Wavelet Packets

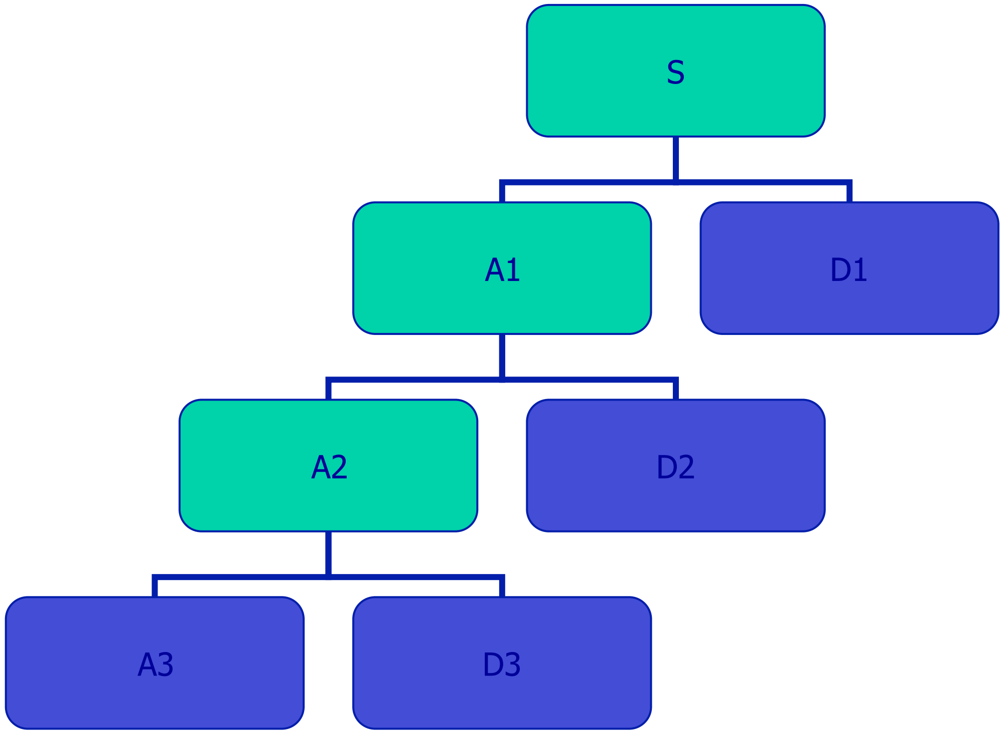
M2009, Chapter 8

Motivation

- Goal
  - Get minimal representation of data relative to particular cost function
- Usage
  - Data compression
  - Noise reduction

Wavelet Transform

- Wavelet transform is applied to low pass results (approximations) only:

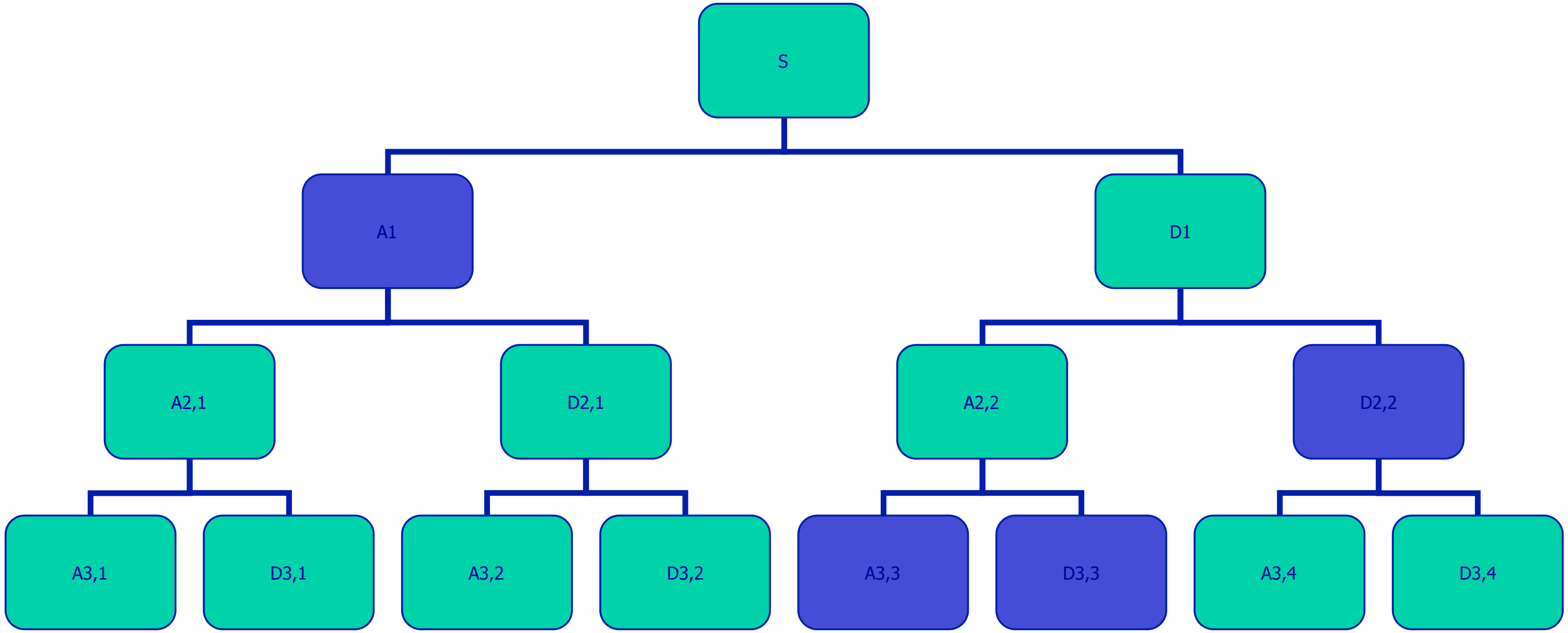


Not optimal

- From the point of view of compression, where we want as many small values as possible, the standard wavelet transform may not produce the best result, since it is limited to wavelet bases that are delated by a power of two with each step.
- It could be that another combination of functions produce a more desirable representation.

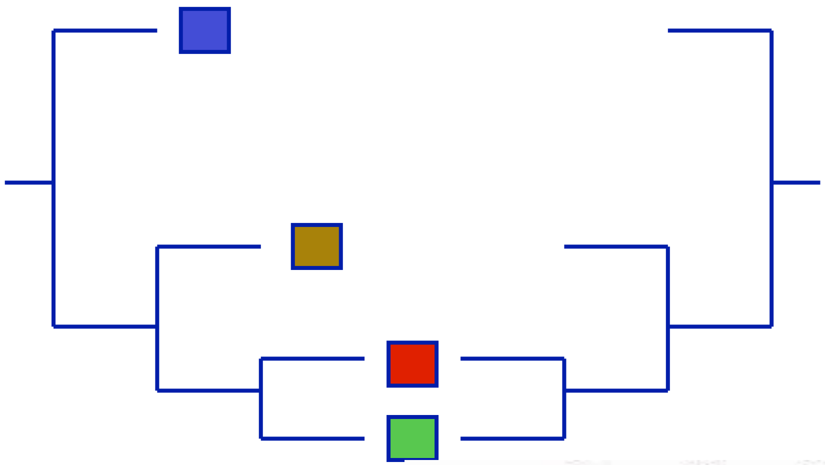
Wavelet Packet Transform

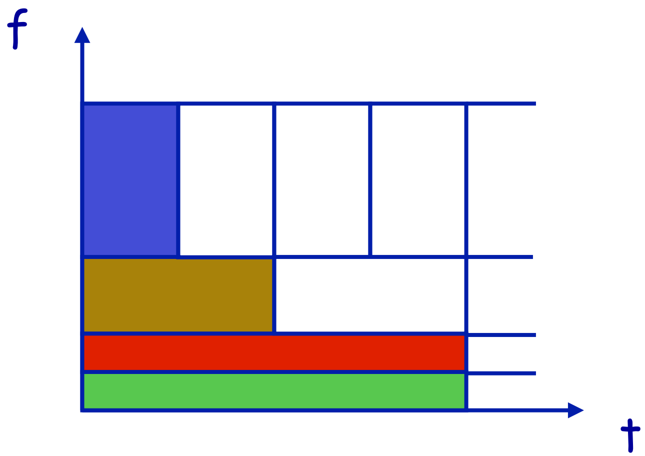
Wavelet packet transform is applied to both low pass results (approximations) and high pass results (details)

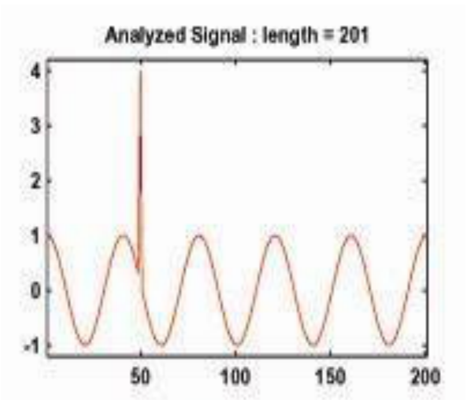


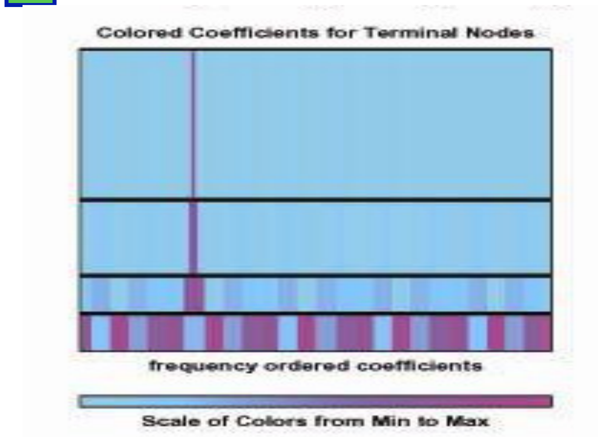
DWT

- Iterate only on the lowpass channel

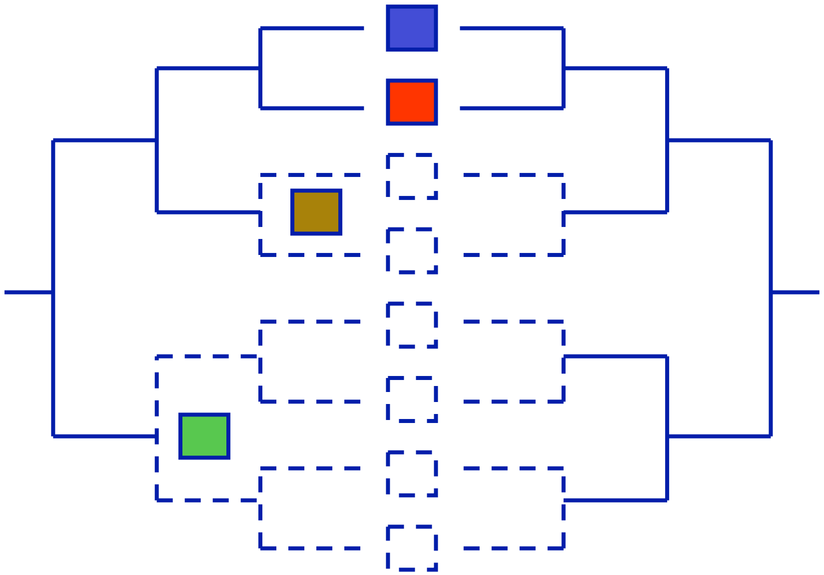


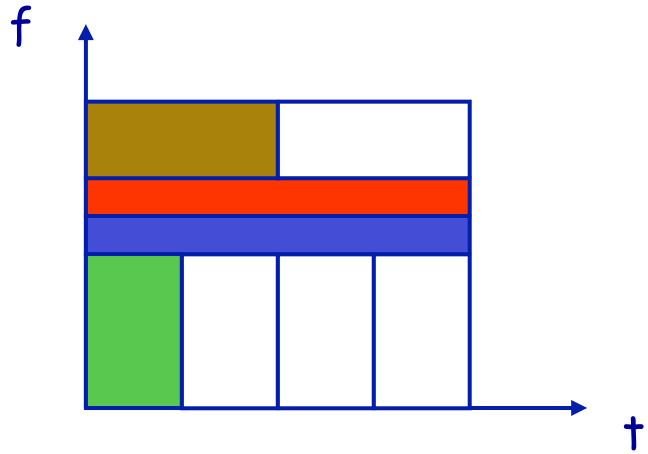






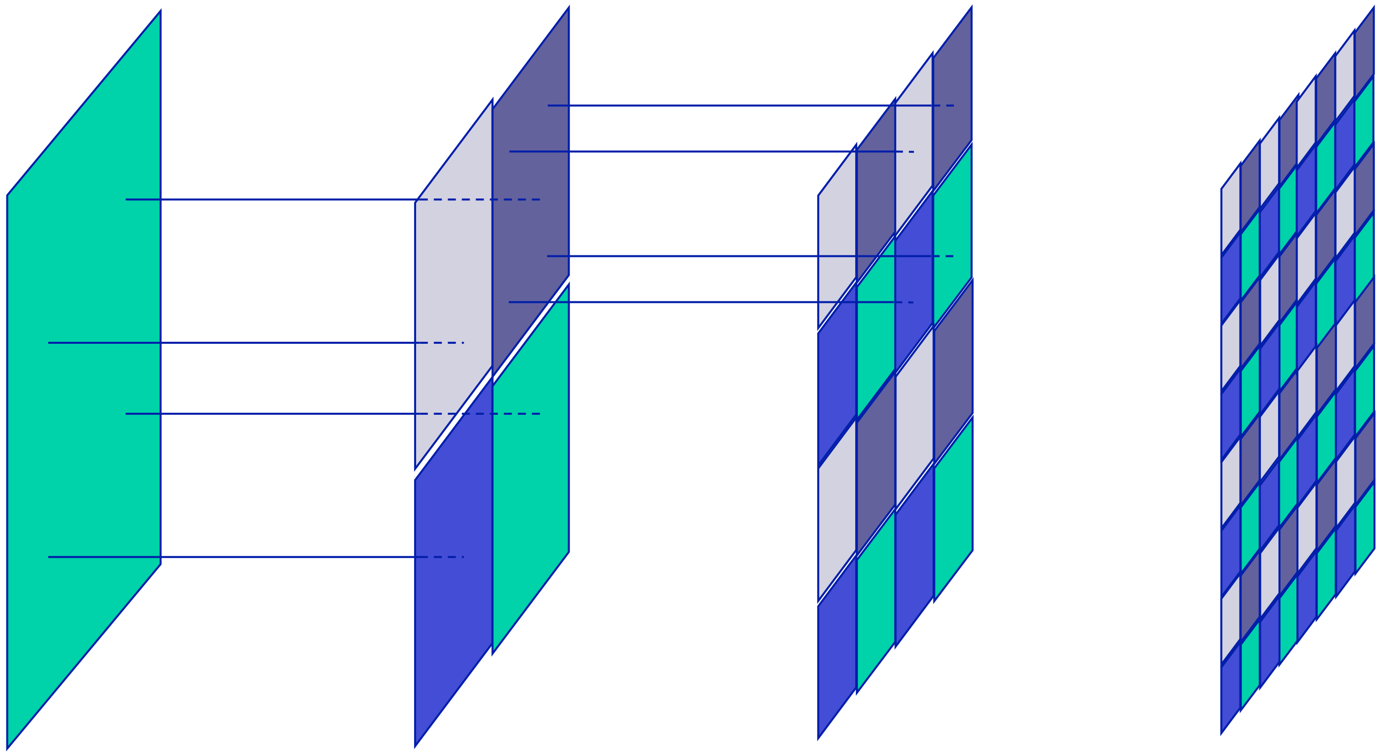
wavelet packet





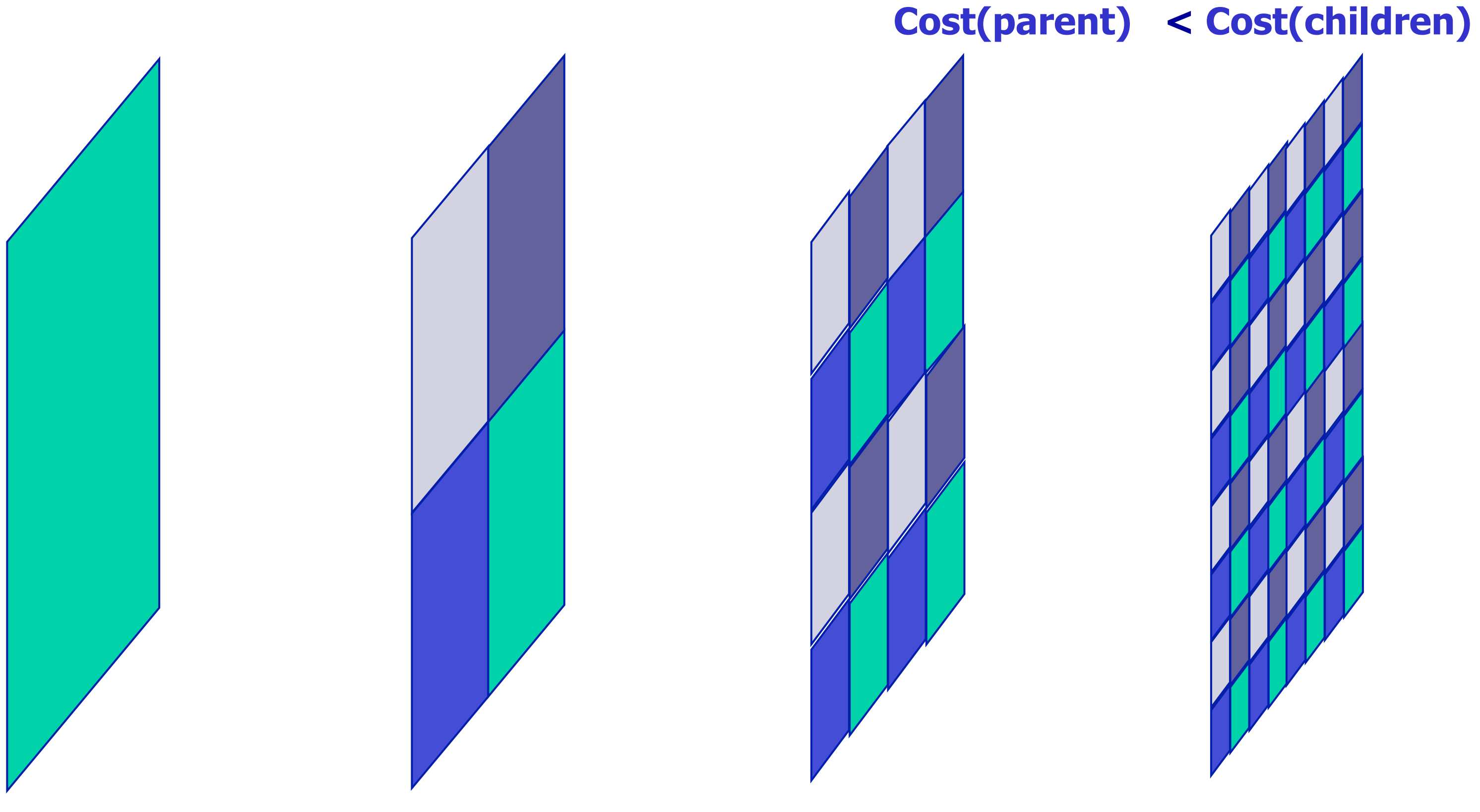
wavelet packet

- First stage: full decomposition



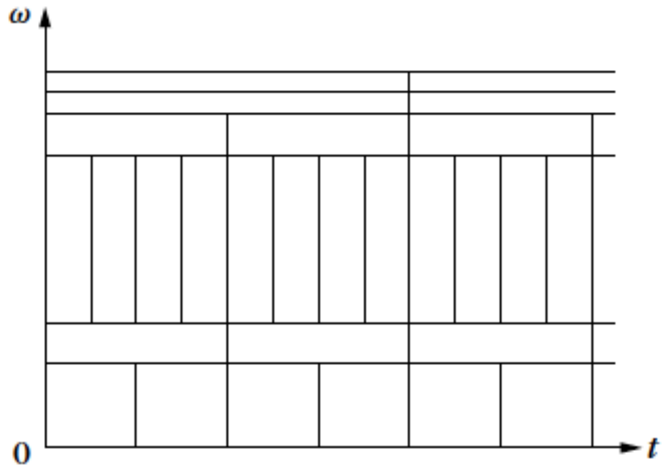
wavelet packet

- Second stage: pruning



Wavelet packets

- Wavelet bases divide the frequency axis into intervals of 1 octave bandwidth. Coifman, Meyer, and Wickerhauser have generalized this construction with bases that split the frequency axis in intervals of bandwidth that may be adjusted.
- Each frequency interval is covered by the Heisenberg time-frequency boxes of wavelet packet functions *translated in time*, in order to cover the whole plane, as shown by Figure 1.7



A wavelet packet basis divides the frequency axis in separate intervals of varying sizes. A tiling is obtained by translating in time the wavelet packets covering each frequency interval.

Wavelet packets (Chapt. 9, M2009)

- Wavelet packets were introduced by Coifman, Meyer, and Wickerhauser [182] by generalizing the link between multiresolution approximations and wavelets.
- A space Vj of a multiresolution approximation is decomposed in a lower-resolution space V_{j+1} plus a detail space W_{j+1} . This is done by dividing the orthogonal basis
- $\{\varphi_j(t-2^j n)\}_{n \in Z}$ of V_j into two new orthogonal bases
- $\{\varphi_{j+1}(t-2^{j+1}n)\}_{n \in Z}$ of V_{j+1} and $\{\psi_{j+1}(t-2^{j+1}n)\}_{n \in Z}$ of W_{j+1}.
- The decomposition is specified by a pair of CMF h[n] and g[n]

$$g[n]=(-1)^{1-n}\, h[1-n]$$

- Theorem 8.1 generalizes this result to any space U_j that admits an orthogonal basis of functions translated by $n2^j$ for $n \in Z$.

Wavelet packets qui

Theorem 8.1: *Coifman, Meyer, Wickerhauser.* Let $\{\theta_j(t-2^j n)\}_{n\in\mathbb{Z}}$ be an orthonormal basis of a space $\mathbf{U}_j$. Let h and g be a pair of conjugate mirror filters. Define

$$\theta^0_{j+1}(t) = \sum_{n=-\infty}^{+\infty} h[n]\,\theta_j(t-2^j n) \quad \text{and} \quad \theta^1_{j+1}(t) = \sum_{n=-\infty}^{+\infty} g[n]\,\theta_j(t-2^j n). \qquad \mathbf{(8.1)}$$

The family

$$\{\theta^0_{j+1}(t-2^{j+1}n),\ \theta^1_{j+1}(t-2^{j+1}n)\}_{n\in\mathbb{Z}}$$

is an orthonormal basis of $\mathbf{U}_j$.

Wavelet packets

Theorem 8.1 proves that conjugate mirror filters transform an orthogonal basis $\{\theta_j(t-2^j n)\}_{n\in\mathbb{Z}}$ in two orthogonal families $\{\theta^0_{j+1}(t-2^{j+1}n)\}_{n\in\mathbb{Z}}$ and $\{\theta^1_{j+1}(t-2^{j+1}n)\}_{n\in\mathbb{Z}}$. Let $\mathbf{U}^0_{j+1}$ and $\mathbf{U}^1_{j+1}$ be the spaces generated by each of these families. Clearly $\mathbf{U}^0_{j+1}$ and $\mathbf{U}^1_{j+1}$ are orthogonal and

$$\mathbf{U}^0_{j+1} \oplus \mathbf{U}^1_{j+1} = \mathbf{U}_j.$$

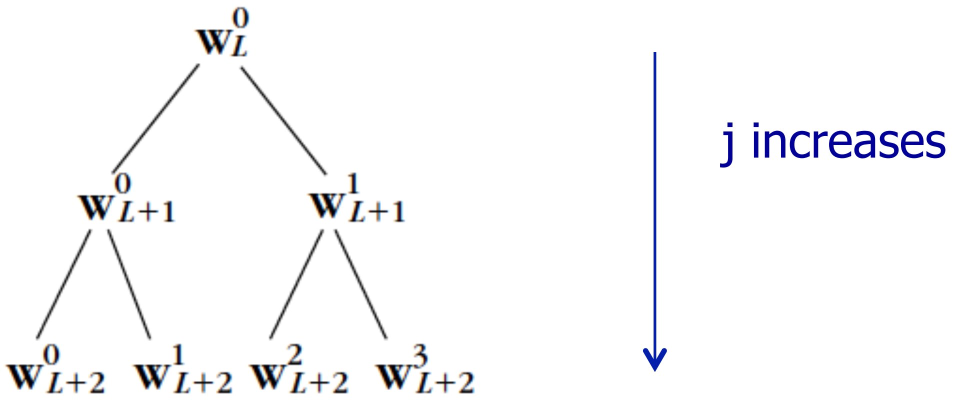
Computing the Fourier transform of (8.1) relates the Fourier transforms of θ^0_{j+1} and θ^1_{j+1} to the Fourier transform of θ_j:

$$\hat{\theta}^0_{j+1}(\omega) = \hat{h}(2^j\omega)\,\hat{\theta}_j(\omega), \quad \hat{\theta}^1_{j+1}(\omega) = \hat{g}(2^j\omega)\,\hat{\theta}_j(\omega). \tag{8.9}$$

Since the transfer functions $\hat{h}(2^j\omega)$ and $\hat{g}(2^j\omega)$ have their energy concentrated in different frequency intervals, this transformation can be interpreted as a division of the frequency support of $\hat{\theta}_j$.

Binary wavelet packet tree

- Instead of dividing only the approximation spaces V_j to construct detail spaces W_j and wavelet bases, Theorem 8.1 proves that we can set $U_j=W_j$ and divide these detail spaces to derive new bases.
- The recursive splitting of vector spaces is represented in a binary tree.
- Any node of the binary tree is labeled by (j, p) , where j≤L is the depth of the node in the tree, and p is the number of nodes that are on its left at the same depth j



Binary wavelet packet tree

- Wavelet packet orthogonal bases at the children node

$$\psi_{j+1}^{2p}(t) = \sum_{n=-\infty}^{+\infty} h[n]\, \psi_j^p(t - 2^j n) \qquad \textbf{(8.10)}$$

and

$$\psi_{j+1}^{2p+1}(t) = \sum_{n=-\infty}^{+\infty} g[n]\, \psi_j^p(t - 2^j n). \qquad \textbf{(8.11)}$$

Since $\{\psi_j^p(t - 2^j n)\}_{n\in\mathbb{Z}}$ is orthonormal,

$$h[n] = \langle \psi_{j+1}^{2p}(u), \psi_j^p(u - 2^j n)\rangle, \quad g[n] = \langle \psi_{j+1}^{2p+1}(u), \psi_j^p(u - 2^j n)\rangle. \qquad \textbf{(8.12)}$$

$$\mathbf{W}_{j+1}^{2p} \oplus \mathbf{W}_{j+1}^{2p+1} = \mathbf{W}_j^p.$$

Example

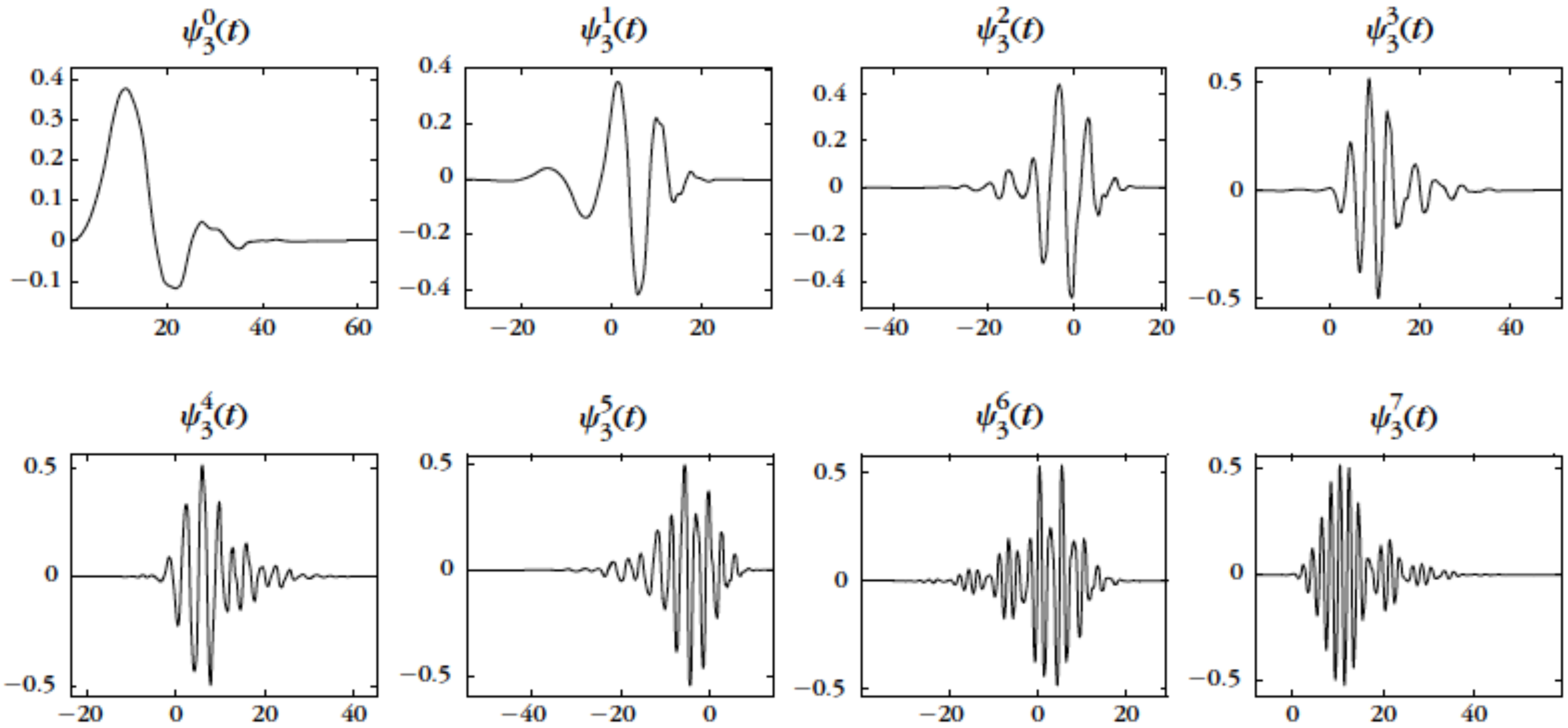


FIGURE 8.2

Wavelet packets computed with a Daubechies 5 filter at the depth $j-L=3$ of the wavelet packet tree, with $L=0$. They are ordered from low to high frequencies.

Admissible tree

Admissible Tree

We call any binary tree where each node has either zero or two children an *admissible tree*, as shown in Figure 8.3. Let $\{j_i, p_i\}_{1 \leq i \leq I}$ be the leaves of an admissible binary tree. By applying the recursive splitting (8.13) along the branches of an admissible tree, we verify that the spaces $\{\mathbf{W}_{j_i}^{p_i}\}_{1 \leq i \leq I}$ are mutually orthogonal and add up to $\mathbf{W}_L^0$:

$$\mathbf{W}_L^0 = \oplus_{i=1}^{I} \mathbf{W}_{j_i}^{p_i}. \tag{8.14}$$

Best basis

- Among the admissible trees, one can select the "best one" with respect to a predefined criterion (cost function)
- The best basis pursuit algorithm finds a set of wavelet bases basically prunes the complete tree under the guidance of the cost function
- A cost function may be chosen to fit a particular application.
  - For example, in a compression algorithm the cost function might be the number of bits needed to represent the result.

Cost function

- The value of the cost function is a real number.
- Given two vectors of finite length, **a** and **b**, we denote their concatenation by [**a b**]. This vector simply consists of the elements in **a** followed by the elements in **b**.
- We require the following two properties:
  - The cost function is additive in the sense that $K([\mathbf{a}\ \mathbf{b}]) = K(\mathbf{a}) + K(\mathbf{b})$ for all finite length vectors **a** and **b**.
  - $K(\mathbf{0}) = 0$, where **0** denotes the zero vector

Cost functions: threshold

- The threshold cost function counts the number of *values* in a wavelet packet tree node whose absolute value is greater than a threshold value ***t***.

$$\text{cost}_{\text{threshold}} = \sum_{i=0}^{N-1} (|s[i]| > t) \; ? \; 1 : 0;$$

➔ Promoting sparsity!

Best basis algorithm

- Compute cost value for each node
- When the wavelet packet tree is constructed, all the leaves are marked with a flag. The best basis calculation is performed bottom up (that is, from the leaves of the tree toward the root):
  - A leaf (a node at the bottom of the tree with no children) returns its cost value.
  - As the calculation recurses up the tree toward the root, if there is a non-leaf node, **v1** is the cost value for that node. The value **v2** is the sum of the cost values of the children of the node.
    - If (**v1** <= **v2**) then we mark the node as part of the best basis set and remove any marks in the nodes in the sub-tree of the current node.
    - If (**v1** > **v2**) then the cost value of the node is replaced with v2.

Best basis contd.

- The best basis set selected by the best basis algorithm is relative to a particular cost function.
- In some cases the best basis set may be the same set yielded by the wavelet transform (in which case we could have used the simpler algorithm).
- In other cases the best basis function may not yield a result that differs from the original data set (e.g., the original data set is already a minimal representation in terms of the cost function).

Other cost functions

- Nonnormalized Shannon (0 log(0)=0)

$$\text{cost}_{shannon} = -\sum_n s[n]^2 \log(s[n]^2)$$

  – The Shannon entropy function provides a measure of the economy of representation of a signal

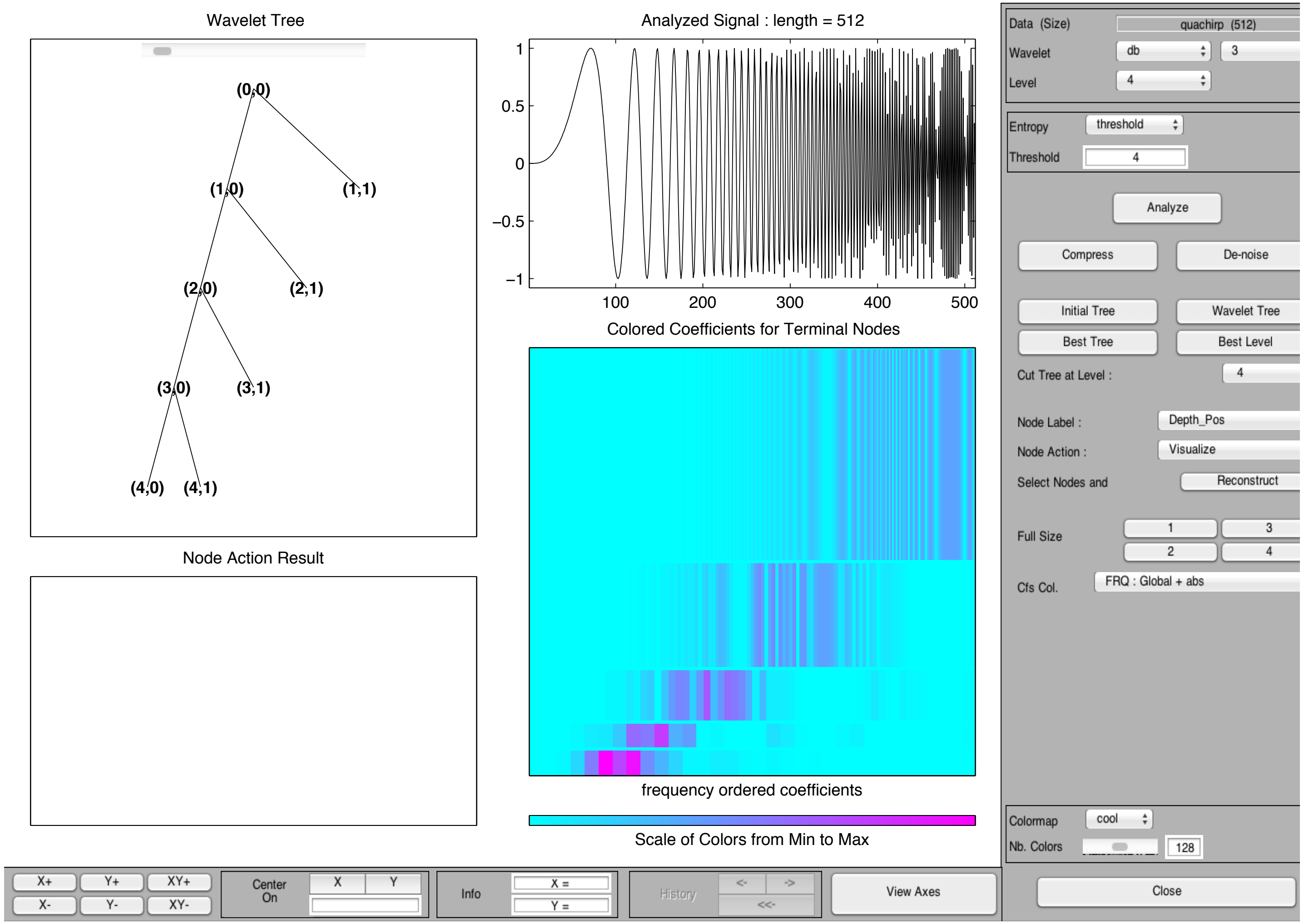
- Concentration in l_p norm $(1 \le p)$

$$\text{cost}_{norm\ p} = \sum_n |s[n]|^p$$

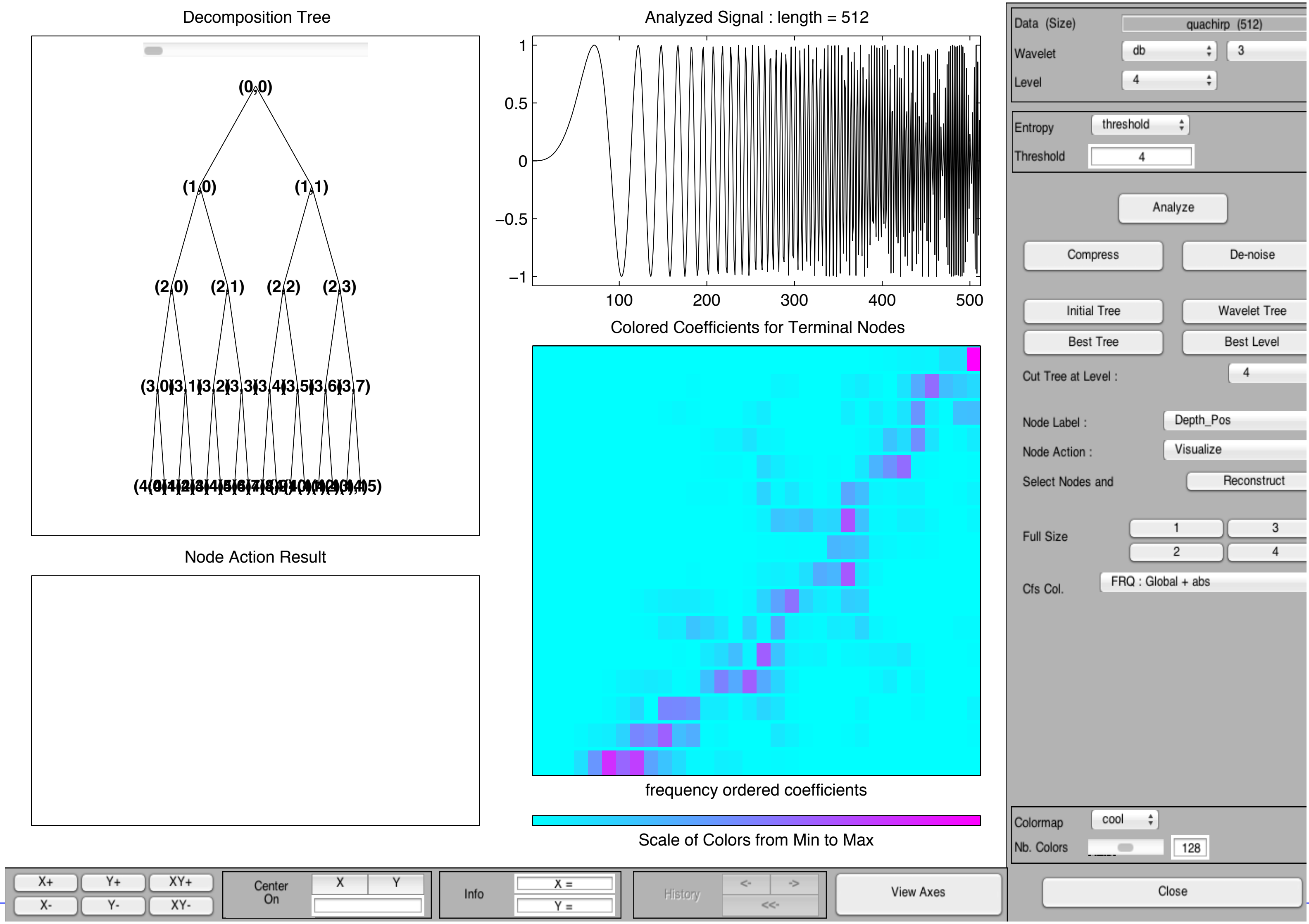
- Logarithm of "energy" (log(0)=0)

$$\text{cost}_{\log} = \sum_n \log(s[n]^2)$$

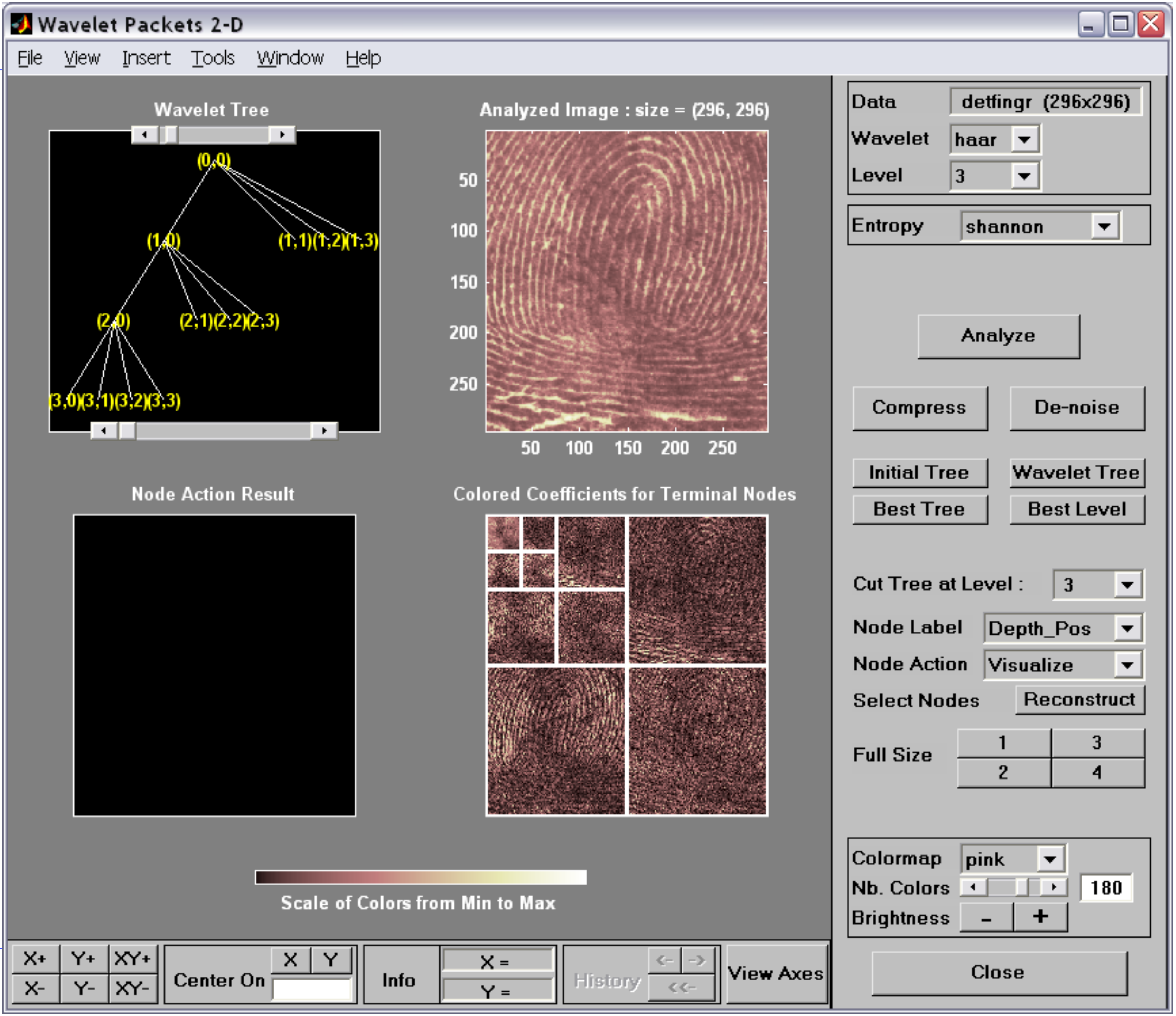
DWT


Wavelet Tree
(0,0)
(1,0)
(1,1)
(2,0)
(2,1)
(3,0)
(3,1)
(4,0)
(4,1)
Node Action Result
Analyzed Signal : length = 512
Colored Coefficients for Terminal Nodes
frequency ordered coefficients
Scale of Colors from Min to Max
Data (Size)
quachirp (512)
Wavelet
db
3
Level
4
Entropy
threshold
Threshold
4
Analyze
Compress
De-noise
Initial Tree
Wavelet Tree
Best Tree
Best Level
Cut Tree at Level :
4
Node Label :
Depth_Pos
Node Action :
Visualize
Select Nodes and
Reconstruct
Full Size
1
3
2
4
Cfs Col.
FRQ : Global + abs
Colormap
cool
Nb. Colors
128
X+
Y+
XY+
X-
Y-
XY-
Center On
X
Y
Info
X =
Y =
History
View Axes
Close

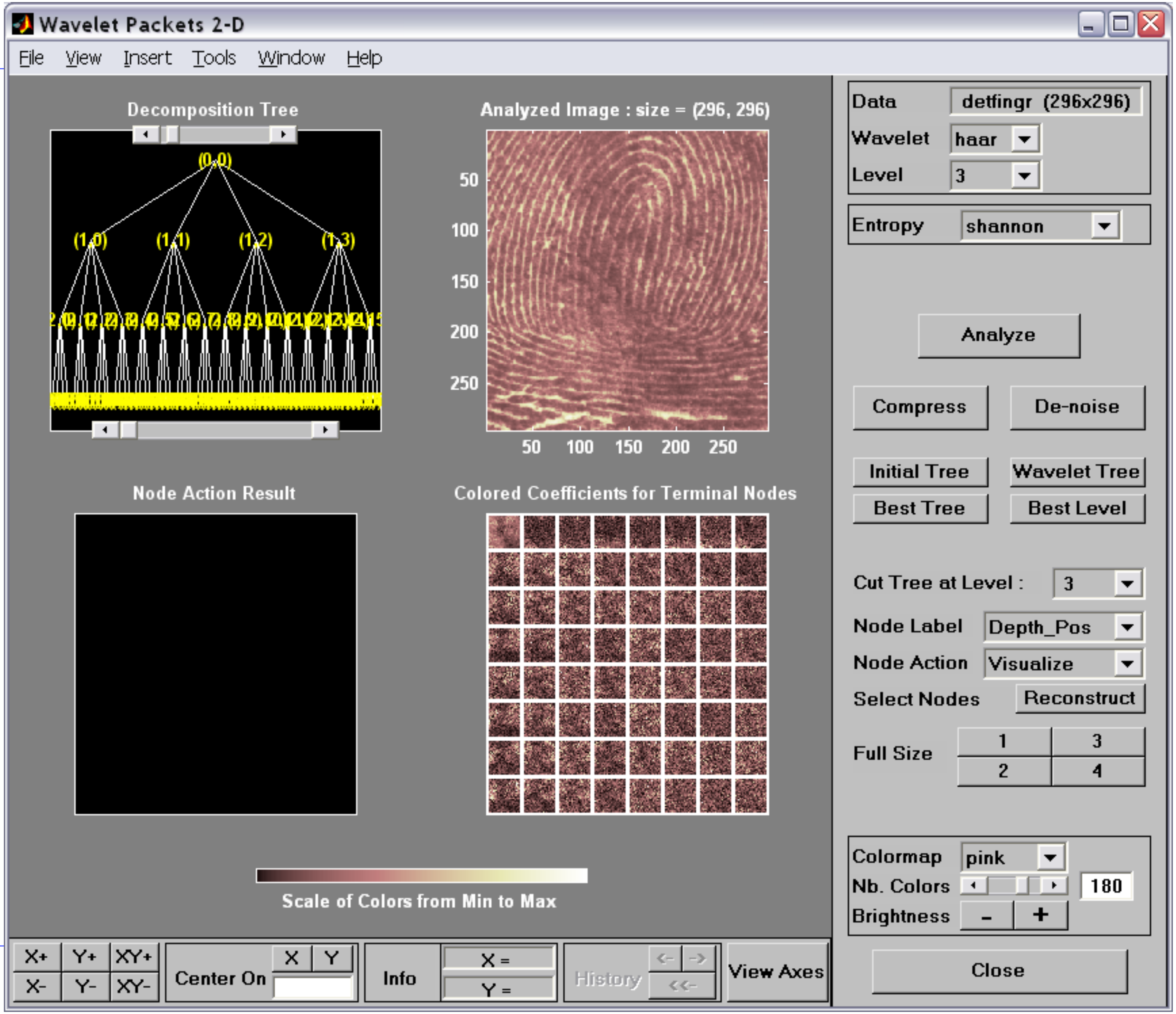
WP



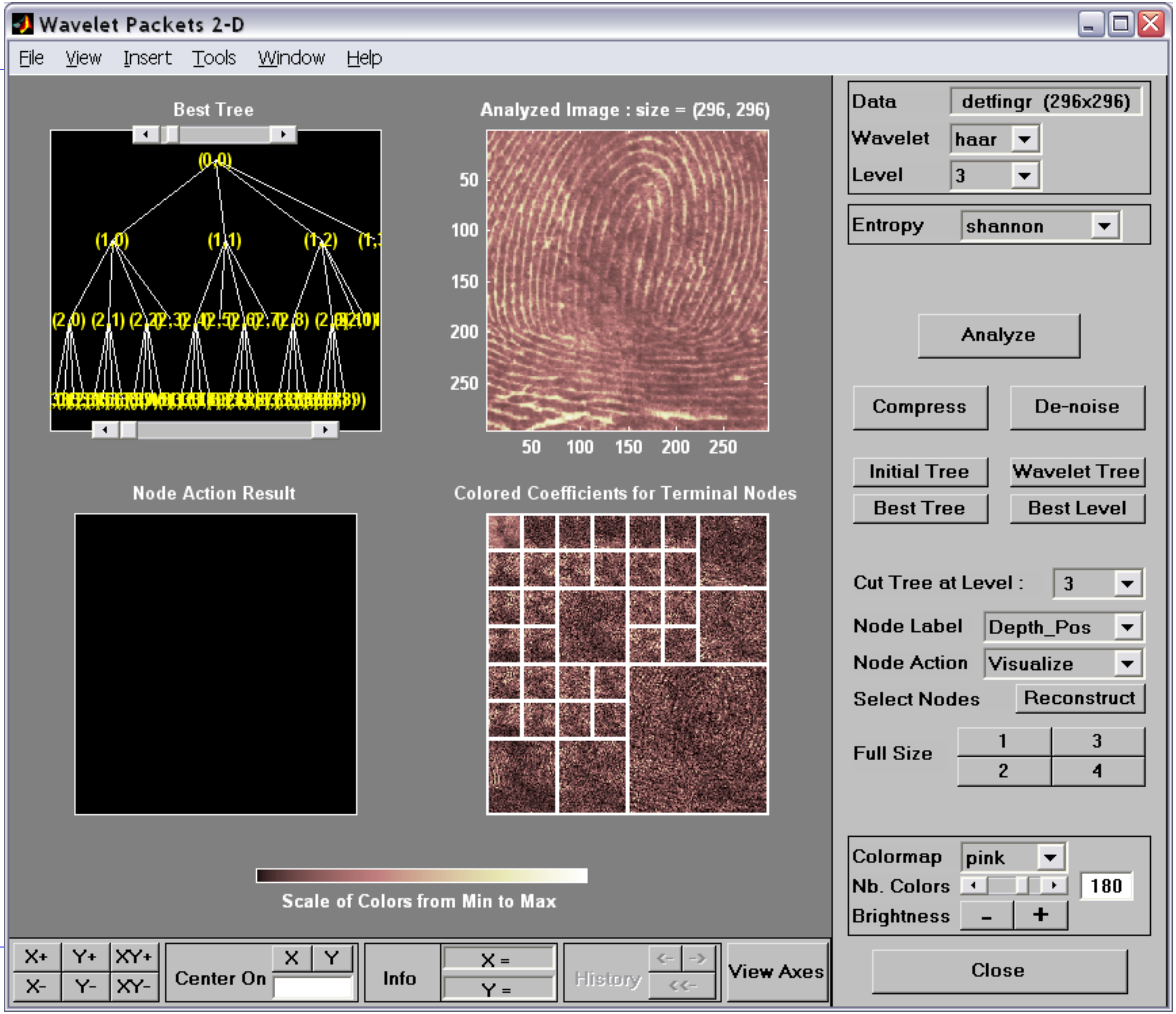
Decomposition Tree
(0,0)
(1,0)
(1,1)
(2,0)
(2,1)
(2,2)
(2,3)
Node Action Result
Analyzed Signal : length = 512
Colored Coefficients for Terminal Nodes
frequency ordered coefficients
Scale of Colors from Min to Max
Data (Size)
quachirp (512)
Wavelet
db
3
Level
4
Entropy
threshold
Threshold
4
Analyze
Compress
De-noise
Initial Tree
Wavelet Tree
Best Tree
Best Level
Cut Tree at Level :
Node Label :
Depth_Pos
Node Action :
Visualize
Select Nodes and
Reconstruct
Full Size
Cfs Col.
FRQ : Global + abs
Colormap
cool
Nb. Colors
128
Center On
Info
History
View Axes
Close

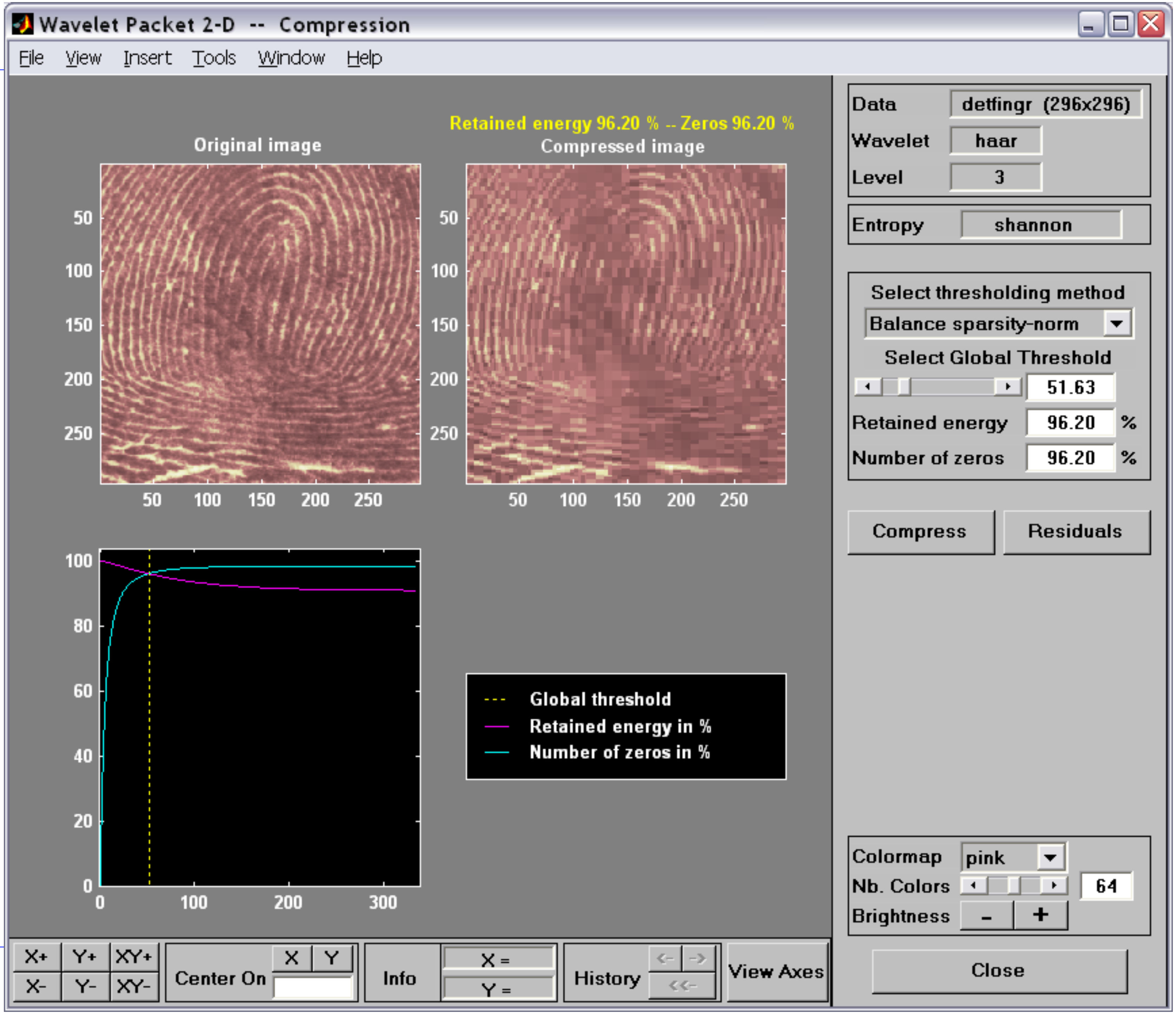


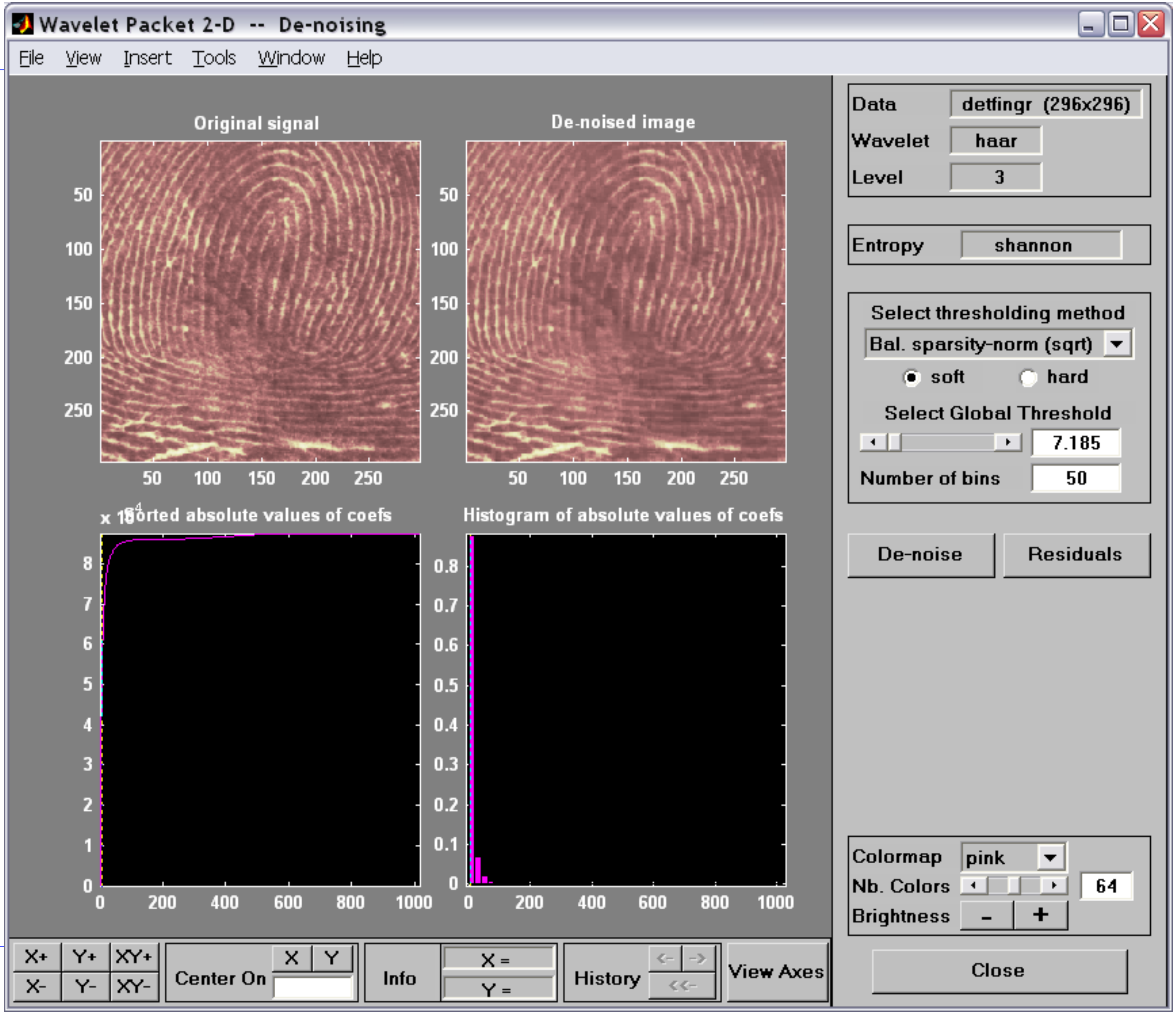
Wavelet Packets 2-D
File View Insert Tools Window Help
Wavelet Tree
(0,0)
(1,0) (1,1)(1,2)(1,3)
(2,0) (2,1)(2,2)(2,3)
(3,0)(3,1)(3,2)(3,3)
Analyzed Image : size = (296, 296)
50 100 150 200 250
Node Action Result
Colored Coefficients for Terminal Nodes
Scale of Colors from Min to Max
Data detfingr (296x296)
Wavelet haar
Level 3
Entropy shannon
Analyze
Compress
De-noise
Initial Tree
Wavelet Tree
Best Tree
Best Level
Cut Tree at Level : 3
Node Label Depth_Pos
Node Action Visualize
Select Nodes Reconstruct
Full Size 1 3 2 4
Colormap pink
Nb. Colors 180
Brightness - +
Close
X+ Y+ XY+
X- Y- XY-
Center On X Y
Info X = Y =
History
View Axes



Wavelet Packets 2-D
File View Insert Tools Window Help
Decomposition Tree
(0,0)
(1,0) (1,1) (1,2) (1,3)
Analyzed Image : size = (296, 296)
50
100
150
200
250
Node Action Result
Colored Coefficients for Terminal Nodes
Scale of Colors from Min to Max
Data detfingr (296x296)
Wavelet haar
Level 3
Entropy shannon
Analyze
Compress
De-noise
Initial Tree
Wavelet Tree
Best Tree
Best Level
Cut Tree at Level : 3
Node Label Depth_Pos
Node Action Visualize
Select Nodes
Reconstruct
Full Size 1 3 2 4
Colormap pink
Nb. Colors 180
Brightness - +
Close
X+ Y+ XY+
X- Y- XY-
Center On X Y
Info
X =
Y =
History
View Axes


Wavelet Packets 2-D
File
View
Insert
Tools
Window
Help
Best Tree
(0,0)
(1,0)
(1,1)
(1,2)
Analyzed Image : size = (296, 296)
50
100
150
200
250
Node Action Result
Colored Coefficients for Terminal Nodes
Scale of Colors from Min to Max
Data
detfingr (296x296)
Wavelet
haar
Level
3
Entropy
shannon
Analyze
Compress
De-noise
Initial Tree
Wavelet Tree
Best Tree
Best Level
Cut Tree at Level :
3
Node Label
Depth_Pos
Node Action
Visualize
Select Nodes
Reconstruct
Full Size
1
2
3
4
Colormap
pink
Nb. Colors
180
Brightness
-
+
Close
X+
Y+
XY+
X-
Y-
XY-
Center On
X
Y
Info
X =
Y =
History
View Axes


Wavelet Packet 2-D -- Compression
File View Insert Tools Window Help
Retained energy 96.20 % -- Zeros 96.20 %
Original image
Compressed image
Global threshold
Retained energy in %
Number of zeros in %
Data detfingr (296x296)
Wavelet haar
Level 3
Entropy shannon
Select thresholding method
Balance sparsity-norm
Select Global Threshold
51.63
Retained energy 96.20 %
Number of zeros 96.20 %
Compress
Residuals
Colormap pink
Nb. Colors 64
Brightness - +
Close
X+ Y+ XY+
X- Y- XY-
Center On X Y
Info
X =
Y =
History
View Axes


Wavelet Packet 2-D -- De-noising
File
View
Insert
Tools
Window
Help
Original signal
De-noised image
Sorted absolute values of coefs
Histogram of absolute values of coefs
Data
detfingr (296x296)
Wavelet
haar
Level
3
Entropy
shannon
Select thresholding method
Bal. sparsity-norm (sqrt)
soft
hard
Select Global Threshold
7.185
Number of bins
50
De-noise
Residuals
Colormap
pink
Nb. Colors
64
Brightness
Close
X+
Y+
XY+
X-
Y-
XY-
Center On
X
Y
Info
X =
Y =
History
View Axes