,
where $\boldsymbol{\alpha}$ is any $\mathscr{L}$-formula and $\mathbf{t}$ is any $\mathscr{L}$-term free for $\mathbf{x}$ in $\boldsymbol{\alpha}$.

(Ax.5) $\mathbf{t}=\mathbf{t}$,
where $\mathbf{t}$ is any $\mathscr{L}$-term.

(Ax.6) $\mathbf{t}_1=\mathbf{t}_{n+1}\to\ldots\to\mathbf{t}_n=\mathbf{t}_{2n}\to\mathbf{f}\mathbf{t}_1\ldots\mathbf{t}_n=\mathbf{f}\mathbf{t}_{n+1}\ldots\mathbf{t}_{2n}$,
where $\mathbf{f}$ is any $n$-ary function symbol of $\mathscr{L}$ and $\mathbf{t}_1,\ldots,\mathbf{t}_{2n}$ are any $\mathscr{L}$-terms.

(Ax.7) $\mathbf{t}_1=\mathbf{t}_{n+1}\to\ldots\to\mathbf{t}_n=\mathbf{t}_{2n}\to\mathbf{P}\mathbf{t}_1\ldots\mathbf{t}_n\to\mathbf{P}\mathbf{t}_{n+1}\ldots\mathbf{t}_{2n}$,
where $\mathbf{P}$ is any $n$-ary predicate symbol of $\mathscr{L}$ and $\mathbf{t}_1,\ldots,\mathbf{t}_{2n}$ are any $\mathscr{L}$-terms.

(Ax.8) All generalizations of axioms of the preceding groups.

As rule of inference we again take *modus ponens*.

If $\mathscr{L}$ is without equality then (Ax.5), (Ax.6) and (Ax.7) are omitted.

1.2. THEOREM. *If* $\Phi \vdash \alpha$ *then* $\Phi \models \alpha$. *In particular, if* $\vdash \alpha$ *then* $\models \alpha$.

PROOF. Similar to Thm. 1.10.2. We first verify that our first-order axioms are logically true. For (Ax.1) this follows from the fact that the propositional axioms are tautologies. For (Ax.2)–(Ax.7) we can use tableaux (see Thm. 2.4.2 and Prob. 2.4.3) or verify directly that they are logically true. For (Ax.8) we merely need to observe that if $\alpha$ is logically true then so is $\forall x \alpha$, and hence so is any generalization of $\alpha$.

The rest is exactly as in the proof of Thm. 1.10.2. ∎

PROOF. Let $\varphi_1, \ldots, \varphi_n$ be a deduction of $\alpha$ from $\Phi$. Thus $\varphi_n = \alpha$. By induction on $k = 1, \ldots, n$ we show that $\Phi \models_0 \varphi_k$. (Thus, for $k = n$, we have $\Phi \models_0 \alpha$.)

If $\varphi_k$ is a propositional axiom, we easily verify that $\varphi_k$ is a tautology (cf. Prob. 6.10). Thus $\varphi_k$ is satisfied by *every* truth valuation and is therefore a tautological consequence of *any* set of formulas.

If $\varphi_k \in \Phi$ then clearly $\Phi \models_0 \varphi_k$.

Finally, if for some $i, j < k$ we have $\varphi_j = \varphi_i \rightarrow \varphi_k$, then $\{\varphi_i, \varphi_j\} \models_0 \varphi_k$ by Lemma 10.1. But by the induction hypothesis $\Phi \models_0 \varphi_i$ and $\Phi \models_0 \varphi_j$. Hence clearly $\Phi \models_0 \varphi_k$. ∎

10.1. LEMMA. *For any formulas* $\alpha$ *and* $\beta$, $\{\alpha, \alpha \rightarrow \beta\} \models_0 \beta$.

1.3. DEDUCTION THEOREM. *Given a deduction of* $\beta$ *from* $\Phi, \alpha$, *we can construct a deduction of* $\alpha \to \beta$ *from* $\Phi$. *(Hence, if* $\Phi, \alpha \vdash \beta$ *then* $\Phi \vdash \alpha \to \beta$.*)*
PROOF. Exactly the same as for the propositional calculus (Thm. 1.10.4). ▮

PROOF. Let $\varphi_1, \ldots, \varphi_n (= \beta)$ be the given deduction of $\beta$ from $\Phi, \alpha$. We show by induction on $k = 1, \ldots, n$ that a deduction of $\alpha \to \varphi_k$ from $\Phi$ can be constructed. The following cases are possible: $\varphi_k$ is an axiom, or $\varphi_k \in \Phi$, or $\varphi_k = \alpha$, or $\varphi_k$ is obtained by *modus ponens* from two earlier formulas $\varphi_i$ and $\varphi_j$.

If $\varphi_k$ is an axiom then the following is a proof of $\alpha \to \varphi_k$ and *a fortiori* a deduction of $\alpha \to \varphi_k$ from $\Phi$:

| | |
|---|---|
| $\varphi_k$, | (Ax.) |
| $\varphi_k \to \alpha \to \varphi_k$, | (Ax. I) |
| $\alpha \to \varphi_k$. | (m.p.) |

If $\varphi_k \in \Phi$, the same sequence of three formulas is a deduction of $\alpha \to \varphi_k$ from $\Phi$ (except that in this case the justification in the first line should read *hyp.* — short for *hypothesis*).

If $\varphi_k = \alpha$, then in the proof of Lemma 10.3 we had a propositional proof[1] of $\alpha \to \alpha$ $(= \alpha \to \varphi_k)$. This is *a fortiori* a deduction of $\alpha \to \varphi_k$ from $\Phi$.

Finally, suppose that for some $i,j<k$ we have $\varphi_j = \varphi_i \to \varphi_k$. Then by the induction hypothesis we have got deductions of $\alpha \to \varphi_i$ and $\alpha \to \varphi_i \to \varphi_k$ from $\Phi$. We concatenate these two deductions and adjoin three new formulas:

$$\begin{array}{ll}
\vdots & \\
\alpha \to \varphi_i, & \\
\vdots & \\
\alpha \to \varphi_i \to \varphi_k, & \\
(\alpha \to \varphi_i \to \varphi_k) \to (\alpha \to \varphi_i) \to \alpha \to \varphi_k, & \text{(Ax. II)} \\
(\alpha \to \varphi_i) \to \alpha \to \varphi_k, & \text{(m.p.)} \\
\alpha \to \varphi_k. & \text{(m.p.)}
\end{array}$$

We thus have a deduction of $\alpha \to \varphi_k$ from $\Phi$. ∎

As in Ch. 2, we say that a variable **x** is free in a set $\Phi$ of formulas, if **x** is free in some formula of $\Phi$. Similarly, we say that **x** is free in a deduction ***D*** if **x** is free in some formula of ***D***.

**è un lemma per dimostrare regola di generalizzazione**

1.4. THEOREM. *Let* **x** *be a variable which is not free in* $\Phi$. *Given a deduction* ***D*** *of* $\alpha$ *from* $\Phi$, *we can construct a deduction* ***D'*** *of* $\forall \mathbf{x} \alpha$ *from* $\Phi$ *such that*

(a) **x** *is not free in* ***D'***,

(b) *every variable free in* ***D'*** *is free in* ***D*** *as well.*

PROOF. Let $\varphi_1,\ldots,\varphi_n$ be the given deduction ***D***. So $\varphi_n = \alpha$. By recursion on $k$ ($k=1,\ldots,n$) we construct a deduction $D_k$ of $\forall \mathbf{x} \varphi_k$ from $\Phi$, such that **x** is not free in $D_k$ and such that each variable free in $D_k$ is free also in ***D***. Then $D_n$ is the required ***D'***.

If $\varphi_k$ is an axiom, then $\forall \mathbf{x} \varphi_k$ is an axiom as well — see (Ax.8) — so $\forall \mathbf{x} \varphi_k$ by itself is the required $D_k$.

If $\varphi_k$ is in $\Phi$, then **x** is not free in $\varphi_k$. We take $D_k$ to be

$$\varphi_k, \qquad \text{(hyp.)}$$

$$\varphi_k \to \forall \mathbf{x} \varphi_k, \qquad \text{(Ax.3)}$$

$$\forall \mathbf{x} \varphi_k. \qquad \text{(m.p.)}$$

**Proposizione mp**: Se $\Phi \vdash \alpha \to \beta$ e $\Psi \models \alpha$ allora $\Phi, \Psi \vdash \beta$

**Proposizione**:
Se $\Phi \vdash \models \alpha_1$ , ..., $\Phi \vdash \alpha_n$, $\Psi, \alpha_1, \ldots, \alpha_n \vdash \beta$
allora $\Phi, \Psi \vdash \beta$

**Dim.**
Per il teorema di deduzione abbiamo che

$$\Psi \vdash \alpha_1 \to \ldots \to \alpha_n \to \beta$$

applicando n volte la proposizione **mp** otteniamo il risultato

Finally, if for some $i,j<k$ we have $\varphi_j = \varphi_i \to \varphi_k$, then by the induction hypothesis we already possess $D_i$ and $D_j$. We let $D_k$ be the deduction obtained by concatenating $D_i$ and $D_j$ and adding three more formulas, as follows:

$$D_i \begin{cases} \vdots \\ \forall \mathbf{x}\varphi_i, \end{cases}$$

$$D_j \begin{cases} \vdots \\ \forall \mathbf{x}(\varphi_i \to \varphi_k), \end{cases}$$

$$\forall \mathbf{x}(\varphi_i \to \varphi_k) \to \forall \mathbf{x}\varphi_i \to \forall \mathbf{x}\varphi_k, \qquad \text{(Ax.2)}$$

$$\forall \mathbf{x}\varphi_i \to \forall \mathbf{x}\varphi_k, \qquad \text{(m.p.)}$$

$$\forall \mathbf{x}\varphi_k. \qquad \text{(m.p.)}$$

This completes the proof. ∎

1.5. REMARK. By Thm. 1.4 we have the following *law of generalization on variables*: if $\Phi \vdash \alpha$ and $\mathbf{x}$ is not free in $\Phi$, then $\Phi \vdash \forall \mathbf{x}\alpha$. Two modified forms of this law are stated in the following problem.

1.6. PROBLEM. Assuming that $\mathbf{x}$ is not free in $\Phi$, $\beta$, show that

(a) if $\Phi \vdash \beta \to \alpha$, then $\Phi \vdash \beta \to \forall \mathbf{x}\alpha$,

(b) if $\Phi \vdash \alpha \to \beta$, then $\Phi \vdash \exists \mathbf{x}\alpha \to \beta$.

(To prove (b), begin by observing that[1] $\alpha \to \beta \vdash_0 \neg\beta \to \neg\alpha$.)

**1.10. Theorem.** *If* $\boldsymbol{\beta} \sim \boldsymbol{\beta}'$ *then* $\boldsymbol{\beta}$ *and* $\boldsymbol{\beta}'$ *are provably equivalent.*

**1.11. Theorem.** *For every formula* $\boldsymbol{\alpha}$, *variable* $\mathbf{x}$ *and term* $\mathbf{t}$,

$$\vdash \forall \mathbf{x} \boldsymbol{\alpha} \to \boldsymbol{\alpha}(\mathbf{x}/\mathbf{t}), \qquad \vdash \boldsymbol{\alpha}(\mathbf{x}/\mathbf{t}) \to \exists \mathbf{x} \boldsymbol{\alpha}.$$

**1.14. Problem.** Using Thm. 1.11 show that if $\mathbf{x}$ does not occur in the term $\mathbf{t}$ then $\vdash \exists \mathbf{x}(\mathbf{t} = \mathbf{x})$.

$$\vdash \underbrace{(t = x)[x/t]}_{t = t} \to \exists x.\, t = x$$

$$\frac{\vdash t = t \qquad \vdash t = t \to \exists x.\, t = x}{\exists x.\, t = x}$$

1.12. THEOREM. *Let* **c** *be a constant which occurs neither in* $\mathbf{\Phi}$ *nor in* $\boldsymbol{\alpha}$. *Given a deduction* ***D*** *of* $\boldsymbol{\alpha}(\mathbf{x}/\mathbf{c})$ *from* $\mathbf{\Phi}$ *we can construct a deduction* ***D'*** *of* $\forall \mathbf{x} \boldsymbol{\alpha}$ from $\mathbf{\Phi}$.

PROOF. We choose a variable **y** which does not occur in ***D***. Let $D_1$ be obtained from ***D*** by replacing every formula $\boldsymbol{\beta}$ by $\boldsymbol{\beta}(\mathbf{c}/\mathbf{y})$.

It is not difficult to verify that if $\boldsymbol{\beta}$ is an axiom then so is $\boldsymbol{\beta}(\mathbf{c}/\mathbf{y})$. The hypotheses (formulas of $\mathbf{\Phi}$) used in ***D*** are left unchanged because **c** does not occur in $\mathbf{\Phi}$. Aiso, every application of *modus ponens* is transformed into an application of *modus ponens.* Since **c** does not occur in $\boldsymbol{\alpha}$,

$$\boldsymbol{\alpha}(\mathbf{x}/\mathbf{c})(\mathbf{c}/\mathbf{y}) = \boldsymbol{\alpha}(\mathbf{x}/\mathbf{y}).$$

Thus $D_1$ is a deduction of $\boldsymbol{\alpha}(\mathbf{x}/\mathbf{y})$ from $\mathbf{\Phi}$. Moreover, if $\mathbf{\Phi}_0$ is the subset of $\mathbf{\Phi}$ consisting of those hypotheses that are actually used in $D_1$, then y does not occur in $\mathbf{\Phi}_0$ and $D_1$ is a deduction of $\boldsymbol{\alpha}(\mathbf{x}/\mathbf{y})$ from $\mathbf{\Phi}_0$. By Thm. 1.4 we get from $D_1$ a deduction $D_2$ of $\forall \mathbf{y} [\boldsymbol{\alpha}(\mathbf{x}/\mathbf{y})]$ from $\mathbf{\Phi}_0$ and hence from $\mathbf{\Phi}$.

Clearly, $\forall \mathbf{y} [\boldsymbol{\alpha}(\mathbf{x}/\mathbf{y})]$ is $\forall \mathbf{x} \boldsymbol{\alpha}$ or is obtained from $\forall \mathbf{x} \boldsymbol{\alpha}$ by alphabetic change. Therefore (either trivially, or as in Thm. 1.10) we obtain a deduction $D_3$ of $\forall \mathbf{x} \boldsymbol{\alpha}$ from $\forall \mathbf{y} [\boldsymbol{\alpha}(\mathbf{x}/\mathbf{y})]$. From $D_2$ and $D_3$ we get the required ***D'***. ∎

1.13. REMARK. By Thm. 1.12 we have the following *law of generalization on constants:*

If $\mathbf{\Phi} \vdash \boldsymbol{\alpha}(\mathbf{x}/\mathbf{c})$ and **c** does not occur in $\mathbf{\Phi}$ nor in $\boldsymbol{\alpha}$, then $\mathbf{\Phi} \vdash \forall \mathbf{x} \boldsymbol{\alpha}$.

# consistency/inconsistency

$\Phi, \alpha$ è insoddisfacibile sse $\Phi \models \sim\alpha$

$\Phi, \sim\alpha$ è insoddisfacibile sse $\Phi \models \alpha$

A set $\Phi$ of formulas is *first-order inconsistent* if for some $\beta$ both $\Phi \vdash \beta$ and $\Phi \vdash \neg\beta$. Otherwise $\Phi$ is *first-order consistent*.

applying soundness ...

1.17. THEOREM. *An inconsistent set of formulas is unsatisfiable.*

1.18. THEOREM. *A set $\Phi$ of formulas is inconsistent iff $\Phi \vdash \alpha$ for every formula $\alpha$.*

PROOF. Similar to that of Thm. 1.10.7. If $\Phi$ is inconsistent, then for some $\beta$ both $\Phi \vdash \beta$ and $\Phi \vdash \neg\beta$. But for every $\alpha$ we have $\{\beta, \neg\beta\} \vdash_0 \alpha$; hence $\Phi \vdash \alpha$. The converse is obvious. ∎

1.19. THEOREM. *For any* $\mathbf{\Phi}$ *and* $\boldsymbol{\alpha}$,

(a) $\mathbf{\Phi}, \neg\boldsymbol{\alpha}$ *is inconsistent iff* $\mathbf{\Phi} \vdash \boldsymbol{\alpha}$,

(b) $\mathbf{\Phi}, \boldsymbol{\alpha}$ *is inconsistent iff* $\mathbf{\Phi} \vdash \neg\boldsymbol{\alpha}$.

PROOF. (a) If $\mathbf{\Phi}, \neg\boldsymbol{\alpha}$ is inconsistent, then, by Thm. 1.18, $\mathbf{\Phi}, \neg\boldsymbol{\alpha} \vdash \boldsymbol{\alpha}$. Therefore, by the Deduction Theorem, $\mathbf{\Phi} \vdash \neg\boldsymbol{\alpha} \rightarrow \boldsymbol{\alpha}$. But one can easily verify (e.g., by a propositional tableau) that $\neg\boldsymbol{\alpha} \rightarrow \boldsymbol{\alpha} \vdash_0 \boldsymbol{\alpha}$. Hence $\mathbf{\Phi} \vdash \boldsymbol{\alpha}$. The converse is obvious.

(b) Similarly, if $\mathbf{\Phi}, \boldsymbol{\alpha}$ is inconsistent, one shows that $\mathbf{\Phi} \vdash \boldsymbol{\alpha} \rightarrow \neg\boldsymbol{\alpha}$. Also, it is readily verified that $\boldsymbol{\alpha} \rightarrow \neg\boldsymbol{\alpha} \vdash_0 \neg\boldsymbol{\alpha}$. Hence $\mathbf{\Phi} \vdash \neg\boldsymbol{\alpha}$. The converse is again obvious. ∎

# verso la completezza

3.14. Strong Completeness Theorem. *If* $\Phi \models \alpha$, *then* $\Phi \vdash \alpha$.

# § 7. Hintikka sets

7.1. DEFINITION. A set $\mathbf{\Psi}$ of $\mathscr{L}$-formulas is a *Hintikka set in* $\mathscr{L}$ if the following conditions hold:

(1) If $\boldsymbol{\varphi}$ is atomic, then $\boldsymbol{\varphi}$ and $\neg\boldsymbol{\varphi}$ do not both belong to $\mathbf{\Psi}$ (i.e., if $\boldsymbol{\varphi}\in\mathbf{\Psi}$, then $\neg\boldsymbol{\varphi}\notin\mathbf{\Psi}$).

(2) If $\neg\neg\boldsymbol{\alpha}\in\mathbf{\Psi}$, then $\boldsymbol{\alpha}\in\mathbf{\Psi}$.

(3) If $\boldsymbol{\alpha}\rightarrow\boldsymbol{\beta}\in\mathbf{\Psi}$, then $\neg\boldsymbol{\alpha}\in\mathbf{\Psi}$ or $\boldsymbol{\beta}\in\mathbf{\Psi}$.

(4) If $\neg(\boldsymbol{\alpha}\rightarrow\boldsymbol{\beta})\in\mathbf{\Psi}$, then both $\boldsymbol{\alpha}\in\mathbf{\Psi}$ and $\neg\boldsymbol{\beta}\in\mathbf{\Psi}$.

(5) If $\forall\mathbf{x}\boldsymbol{\alpha}\in\mathbf{\Psi}$, then $\boldsymbol{\alpha}(\mathbf{x}/\mathbf{t})\in\mathbf{\Psi}$ for every $\mathscr{L}$-term $\mathbf{t}$.

(6) If $\neg\forall\mathbf{x}\boldsymbol{\alpha}\in\mathbf{\Psi}$, then $\neg\boldsymbol{\alpha}(\mathbf{x}/\mathbf{t})\in\mathbf{\Psi}$ for some $\mathscr{L}$-term $\mathbf{t}$.

If $\mathscr{L}$ is a language with equality, we also require:

(7) For every $\mathscr{L}$-term $\mathbf{t}$, $(\mathbf{t}=\mathbf{t}) \in \mathbf{\Psi}$.

(8) If $\mathbf{f}$ is an $n$-ary function symbol of $\mathscr{L}$ and $\mathbf{t}_1,\ldots,\mathbf{t}_{2n}$ are $\mathscr{L}$-terms, then the formula

$$\mathbf{t}_1=\mathbf{t}_{n+1}\rightarrow \mathbf{t}_2=\mathbf{t}_{n+2}\rightarrow \ldots \rightarrow \mathbf{t}_n=\mathbf{t}_{2n}\rightarrow \mathbf{f}\mathbf{t}_1\ldots\mathbf{t}_n=\mathbf{f}\mathbf{t}_{n+1}\ldots\mathbf{t}_{2n}$$

belongs to $\mathbf{\Psi}$.

(9) If $\mathbf{P}$ is an $n$-ary predicate symbol of $\mathscr{L}$ and $\mathbf{t}_1,\ldots,\mathbf{t}_{2n}$ are terms then the formula

$$\mathbf{t}_1=\mathbf{t}_{n+1}\rightarrow \mathbf{t}_2=\mathbf{t}_{n+2}\rightarrow \ldots \rightarrow \mathbf{t}_n=\mathbf{t}_{2n}\rightarrow \mathbf{P}\mathbf{t}_1\ldots\mathbf{t}_n\rightarrow \mathbf{P}\mathbf{t}_{n+1}\ldots\mathbf{t}_{2n}$$

belongs to $\mathbf{\Psi}$. (For $n=2$, $\mathbf{P}$ can be $=$.)

Throughout the present section we let $\boxed{\Psi}$ be a fixed (but arbitrary) Hintikka set. Also, throughout this section we shall refer to the nine conditions of Def. 7.1 simply as (1), (2) etc. instead of 7.1.(1), 7.1.(2) etc.

Our aim is to show that $\Psi$ is satisfiable; but this will require some preliminary work.

We begin by defining a binary relation $E$ between $\mathscr{L}$-terms.

If $\mathscr{L}$ is a language without equality, we simply take $E$ to be the identity relation. In other words, $\mathbf{s}E\mathbf{t}$ iff $\mathbf{s}$ and $\mathbf{t}$ are the same term.[1]

If $\mathscr{L}$ has equality, we define $\mathbf{s}E\mathbf{t}$ to mean that the equation $\mathbf{s}=\mathbf{t}$ belongs to $\mathbf{\Psi}$.

7.2. LEMMA. *$E$ is an equivalence relation (i.e., it is reflexive, symmetric and transitive).*