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Master's Program in Applied Mathematics
Written test of Functional Analysis
January 6, 2013

Solve some of the following problems. Justify your conclusions. Time: 120 min.

Pb 1. Show that the function $u(x) = \arctan(1/x)$ is in $L^p(0, +\infty)$ for every $p > 1$. Deduce that the same holds for $u_n(x) = \pi/2 - \arctan(nx)$, and study the convergence of u_n in $L^p(0, +\infty)$.

Pb 2. In $\mathbb{R}^2$ consider the subspace

$$V = \{(x, y) \in \mathbb{R}^2 : 3y - x = 0\},$$

and the linear functional $\varphi : V \to \mathbb{R}$ defined by $\varphi(x, y) = y$. Find the unique linear extension $\Phi : \mathbb{R}^2 \to \mathbb{R}$ of φ, having the same norm as φ.

Pb 3. Let (u_n) be a sequence in $C^1([0, 1])$, bounded in $C([0, 1])$ and such that (u_n') is bounded in $L^3(0, 1)$. Prove that (u_n) has a convergent subsequence in $C(0, 1)$.

Pb 4. Let $f : [0, 1] \to [0, 1]$ be a continuous function such that $1 \le f(x) \le 2$ for all $x \in [0, 1]$. Consider the operator

$$T : C([0, 1]) \to C([0, 1])$$

defined by $(Tu)(x) = f(x)u(x)$. Is T continuous, invertible, compact? Is T continuous and/or compact in the case $f(x) = x$?

Pb 5. Consider the convex closed set

$$C = \Big\{u \in L^2(0, 2) : \int_0^2 u(s)ds = 1\Big\}.$$

Show that $C^\perp$ is a *very* small subspace of $L^2(0, 2)$ and discuss why the equality $L^2(0, 2) = C \oplus C^\perp$ does *not* hold.

Pb 6. Let $\gamma > 0$ and $p \ge 1$. Consider $T : \ell^p \to \ell^p$ defined by

$$(Tx)_j := \frac{x_j}{j^\gamma}, \quad j \ge 1.$$

Is it a bounded linear operator? Is it compact? (*Hint: use finite-dimensional approximation.*) Furthermore, consider the same questions when $T : \ell^p \to \ell^q$ for $1 \le q < p$.

Pb 7. Study the convergence of the sequence $u_n(x) = (x + 1/n)^{1/3}$ in the Sobolev spaces $W^{1,p}(0, 1)$, $p \ge 1$.