An approximation tour
Approximation with pursuits

Mallat 1999 Ch. 9 Section 9.4

Approximations with pursuits

- To optimize non-linear signal approximations, one can adaptively choose the basis depending on the signal
- The set of orthogonal bases is much smaller than the set of non-orthogonal bases that could be constructed by choosing N linearly independent vectors from these P.
- To improve the approximation of complex signals such as music recordings, we study general non-orthogonal signal decompositions.
- Consider the space of signals of size N. Let

$$D = \left\{ g_p \right\}_{0 \le p < P}$$

- be a redundant dictionary of P>N vectors which includes at least N linearly independent vectors

Approximations with pursuits

- For M≥1, an approximation fM of f can be calculated with a linear combination of any M dictionary vectors

$$f_M = \sum_{m=0}^{M-1} a[p_m]\, g_{p_m}.$$

- The freedom of choice opens the door to a considerable combinatorial explosion.
- For general dictionaries of P > N vectors, computing the approximation f~ that minimizes ||f – fM II is an NP hard problem.
  - This means that there is no known polynomial time algorithm that can solve this optimization.
- Pursuit algorithms reduce the computational complexity by searching for efficient but non-optimal approximations.

Basis pursuits

- A basis pursuit formulates the search as a linear programming problem, providing remarkably good approximations with 0(N3.5log23.5N) operations.
- For large signals, this remains prohibitive. Matching pursuits are faster greedy algorithms that make the problem tractable

- **We study the construction of a “be**st” basis B, not necessarily orthogonal, for efficiently approximating a signal **f**
- The N vectors of the basis

$$\mathcal{B} = \{g_{p_m}\}_{0\leq m<N}$$

- are selected with a pursuit.

Basis pursuit

- Let us decompose f in the basis

$$f = \sum_{m=0}^{N-1} a[p_m] \, g_{p_m}.$$

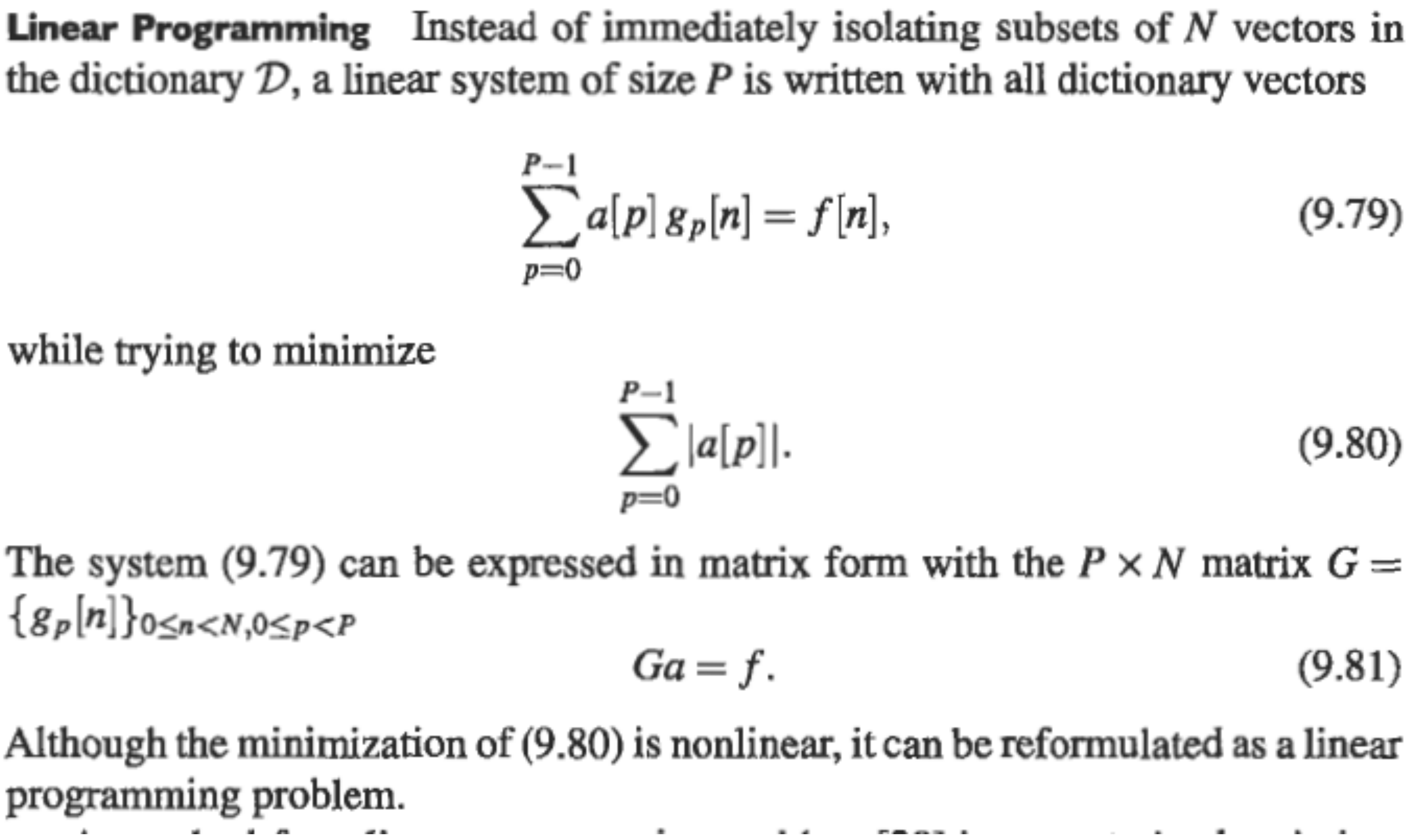
- A basis pursuit searches for a basis that minimizes

$$C(f, \mathcal{B}) = \frac{1}{\|f\|} \sum_{m=0}^{N-1} |a[p_m]|.$$

Minimizing the l^1 norm of the decomposition coefficients avoids diffusing the energy of f among many vectors. It reduces cancellations between the vectors $a[p_m]g_{pm}$, that decompose f, because such cancellations increase $|a[p_m]|$ and thus increase the cost.

The minimization of an l1 norm is also related to linear programming, which leads to fast computational algorithms.

Linear programming



Linear Programming Instead of immediately isolating subsets of N vectors in the dictionary $\mathcal{D}$, a linear system of size P is written with all dictionary vectors

$$\sum_{p=0}^{P-1} a[p]\, g_p[n] = f[n], \tag{9.79}$$

while trying to minimize

$$\sum_{p=0}^{P-1} |a[p]|. \tag{9.80}$$

The system (9.79) can be expressed in matrix form with the $P \times N$ matrix $G = \{g_p[n]\}_{0\leq n<N, 0\leq p<P}$

$$Ga = f. \tag{9.81}$$

Although the minimization of (9.80) is nonlinear, it can be reformulated as a linear programming problem.

Linear programming

- It can be shown that the solution has at most N non zero coefficients
- In the non degenerate cases, which are most often encountered, the non zero coefficients correspond to N indicis $\{p_m\}_{0 \leq m < M}$ such that

$$\{g_{p_m}\}_{0 \leq m < N}$$

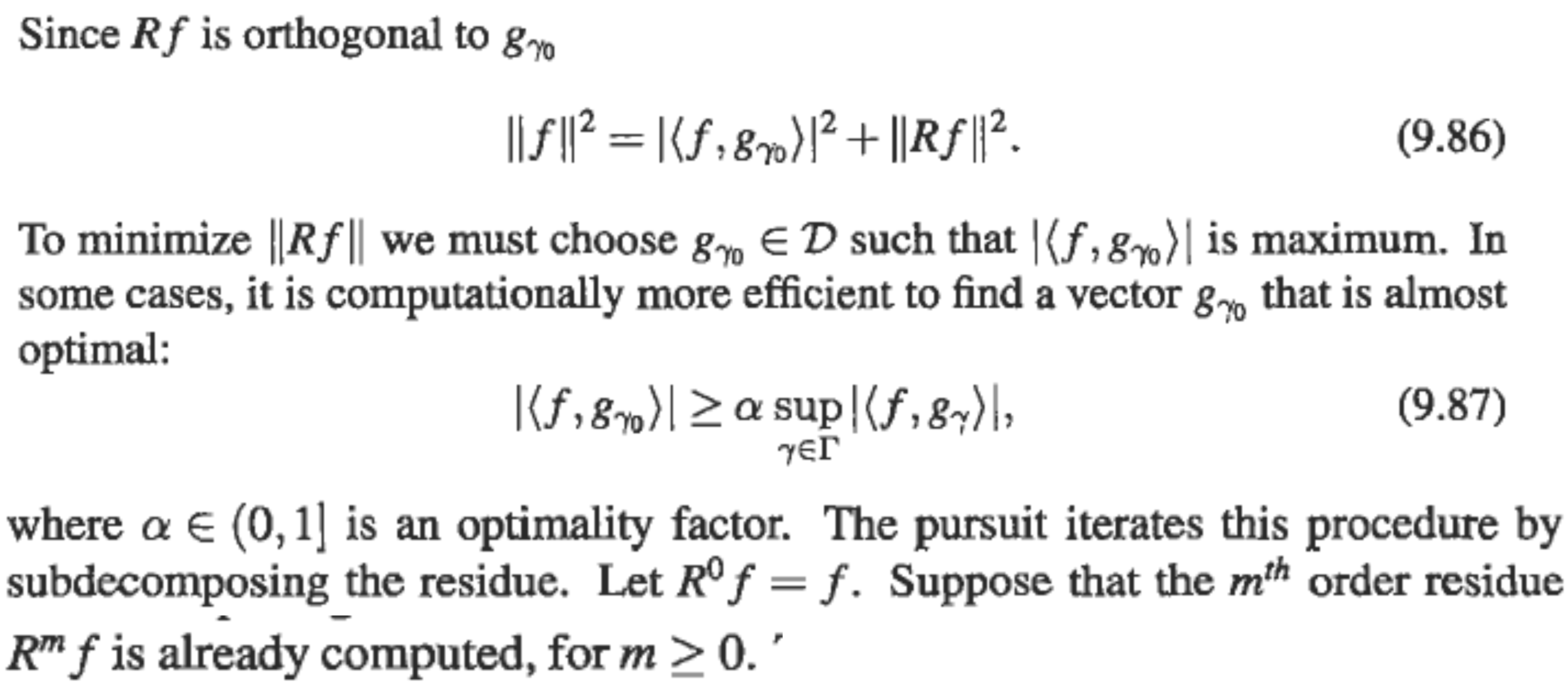
- are linearly independent.
- This is the best basis of R^N that minimizes the cost.

Matching pursuit

- Despite the linear programming approach, a basis pursuit is computationally expensive because it minimizes a global cost function over all dictionary vectors.
- The matching pursuit introduced by Mallat and Zhang [259] reduces the computational complexity with a greedy strategy.
- Let $\mathcal{D} = \{g_\gamma\}_{\gamma \in \Gamma}$ be a dictionary of P>N vectors having unit norm.
- This dictionary includes N linearly independent vectors that define a basis of the space C^N of signals of size N.
- A matching pursuit begins by projecting f on a vector $g_{\gamma_0} \in \mathcal{D}$ and computing the residue Rf

$$f = \langle f, g_{\gamma_0} \rangle g_{\gamma_0} + Rf.$$

Matching pursuit



Since Rf is orthogonal to g_{γ_0}

$$\|f\|^2 = |\langle f, g_{\gamma_0} \rangle|^2 + \|Rf\|^2. \tag{9.86}$$

To minimize $\|Rf\|$ we must choose $g_{\gamma_0} \in \mathcal{D}$ such that $|\langle f, g_{\gamma_0} \rangle|$ is maximum. In some cases, it is computationally more efficient to find a vector g_{γ_0} that is almost optimal:

$$|\langle f, g_{\gamma_0} \rangle| \geq \alpha \sup_{\gamma \in \Gamma} |\langle f, g_\gamma \rangle|, \tag{9.87}$$

where $\alpha \in (0,1]$ is an optimality factor. The pursuit iterates this procedure by subdecomposing the residue. Let $R^0 f = f$. Suppose that the m^{th} order residue $R^m f$ is already computed, for $m \geq 0$.

Matching pursuit

The next iteration chooses $g_{\gamma_m} \in \mathcal{D}$ such that

$$|\langle R^m f, g_{\gamma_m}\rangle| \geq \alpha \sup_{\gamma\in\Gamma} |\langle R^m f, g_\gamma\rangle|,$$

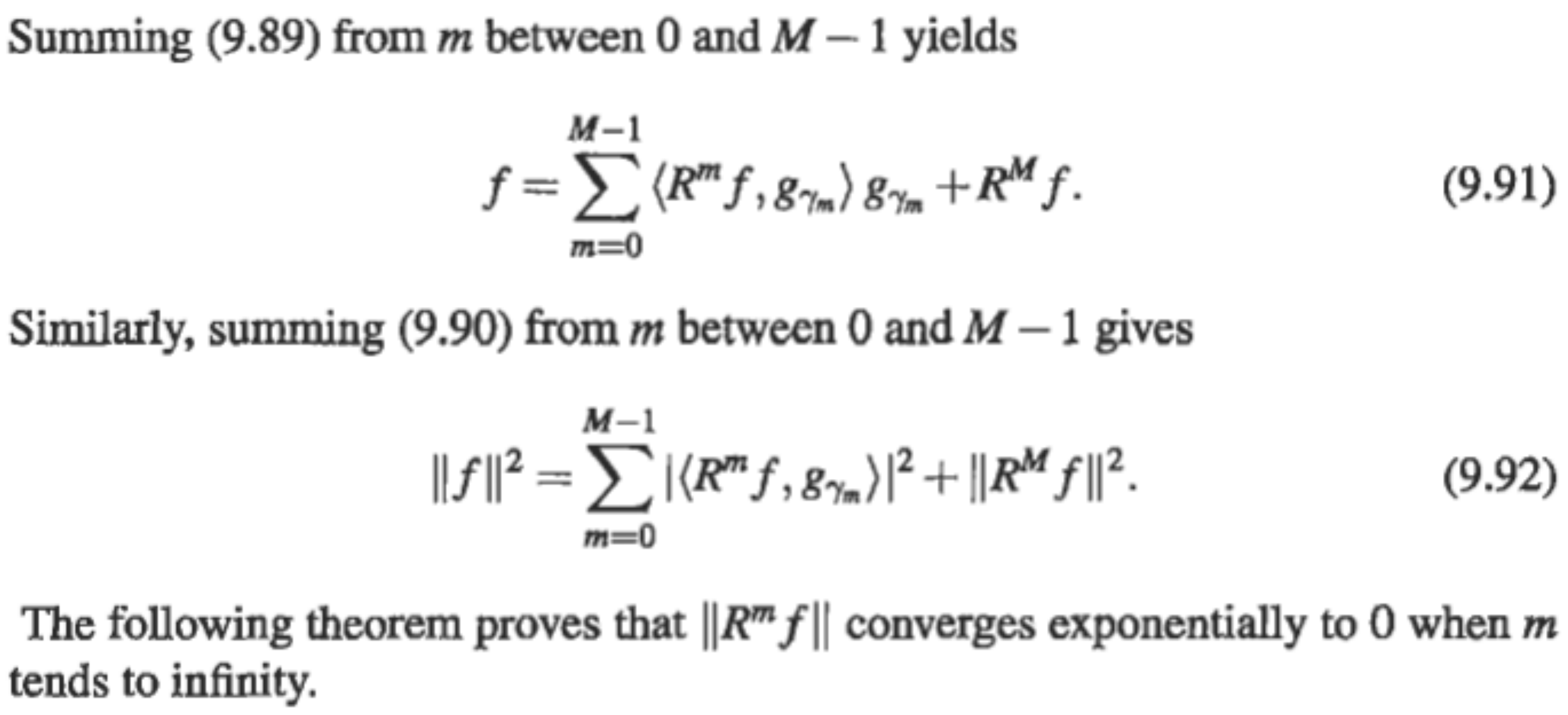
and projects $R^m f$ on g_{γ_m}:

$$R^m f = \langle R^m f, g_{\gamma_m}\rangle\, g_{\gamma_m} + R^{m+1} f. \tag{9.89}$$

The orthogonality of $R^{m+1} f$ and g_{γ_m} implies

$$\|R^m f\|^2 = |\langle R^m f, g_{\gamma_m}\rangle|^2 + \|R^{m+1} f\|^2.$$

Matching pursuit



Summing (9.89) from m between 0 and $M-1$ yields

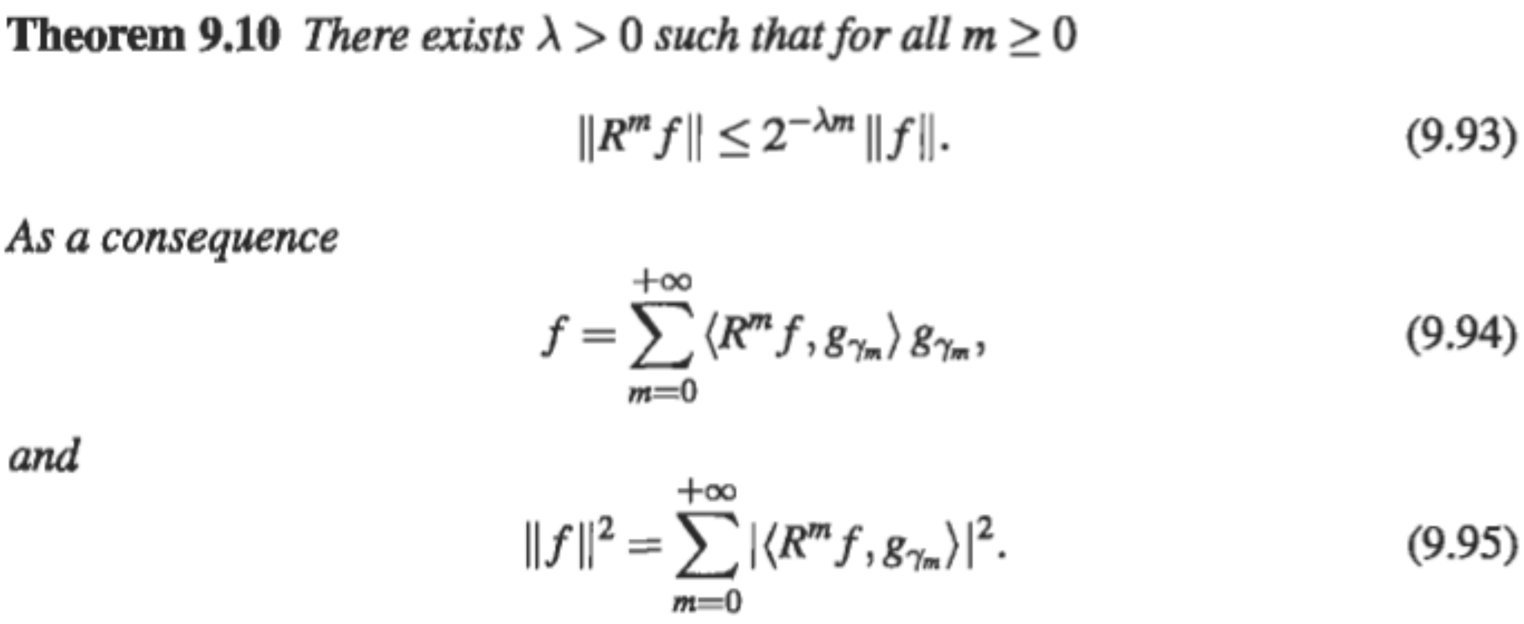
$$f = \sum_{m=0}^{M-1} \langle R^m f, g_{\gamma_m} \rangle \, g_{\gamma_m} + R^M f. \tag{9.91}$$

Similarly, summing (9.90) from m between 0 and $M-1$ gives

$$\|f\|^2 = \sum_{m=0}^{M-1} |\langle R^m f, g_{\gamma_m} \rangle|^2 + \|R^M f\|^2. \tag{9.92}$$

The following theorem proves that $\|R^m f\|$ converges exponentially to 0 when m tends to infinity.

Theorem



Theorem 9.10 *There exists $\lambda > 0$ such that for all $m \geq 0$*

$$\|R^m f\| \leq 2^{-\lambda m} \|f\|. \tag{9.93}$$

As a consequence

$$f = \sum_{m=0}^{+\infty} \langle R^m f, g_{\gamma_m} \rangle \, g_{\gamma_m}, \tag{9.94}$$

and

$$\|f\|^2 = \sum_{m=0}^{+\infty} |\langle R^m f, g_{\gamma_m} \rangle|^2. \tag{9.95}$$

Matching pursuit

- The convergence rate X decreases when the size N of the signal space increases.
- In the limit of infinite dimensional spaces, Jones' theorem proves that the algorithm still converges but the convergence is not exponential [230,259]**.**
- Observe that even in finite dimensions, an infinite number of iterations is necessary to completely reduce the residue.
- In most signal processing applications, this is not an issue because many fewer than N iterations are needed to obtain sufficiently precise signal approximations.

Fast network calculations

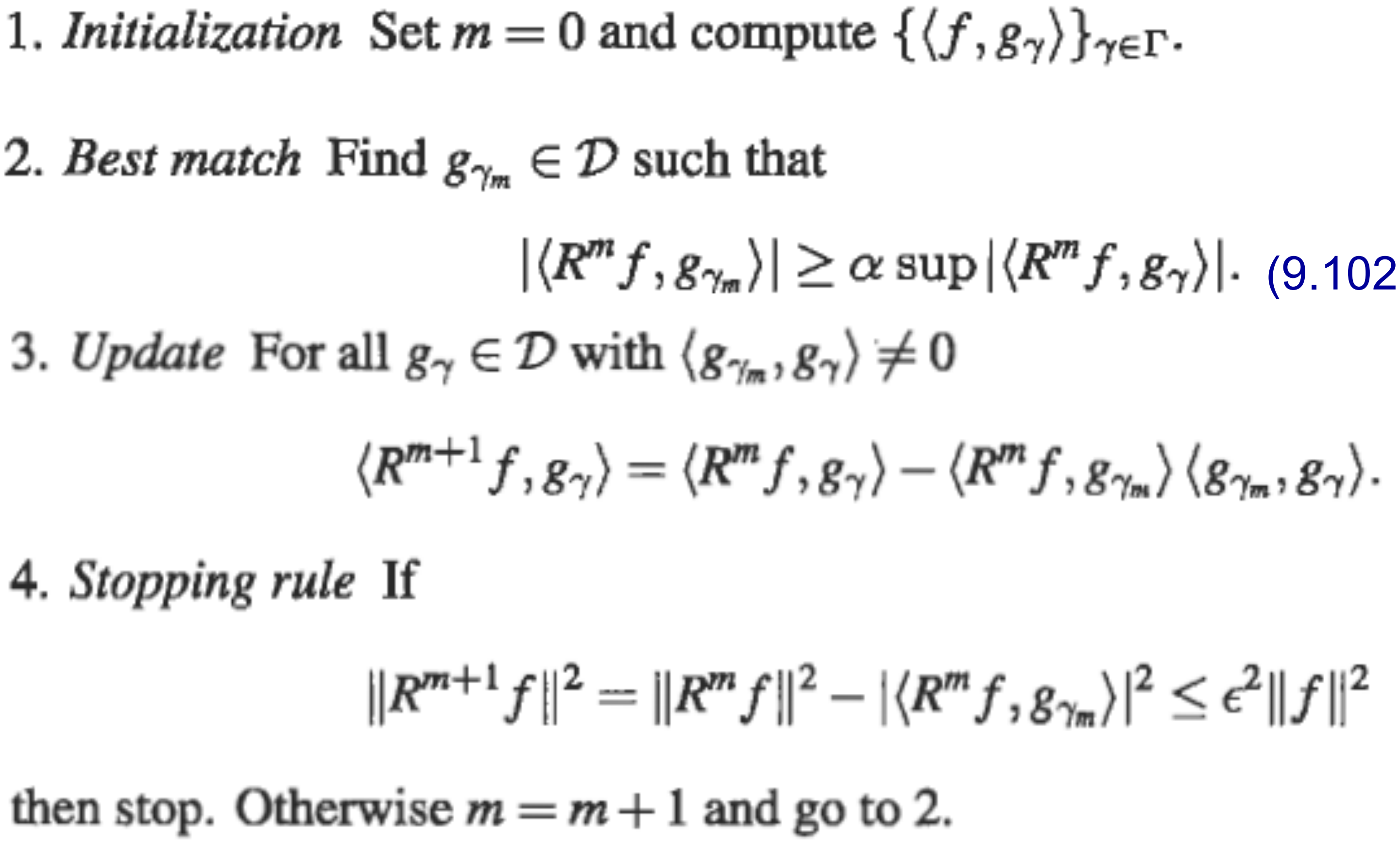
- A matching pursuit is implemented with a fast algorithm that computes $\left\langle R^{m+1} f, g_\gamma \right\rangle$ from $\left\langle R^{m} f, g_\gamma \right\rangle$ with a simple updating formula

$$\left\langle R^{m+1} f, g_\gamma \right\rangle = \left\langle R^{m+1} f, g_\gamma \right\rangle - \left\langle R^{m+1} f, g_{\gamma_m} \right\rangle \left\langle g_{\gamma_m}, g_\gamma \right\rangle$$

- To reduce the computational load, it is necessary to construct dictionaries with vectors having a sparse interaction. This means that each g_γ has non-zero inner products with only a small fraction of all other dictionary vectors
- Dictionaries are designed so that non-zero weights $\left\langle g_\alpha, g_\gamma \right\rangle$ can be retrieved from memory or computed with O(1) operations

Matching pursuit

- A matching pursuit with a relative precision ε is implemented as follows



1. *Initialization* Set $m = 0$ and compute $\{\langle f, g_\gamma \rangle\}_{\gamma \in \Gamma}$.

2. *Best match* Find $g_{\gamma_m} \in \mathcal{D}$ such that

$$|\langle R^m f, g_{\gamma_m} \rangle| \geq \alpha \sup |\langle R^m f, g_\gamma \rangle|. \qquad (9.102)$$

3. *Update* For all $g_\gamma \in \mathcal{D}$ with $\langle g_{\gamma_m}, g_\gamma \rangle \neq 0$

$$\langle R^{m+1} f, g_\gamma \rangle = \langle R^m f, g_\gamma \rangle - \langle R^m f, g_{\gamma_m} \rangle \langle g_{\gamma_m}, g_\gamma \rangle.$$

4. *Stopping rule* If

$$\|R^{m+1} f\|^2 = \|R^m f\|^2 - |\langle R^m f, g_{\gamma_m} \rangle|^2 \leq \epsilon^2 \|f\|^2$$

then stop. Otherwise $m = m + 1$ and go to 2.

Matching pursuit

- If D is highly redundant, computations at steps 2 and 3 are reduced by performing the calculation on a subdictionary Ds

$$\mathcal{D}_s = \{g_\gamma\}_{\gamma\in\Gamma_s} \subset \mathcal{D}.$$

- The sub-dictionary Ds is constructed so that

$$\text{if } g_{\tilde{\gamma}_m} \in \mathcal{D}_s \text{ maximizes } |\langle f, g_\gamma\rangle| \text{ in } \mathcal{D}_s$$

- then there exists $g_{\gamma_m} \in \mathcal{D}$ which minimizes (9.102) and whos γ_m is close to $\tilde{\gamma}_m$
- The index γ_m is found by a local search
- This is done in time-frequency dictionaries where a sub-dictionary can be sufficient to indicate a time-frequency region where an almost best match is located.

Translation invariance

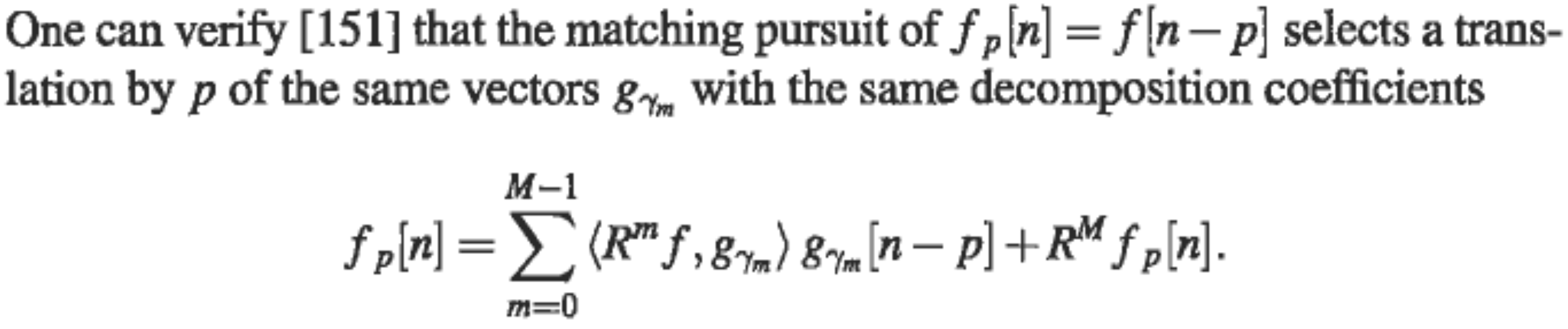
- Decompositions in orthogonal bases lack translation invariance and are thus difficult to use for pattern recognition.
- Matching pursuits are translation invariant if calculated in translation invariant dictionaries
- A dictionary is translation invariant if for any

$$g_\gamma \in D \text{ and } n \in [0, N-1] \rightarrow g_\gamma [n-p] \in D$$

- Suppose that the matching decomposition of f in D is

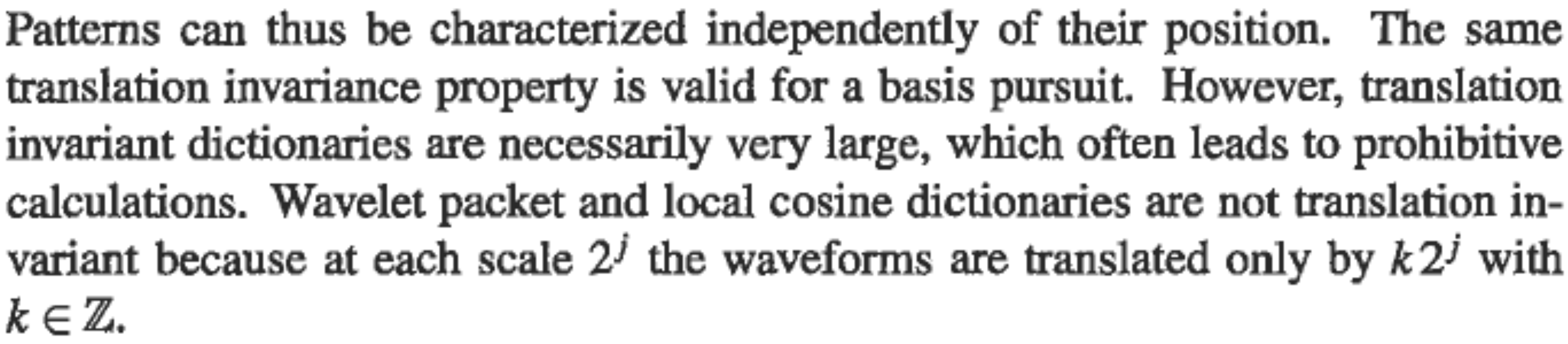
$$f[n] = \sum_{m=0}^{M-1} \langle R^m f, g_{\gamma_m} \rangle \, g_{\gamma_m}[n] + R^M f[n].$$

Translation invariance



One can verify [151] that the matching pursuit of $f_p[n] = f[n-p]$ selects a translation by p of the same vectors g_{γ_m} with the same decomposition coefficients

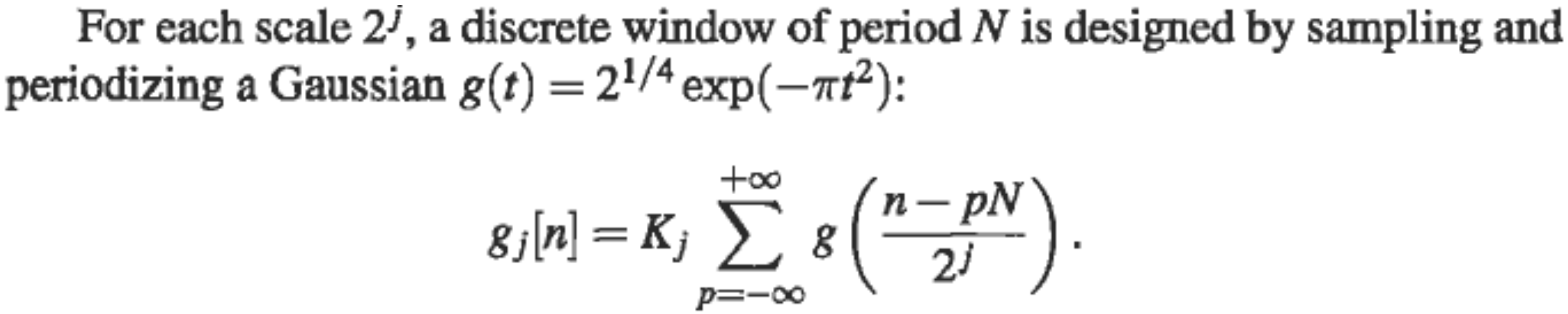
$$f_p[n] = \sum_{m=0}^{M-1} \langle R^m f, g_{\gamma_m} \rangle \, g_{\gamma_m}[n-p] + R^M f_p[n].$$



Patterns can thus be characterized independently of their position. The same translation invariance property is valid for a basis pursuit. However, translation invariant dictionaries are necessarily very large, which often leads to prohibitive calculations. Wavelet packet and local cosine dictionaries are not translation invariant because at each scale 2^j the waveforms are translated only by $k 2^j$ with $k \in \mathbb{Z}$.

Gabor dictionaries

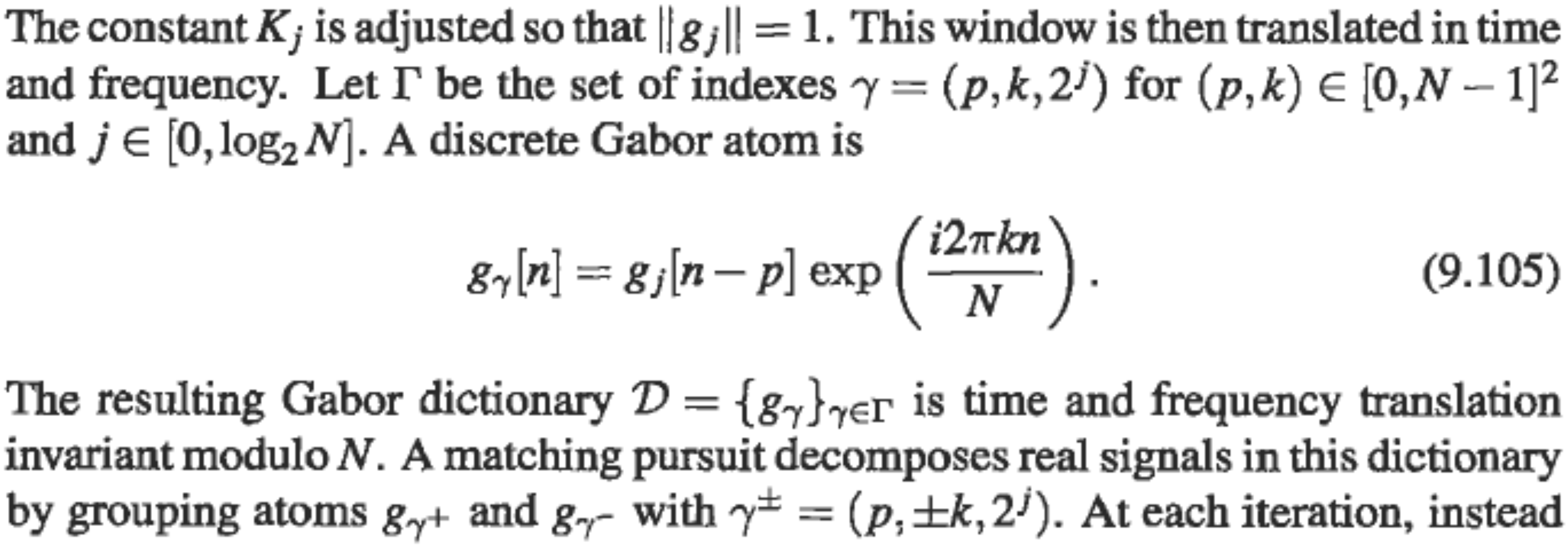
- A time and frequency translation invariant Gabor dictionary is constructed by Qian and Chen **[287]** as well as Mallat and Zhong [259], by scaling, translating and modulating a Gaussian window.
- Gaussian windows are used because of their optimal time and frequency energy concentration, proved by the uncertainty Theorem



For each scale 2^j, a discrete window of period N is designed by sampling and periodizing a Gaussian $g(t) = 2^{1/4} \exp(-\pi t^2)$:

$$g_j[n] = K_j \sum_{p=-\infty}^{+\infty} g\left(\frac{n-pN}{2^j}\right).$$

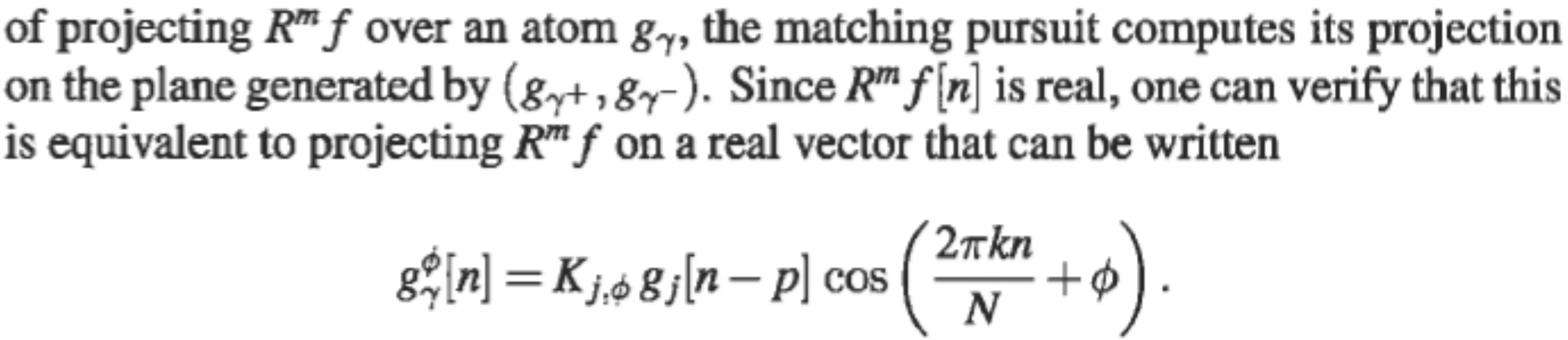
Gabor dictionaries



The constant K_j is adjusted so that $\|g_j\| = 1$. This window is then translated in time and frequency. Let Γ be the set of indexes $\gamma = (p, k, 2^j)$ for $(p, k) \in [0, N-1]^2$ and $j \in [0, \log_2 N]$. A discrete Gabor atom is

$$g_\gamma[n] = g_j[n-p] \exp\left(\frac{i2\pi kn}{N}\right). \tag{9.105}$$

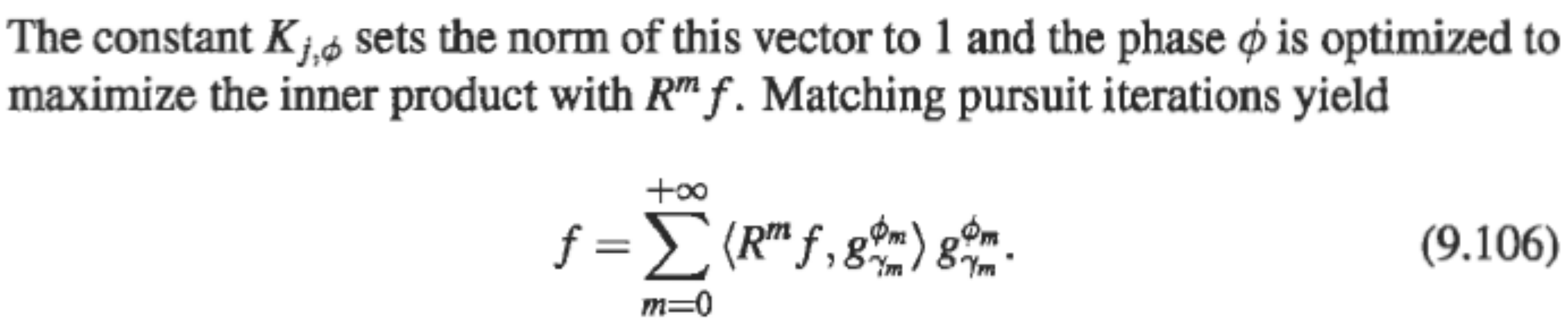
The resulting Gabor dictionary $\mathcal{D} = \{g_\gamma\}_{\gamma\in\Gamma}$ is time and frequency translation invariant modulo N. A matching pursuit decomposes real signals in this dictionary by grouping atoms g_{γ^+} and g_{γ^-} with $\gamma^\pm = (p, \pm k, 2^j)$. At each iteration, instead



of projecting $R^m f$ over an atom g_γ, the matching pursuit computes its projection on the plane generated by $(g_{\gamma^+}, g_{\gamma^-})$. Since $R^m f[n]$ is real, one can verify that this is equivalent to projecting $R^m f$ on a real vector that can be written

$$g_\gamma^\phi[n] = K_{j,\phi}\, g_j[n-p] \cos\left(\frac{2\pi kn}{N} + \phi\right).$$

Gabor dictionaries

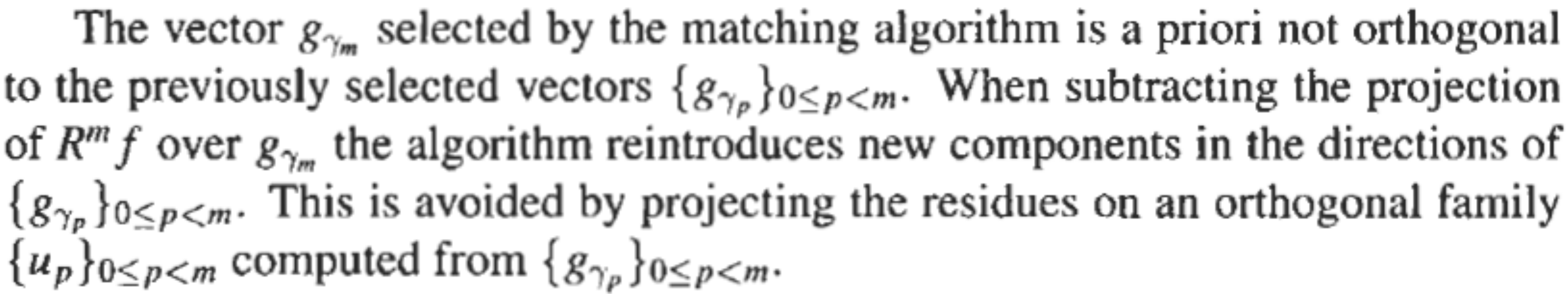


The constant $K_{j,\phi}$ sets the norm of this vector to 1 and the phase ϕ is optimized to maximize the inner product with $R^m f$. Matching pursuit iterations yield

$$f = \sum_{m=0}^{+\infty} \langle R^m f, g_{\gamma_m}^{\phi_m} \rangle \, g_{\gamma_m}^{\phi_m}. \qquad (9.106)$$

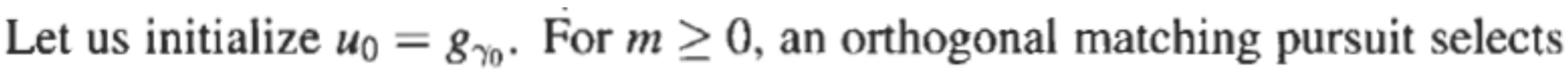
Orthogonal MP

- The approximations of a matching pursuit are improved by orthogonalizing the directions of projection, with a Gram-Schmidt procedure
- The resulting orthogonal pursuit converges with a finite number of iterations, which is not the case for a non-orthogonal pursuit.
- The price to be paid is the important computational cost of the Gram-Schmidt orthogonalization.

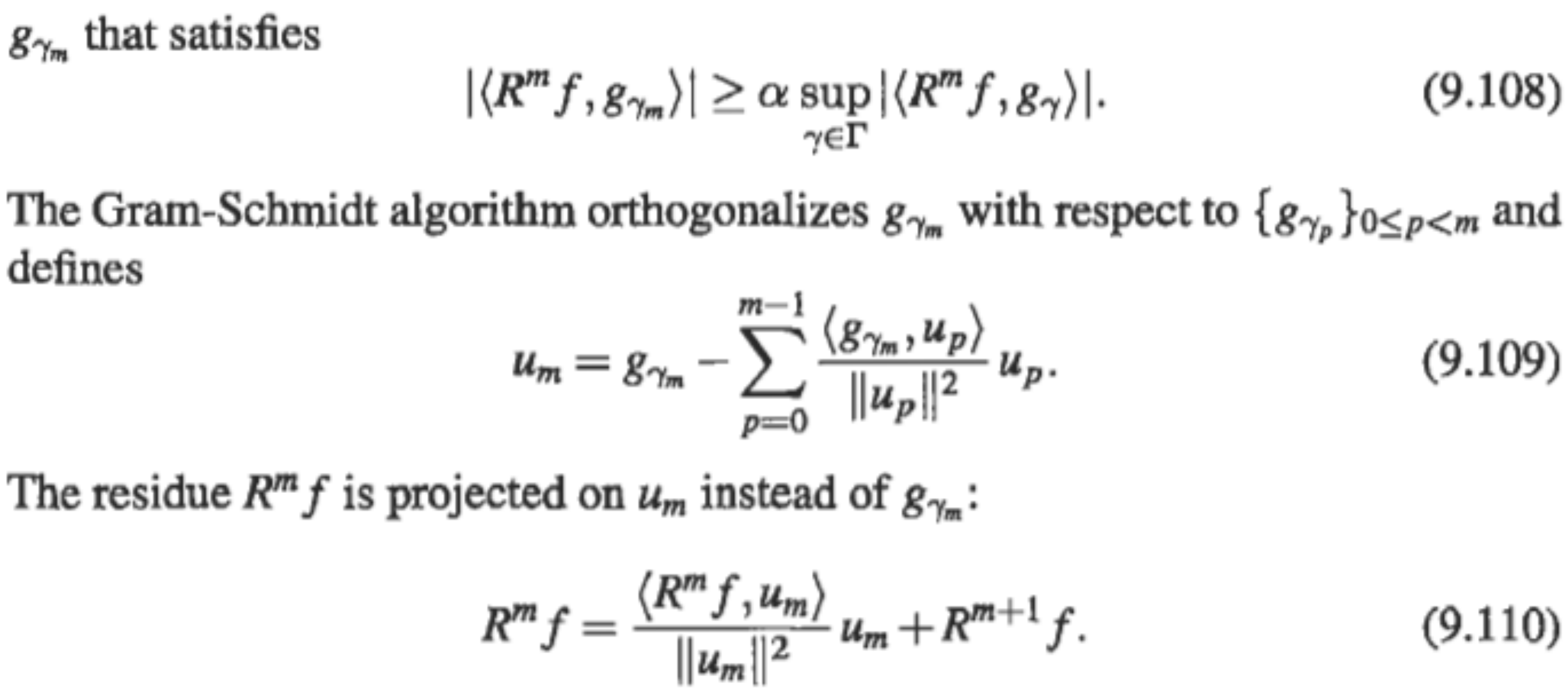


The vector g_{γ_m} selected by the matching algorithm is a priori not orthogonal to the previously selected vectors $\{g_{\gamma_p}\}_{0\leq p<m}$. When subtracting the projection of $R^m f$ over g_{γ_m} the algorithm reintroduces new components in the directions of $\{g_{\gamma_p}\}_{0\leq p<m}$. This is avoided by projecting the residues on an orthogonal family $\{u_p\}_{0\leq p<m}$ computed from $\{g_{\gamma_p}\}_{0\leq p<m}$.

Orthogonal MP



Let us initialize $u_0 = g_{\gamma_0}$. For $m \geq 0$, an orthogonal matching pursuit selects



g_{γ_m} that satisfies

$$|\langle R^m f, g_{\gamma_m} \rangle| \geq \alpha \sup_{\gamma \in \Gamma} |\langle R^m f, g_\gamma \rangle|. \tag{9.108}$$

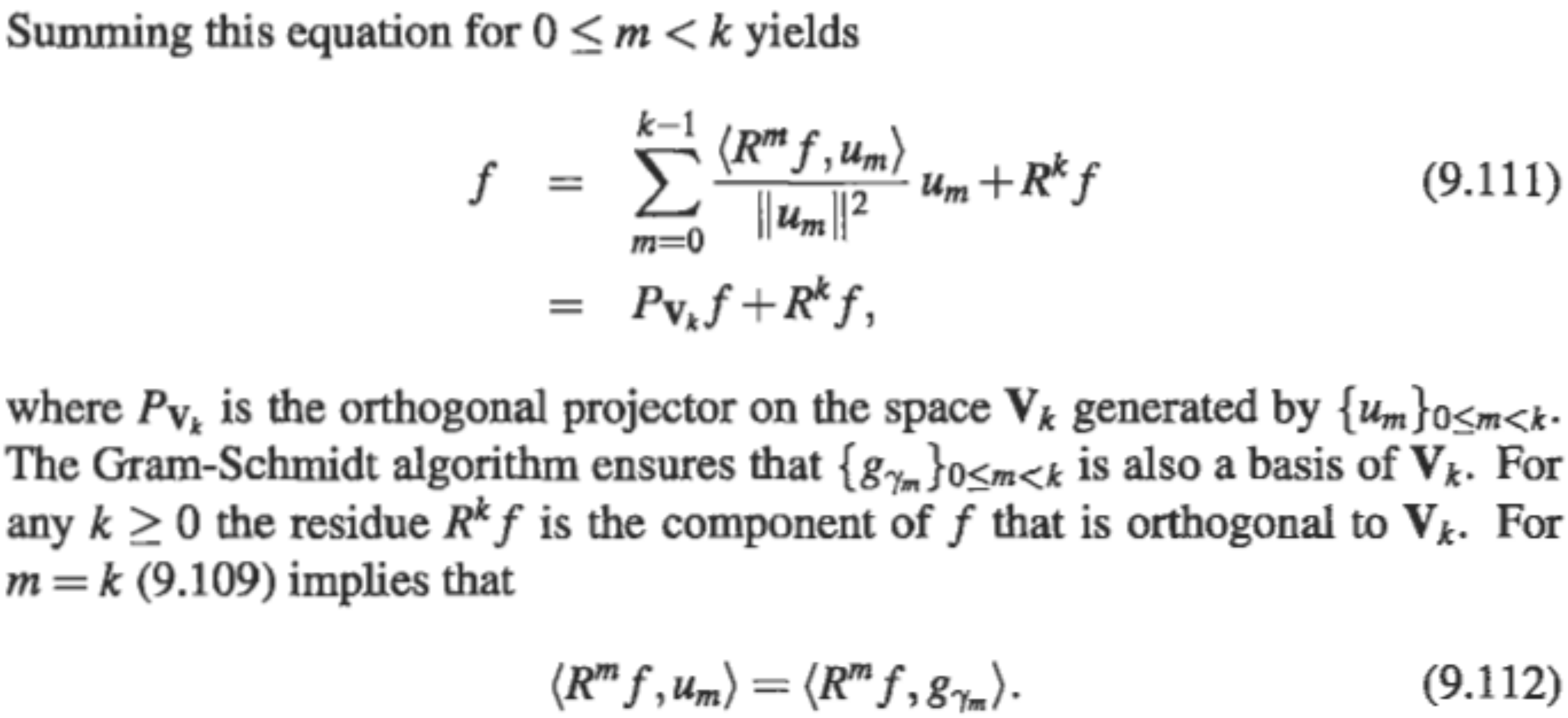
The Gram-Schmidt algorithm orthogonalizes g_{γ_m} with respect to $\{g_{\gamma_p}\}_{0 \leq p < m}$ and defines

$$u_m = g_{\gamma_m} - \sum_{p=0}^{m-1} \frac{\langle g_{\gamma_m}, u_p \rangle}{\|u_p\|^2} u_p. \tag{9.109}$$

The residue $R^m f$ is projected on u_m instead of g_{γ_m}:

$$R^m f = \frac{\langle R^m f, u_m \rangle}{\|u_m\|^2} u_m + R^{m+1} f. \tag{9.110}$$

Orthogonal MP



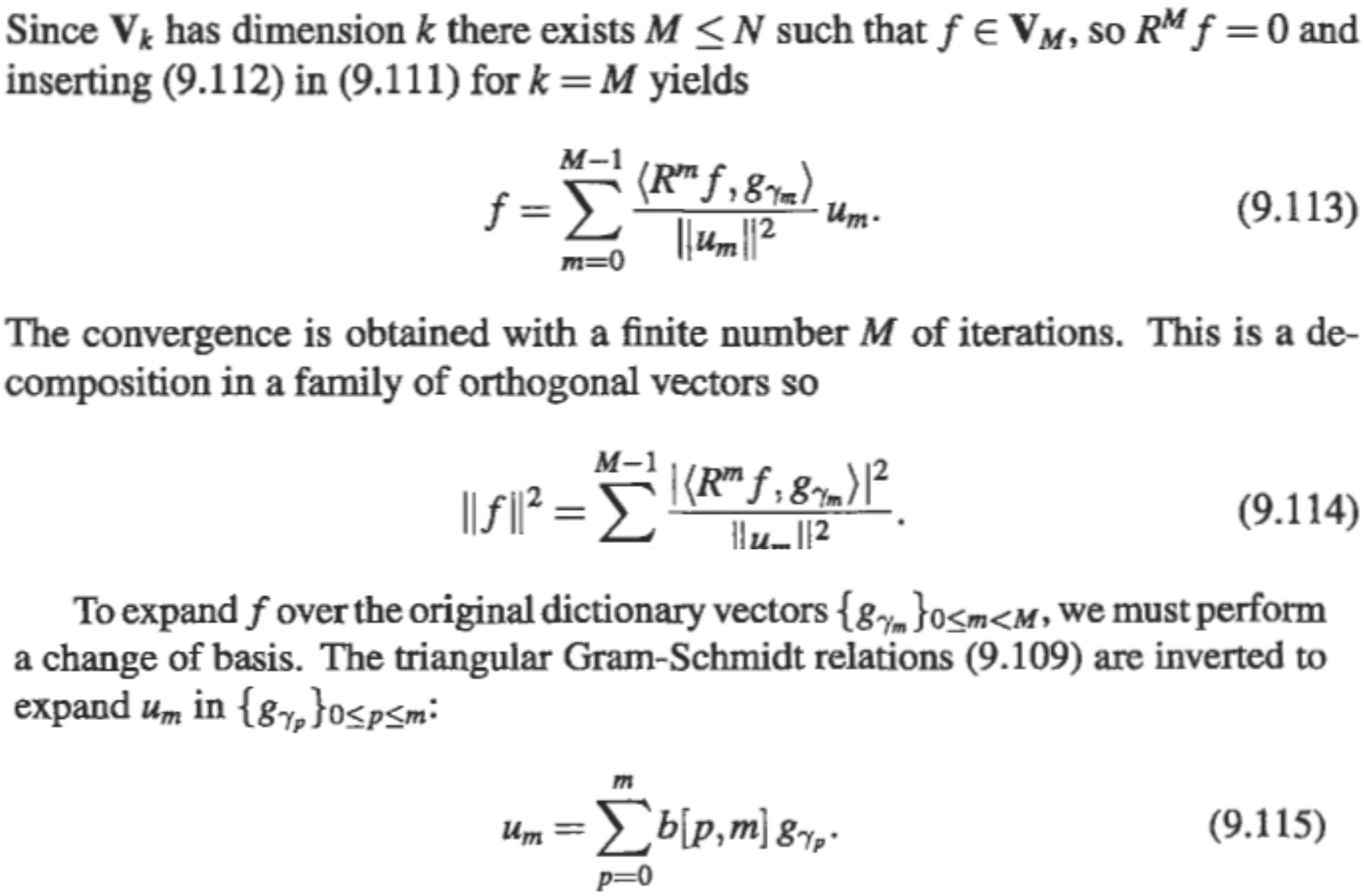
Summing this equation for $0 \leq m < k$ yields

$$\begin{aligned} f &= \sum_{m=0}^{k-1} \frac{\langle R^m f, u_m \rangle}{\|u_m\|^2} u_m + R^k f \qquad (9.111) \\ &= P_{\mathbf{V}_k} f + R^k f, \end{aligned}$$

where $P_{\mathbf{V}_k}$ is the orthogonal projector on the space $\mathbf{V}_k$ generated by $\{u_m\}_{0 \leq m < k}$. The Gram-Schmidt algorithm ensures that $\{g_{\gamma_m}\}_{0 \leq m < k}$ is also a basis of $\mathbf{V}_k$. For any $k \geq 0$ the residue $R^k f$ is the component of f that is orthogonal to $\mathbf{V}_k$. For $m = k$ (9.109) implies that

$$\langle R^m f, u_m \rangle = \langle R^m f, g_{\gamma_m} \rangle. \qquad (9.112)$$

Orthogonal MP



Since $\mathbf{V}_k$ has dimension k there exists $M \leq N$ such that $f \in \mathbf{V}_M$, so $R^M f = 0$ and inserting (9.112) in (9.111) for $k = M$ yields

$$f = \sum_{m=0}^{M-1} \frac{\langle R^m f, g_{\gamma_m} \rangle}{\|u_m\|^2} u_m. \tag{9.113}$$

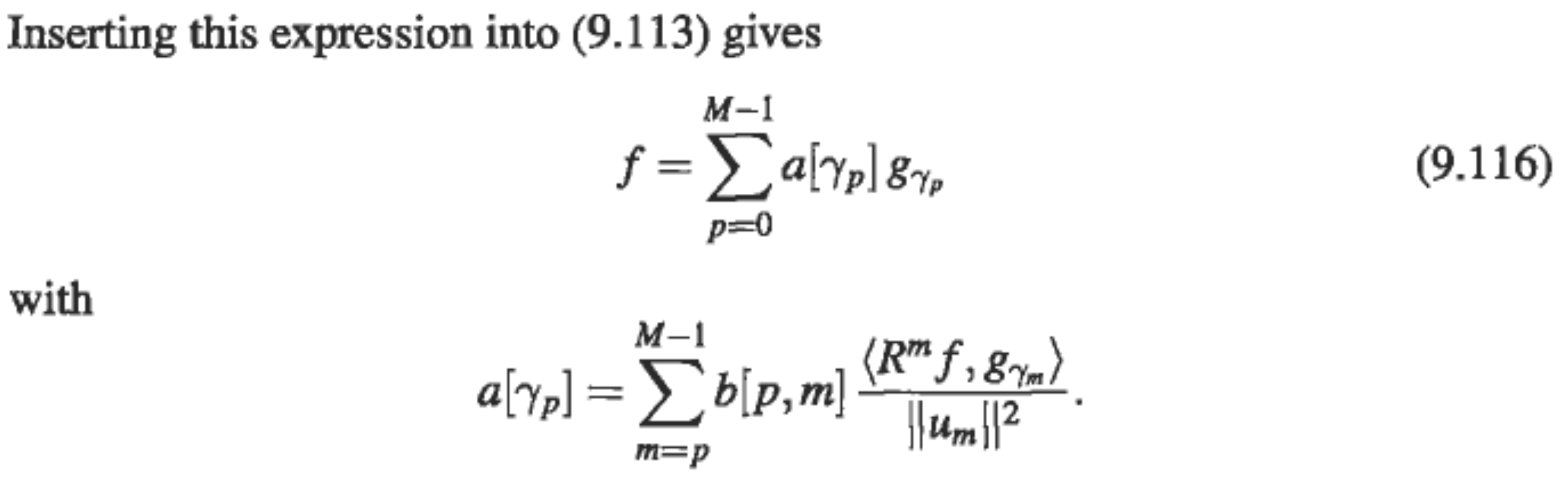
The convergence is obtained with a finite number M of iterations. This is a decomposition in a family of orthogonal vectors so

$$\|f\|^2 = \sum^{M-1} \frac{|\langle R^m f, g_{\gamma_m} \rangle|^2}{\|u_m\|^2}. \tag{9.114}$$

To expand f over the original dictionary vectors $\{g_{\gamma_m}\}_{0 \leq m < M}$, we must perform a change of basis. The triangular Gram-Schmidt relations (9.109) are inverted to expand u_m in $\{g_{\gamma_p}\}_{0 \leq p \leq m}$:

$$u_m = \sum_{p=0}^{m} b[p,m]\, g_{\gamma_p}. \tag{9.115}$$

Orthogonal MP



Inserting this expression into (9.113) gives

$$f = \sum_{p=0}^{M-1} a[\gamma_p]\, g_{\gamma_p} \tag{9.116}$$

with

$$a[\gamma_p] = \sum_{m=p}^{M-1} b[p,m]\, \frac{\langle R^m f, g_{\gamma_m} \rangle}{\|u_m\|^2}.$$

Orthogonal MP

- During the first few iterations, the pursuit often selects nearly orthogonal vectors, so the Gram-Schmidt orthogonalization is not needed.
- The orthogonal and nonorthogonal pursuits are then nearly the same.
- When the number of iterations increases and gets close to N, the residues of an orthogonal pursuit have norms that decrease faster than for a non-orthogonal pursuit.