Teorie e Modelli

let $\mathscr{L}$ be a *countable* first-order language.

By a *theory* in $\mathscr{L}$ we shall mean a set $\mathbf{\Sigma}$ of $\mathscr{L}$-sentences which is closed under deducibility, i.e. such that for each $\mathscr{L}$-sentence $\boldsymbol{\sigma}$, if $\mathbf{\Sigma} \vdash \boldsymbol{\sigma}$, then $\boldsymbol{\sigma} \in \mathbf{\Sigma}$. A subset $\mathbf{\Gamma}$ of a theory $\mathbf{\Sigma}$ is called a *set of postulates* for $\mathbf{\Sigma}$ if $\mathbf{\Gamma} \vdash \boldsymbol{\sigma}$ for every $\boldsymbol{\sigma} \in \mathbf{\Sigma}$. It is clear that each set of sentences $\mathbf{\Gamma}$ is a set of postulates for a unique theory $\mathbf{\Sigma}$, namely,

$$\mathbf{\Sigma} = \{\boldsymbol{\sigma} : \boldsymbol{\sigma} \text{ is an } \mathscr{L}\text{-sentence and } \mathbf{\Gamma} \vdash \boldsymbol{\sigma}\}.$$

A property P of $\mathscr{L}$-structures is called a *first-order* property if there is an $\mathscr{L}$-sentence $\boldsymbol{\sigma}$ such that, for any $\mathscr{L}$-structure $\mathfrak{A}$,

$$\mathfrak{A} \text{ has property } P \Leftrightarrow \mathfrak{A} \models \boldsymbol{\sigma}.$$

Consider, for example, the (first-order) theory of partially ordered sets, **PO**. This theory is formulated in a language $\mathscr{L}$ having one binary predicate symbol **R**. Its postulates are

$$\forall \mathbf{x} \mathbf{R}\mathbf{x}\mathbf{x},$$

$$\forall \mathbf{x} \forall \mathbf{y} [\mathbf{R}\mathbf{x}\mathbf{y} \wedge \mathbf{R}\mathbf{y}\mathbf{x} \rightarrow \mathbf{x} = \mathbf{y}],$$

$$\forall \mathbf{x} \forall \mathbf{y} \forall \mathbf{z} [\mathbf{R}\mathbf{x}\mathbf{y} \wedge \mathbf{R}\mathbf{y}\mathbf{z} \rightarrow \mathbf{R}\mathbf{x}\mathbf{z}].$$

An $\mathscr{L}$-structure $\langle A, R \rangle$ is then a model of **PO** iff it is a partially ordered set. Since a partially ordered set can have many different first-order properties, e.g. it can be a lattice, or a Boolean algebra, or a totally ordered set, etc., it is clear that **PO** does not precisely determine the first-order properties of its models.

Let us call a theory Σ *complete* if it is consistent and the first-order properties of any model of Σ are just those determined by the sentences in Σ. More precisely, if for each $\mathscr{L}$-structure $\mathfrak{A}$ we define $\mathbf{Th}(\mathfrak{A})$, *the theory of* $\mathfrak{A}$, to be the set of all $\mathscr{L}$-sentences $\boldsymbol{\sigma}$ such that $\mathfrak{A} \models \boldsymbol{\sigma}$, then Σ is complete iff Σ is consistent and $\mathbf{Th}(\mathfrak{A}) = \Sigma$ for each model $\mathfrak{A}$ of Σ.

4.1. LEMMA. *The following conditions on a consistent theory* Σ *are equivalent:*

(i) Σ *is complete.*

(iii) *For any* $\mathscr{L}$*-sentence* $\boldsymbol{\sigma}$, *either* $\boldsymbol{\sigma} \in \Sigma$ *or* $\neg\boldsymbol{\sigma} \in \Sigma$.

Let α be an infinite cardinal. A theory Σ is said to be *α-categorical* if any pair of models of Σ of cardinality α are isomorphic. We now give some examples.

4.4. THEOREM. *Let* Σ *be a consistent theory with no finite models, and which is* α*-categorical for some infinite* α. *Then* Σ *is complete.*

UDO

(UDO$_1$) $\forall x Rxx \wedge \forall x \forall y\,[Rxy \wedge Ryx \rightarrow x=y] \wedge \forall x \forall y \forall z [Rxy \wedge Ryz \rightarrow Rxz]$
$\wedge \forall x \forall y\,[Rxy \vee Ryx]$

(UDO$_2$) $\forall x \forall y\,[Rxy \wedge x \neq y \rightarrow \exists z\,[x \neq z \wedge y \neq z \wedge Rxz \wedge Rzy]]$

(UDO$_3$) $\forall x \exists y \exists z\,[x \neq y \wedge x \neq z \wedge Ryx \wedge Rxz]$.

UDO$_1$ asserts that R is a total ordering, UDO$_2$ asserts that R is dense, and UDO$_3$ that it is unbounded both below and above. Natural examples of models of Σ are $\mathfrak{Q}$ and $\mathfrak{R}$, the sets of rational numbers and real numbers, with their natural orderings.

4.5. THEOREM. **UDO** *is* $\aleph_0$-*categorical.*

4.6. COROLLARY. **UDO** *is a complete theory.*

Axiomatic theories

5.1. Definition. For any set Σ of sentences we let T_Σ be the property such that $T_\Sigma(x)$ holds iff x is a SENTENCE belonging to Σ.

5.2. Definition. A theory Σ is *axiomatizable* if there exists a set Γ of postulates for Σ such that T_Γ is recursively enumerable. If such a set Γ of postulates is actually *given* to us so that we can find an r.e. index for T_Γ (in the sense of §11 of Ch. 6), then we say that Σ is *axiomatic.*

5.4. Theorem. *A theory Σ is axiomatizable iff T_Σ is recursively enumerable.*

Arithmetic

the finitely axiomatized theory $\mathbf{\Pi_2}$

(8.1) $\forall v_1(sv_1 \neq s_0)$,

(8.2) $\forall v_1 \forall v_2(sv_1 = sv_2 \rightarrow v_1 = v_2)$,

(8.3) $\forall v_1(v_1 + s_0 = v_1)$,

(8.4) $\forall v_1 \forall v_2[v_1 + sv_2 = s(v_1 + v_2)]$,

(8.5) $\forall v_1(v_1 \times s_0 = s_0)$,

(8.6) $\forall v_1 \forall v_2(v_1 \times sv_2 = v_1 \times v_2 + v_1)$,

(8.7) $\forall v_1(v_1 \leqslant s_0 \rightarrow v_1 = s_0)$,

(8.8) $\forall v_1 \forall v_2(v_1 \leqslant sv_2 \rightarrow v_1 \leqslant v_2 \vee v_1 = sv_2)$,

(8.9) $\forall v_1 \forall v_2(v_1 \leqslant v_2 \vee v_2 \leqslant v_1)$.

first order Peano Arithmetic

We let $\mathbf{\Phi}_n$ be the set of all $\mathscr{L}$-formulas whose free variables are among $\mathbf{v}_1,\ldots,\mathbf{v}_n$ (i.e., the first n variables of $\mathscr{L}$). In particular, $\mathbf{\Phi}_0$ is the set of all $\mathscr{L}$-sentences.

The theory $\mathbf{\Pi}$ which we shall now begin to study has as its postulates the six sentences (8.1)–(8.6) and all sentences of the form

$$(9.1) \qquad \forall \mathbf{v}_2 \ldots \forall \mathbf{v}_k \, [\boldsymbol{\alpha}(\mathbf{s}_0) \rightarrow \forall \mathbf{v}_1 \{\boldsymbol{\alpha} \rightarrow \boldsymbol{\alpha}(\mathbf{s}\mathbf{v}_1)\} \rightarrow \forall \mathbf{v}_1 \boldsymbol{\alpha}],$$

where $k \geqslant 1$ and $\boldsymbol{\alpha} \in \mathbf{\Phi}_k$. Postulates (9.1) are called *induction* postulates.

The postulates of $\mathbf{\Pi}$ were obviously obtained by trying to formalize in $\mathscr{L}$ the seven Peano postulates listed in §2. For this reason $\mathbf{\Pi}$ is usually called *first-order Peano arithmetic*.

A theory Σ is *complete* if it is consistent and for every sentence α we have $\alpha \in \Sigma$ or $\neg\alpha \in \Sigma$.

Baby arithmetic Π_0

(6.1) $s_n + s_0 = s_n$,

(6.2) $s_n + s_{m+1} = s(s_n + s_m)$,

(6.3) $s_n \times s_0 = s_0$,

(6.4) $s_n \times s_{m+1} = (s_n \times s_m) + s_n$,

for all numbers n and m.

Junior arithmetic $\boldsymbol{\Pi}_1$

The theory $\boldsymbol{\Pi}_1$ — which we shall call *junior arithmetic* — is based on postulates (6.1)–(6.4) plus the following:

(7.2) $\quad \mathbf{s}_n \neq \mathbf{s}_m,$

(7.3) $\quad \forall \mathbf{v}_1(\mathbf{v}_1 \leqslant \mathbf{s}_n \leftrightarrow \mathbf{v}_1 = \mathbf{s}_0 \vee \ldots \vee \mathbf{v}_1 = \mathbf{s}_n),$

(7.4) $\quad \forall \mathbf{v}_1(\mathbf{s}_n \leqslant \mathbf{v}_1 \vee \mathbf{v}_1 \leqslant \mathbf{s}_n),$

for all n and all $m \neq n$.

Obviously, $\boldsymbol{\Pi}_1$ is an extension of $\boldsymbol{\Pi}_0$; it is a *proper* extension because, e.g., $\mathbf{s}_0 \neq \mathbf{s}_1$ belongs to $\boldsymbol{\Pi}_1$ but not to $\boldsymbol{\Pi}_0$.

8.11. THEOREM. $\boldsymbol{\Pi}_1 \subseteq \boldsymbol{\Pi}_2$.

11.7. THEOREM. *Every consistent axiomatizable theory that includes* $\mathbf{\Pi}_1$ *is incomplete.*

11.8. FIRST INCOMPLETENESS THEOREM. *Given any consistent axiomatic theory* Σ *that includes* $\mathbf{\Pi}_1$, *we can find a formula* $\gamma \in \mathbf{\Phi}_1$ *(of the form described in Lemma 7.9) such that the sentences* $\gamma(\mathbf{s}_{\#\gamma})$ *and* $\neg\gamma(\mathbf{s}_{\#\gamma})$ *do not belong to* Σ.

(#γ is the code number of γ)

11.9. SECOND INCOMPLETENESS THEOREM. *Let* Σ *be an axiomatizable theory that includes first-order Peano arithmetic. If* Σ *is consistent, then the* $\mathscr{L}$*-sentence asserting this is not in* Σ. ∎

Introduction to Model Theory

any $\mathscr{L}$-structure whose domain is a set may be regarded as an ordered triple

$$\mathfrak{A} = \langle A, \mathscr{R}, \mathfrak{c} \rangle,$$

where

(1) A is a non-empty set called the *domain* or *universe* of $\mathfrak{A}$;

(2) $\mathscr{R}$ is a mapping of I into the set of all relations on A such that for each $i \in I$, $\mathscr{R}(i)$ is a $\lambda(i)$-ary relation;

(3) $\mathfrak{c}$ is a mapping of J into A.

For each $i \in I$ and each $j \in J$ we often write R_i for $\mathscr{R}(i)$ and c_j for $\mathfrak{c}(j)$, and we also write

(4) $\mathfrak{A} = \langle A, \langle R_i \rangle_{i \in I}, \langle c_j \rangle_{j \in J} \rangle$.

The R_i and the c_j are called the *relations* and *designated individuals* of $\mathfrak{A}$, respectively. We shall sometimes write $\mathbf{R}_i^{\mathfrak{A}}$ for R_i and $\mathbf{c}_j^{\mathfrak{A}}$ for c_j, in order to emphasize the fact that R_i is the interpretation of $\mathbf{R}_i$, and c_j that of $\mathbf{c}_j$, in $\mathfrak{A}$.

If $\mathfrak{A}$ is an $\mathscr{L}$-structure, we often call $\mathscr{L}$ the *language for* $\mathfrak{A}$.

Given an $\mathscr{L}$-structure of the form (4), we obtain an *$\mathscr{L}$-valuation* (Chapter 2, §1) by further specifying a sequence

$$\mathfrak{a} = \langle a_0, a_1, \ldots \rangle$$

of members of A as an assignment of values to the variables $\mathbf{v}_0, \mathbf{v}_1, \ldots$ of $\mathscr{L}$. We shall call such a sequence an *assignment in* $\mathfrak{A}$.

$$\mathfrak{a}: \mathrm{VAR} \to \mathrm{A}$$

If $\mathfrak{a}$ is an assignment in an $\mathscr{L}$-structure $\mathfrak{A}$ and $b \in A$, we define $\mathfrak{a}(n|b)$ to be the assignment which assigns the same values to the variables as does $\mathfrak{a}$, *except* that it assigns the value b to the variable $\mathbf{v}_n$. Thus

$$\mathfrak{a}(n|b) = \langle a_0, a_1, \ldots, a_{n-1}, b, a_{n+1}, \ldots \rangle.$$

Let $\mathscr{L}$ be a language with predicate symbols $\{\mathbf{R}_i\colon\ i\in I\}$, constant symbols $\{\mathbf{c}_j\colon\ j\in J\}$ and signature λ. Let

$$\mathfrak{A}=\langle A, \langle R_i\rangle_{i\in I}, \langle c_j\rangle_{j\in J}\rangle$$

be an $\mathscr{L}$-structure, and let $\mathfrak{a}=\langle a_0,a_1,\ldots\rangle$ be an assignment in $\mathfrak{A}$. For all $\mathscr{L}$-formulas $\boldsymbol{\varphi}$ we define the relation $\mathfrak{a}$ *satisfies* $\boldsymbol{\varphi}$ *in* $\mathfrak{A}$, which we write $\mathfrak{A}\models_{\mathfrak{a}}\boldsymbol{\varphi}$, by induction on deg $\boldsymbol{\varphi}$:

(1) For terms $\mathbf{t}_1, \mathbf{t}_2$ of $\mathscr{L}$,

$$\mathfrak{A}\models_{\mathfrak{a}} \mathbf{t}_1\mathbf{=}\mathbf{t}_2 \ \Leftrightarrow\ b_1=b_2,$$

where if $\mathbf{t}_n$ $(n=1, 2)$ is the variable $\mathbf{v}_k$ then b_n is a_k, while if $\mathbf{t}_n$ is the constant $\mathbf{c}_j$ then b_n is c_j.

(2) For $i\in I$ and terms $\mathbf{t}_1,\ldots,\mathbf{t}_{\lambda(i)}$ of $\mathscr{L}$,

$$\mathfrak{A}\models_{\mathfrak{a}} \mathbf{R}_i\mathbf{t}_1\ldots\mathbf{t}_{\lambda(i)} \ \Leftrightarrow\ \langle b_1,\ldots,b_{\lambda(i)}\rangle\in R_i,$$

where if $\mathbf{t}_n$ $(n=1,\ldots,\lambda(i))$ is the variable $\mathbf{v}_k$ then b_n is a_k, while if $\mathbf{t}_n$ is the constant $\mathbf{c}_j$ then b_n is c_j.

(3) $\mathfrak{A}\models_{\mathfrak{a}} \neg\boldsymbol{\varphi} \ \Leftrightarrow$ not $\mathfrak{A}\models_{\mathfrak{a}}\boldsymbol{\varphi}$.

(4) $\mathfrak{A}\models_{\mathfrak{a}} \boldsymbol{\varphi}\wedge\boldsymbol{\psi} \ \Leftrightarrow\ \mathfrak{A}\models_{\mathfrak{a}}\boldsymbol{\varphi}$ and $\mathfrak{A}\models_{\mathfrak{a}}\boldsymbol{\psi}$.

(5) $\mathfrak{A}\models_{\mathfrak{a}} \exists v_n\boldsymbol{\varphi} \ \Leftrightarrow\ \mathfrak{A}\models_{\mathfrak{a}(n|b)}\boldsymbol{\varphi}$ for some $b\in A$.

It should be clear that the above definition does not differ essentially from that given in 2.1.1. In fact, it is easy to verify that if $\mathfrak{A}$ is an $\mathscr{L}$-structure and $\mathfrak{a}$ is an assignment in $\mathfrak{A}$, then, if σ is the $\mathscr{L}$-valuation determined by $\mathfrak{A}$, $\mathfrak{a}$, we have, for each $\mathscr{L}$-formula φ,

$$\mathfrak{A} \models_{\mathfrak{a}} \varphi \iff \varphi^{\sigma} = \top .$$

The following facts are clear:

(a) $\mathfrak{A} \models_{\mathfrak{a}} \forall \mathbf{v}_n \, \varphi \iff \mathfrak{A} \models_{\mathfrak{a}(n|b)} \varphi$ for all $b \in A$;

(b) if φ is a formula and $\mathfrak{a}$, $\mathfrak{a}'$ are assignments in $\mathfrak{A}$ such that $a_n = a'_n$ whenever $\mathbf{v}_n$ occurs free in φ, then $\mathfrak{A} \models_{\mathfrak{a}} \varphi \iff \mathfrak{A} \models_{\mathfrak{a}'} \varphi$. (See Thm. 2.2.3.)

In view of fact (b), the truth of $\mathfrak{A} \models_{\mathfrak{a}} \varphi$, insofar as it depends on $\mathfrak{a}$, depends only on the values $\mathfrak{a}$ assigns to the *free* variables of φ. Accordingly we make the following definition: if φ is a formula all of whose free variables are among $\mathbf{v}_0, \ldots, \mathbf{v}_n$ and $a_0, \ldots, a_n \in A$, we say that the finite sequence $a_0, \ldots, a_n$ *satisfies* φ in $\mathfrak{A}$ and write

$$\mathfrak{A} \models \varphi \, [a_0, \ldots, a_n]$$

if $\mathfrak{A} \models_{\mathfrak{a}'} \varphi$ for some assignment $\mathfrak{a}'$ in A such that $a'_0 = a_0, \ldots, a'_n = a_n$. It follows immediately from (b) that $\mathfrak{A} \models \varphi \, [a_0, \ldots, a_n]$ iff $\mathfrak{A} \models_{\mathfrak{a}'} \varphi$ for *all* assignments $\mathfrak{a}'$ in $\mathfrak{A}$ such that $a'_0 = a_0, \ldots, a'_n = a_n$.

If $\boldsymbol{\sigma}$ is a *sentence*, i.e. a formula *without* free variables, we say that $\boldsymbol{\sigma}$ is *valid* or *holds* in $\mathfrak{A}$, or that $\mathfrak{A}$ is a *model* of $\boldsymbol{\sigma}$, and write

$$\mathfrak{A} \models \boldsymbol{\sigma},$$

if $\mathfrak{A} \models_{\mathfrak{a}} \boldsymbol{\sigma}$ for some assignment — and hence, in view of (b), all assignments — $\mathfrak{a}$ in $\mathfrak{A}$. If $\boldsymbol{\Sigma}$ is a set of sentences, we say that $\mathfrak{A}$ is a *model* of $\boldsymbol{\Sigma}$ and write

$$\mathfrak{A} \models \boldsymbol{\Sigma}$$

if $\mathfrak{A}$ is a model of each sentence in $\boldsymbol{\Sigma}$.

Let $\mathscr{L}'$ be a language which is an extension of $\mathscr{L}$, so that, in addition to the predicate symbols and constant symbols of $\mathscr{L}$, $\mathscr{L}'$ contains a set $\{\mathbf{R}_i\colon\ i \in I'\}$ of predicate symbols and a set $\{\mathbf{c}_j\colon\ j \in J'\}$ of constants. Given an $\mathscr{L}'$-structure

$$\mathfrak{A}' = \langle A, \langle R_i \rangle_{i \in I \cup I'}, \langle c_j \rangle_{j \in J \cup J'} \rangle,$$

the $\mathscr{L}$-structure

$$\mathfrak{A} = \langle A, \langle R_i \rangle_{i \in I}, \langle c_j \rangle_{j \in J} \rangle$$

is called the *$\mathscr{L}$-reduction of* $\mathfrak{A}'$, and $\mathfrak{A}'$ is called an *$\mathscr{L}'$-expansion* of $\mathfrak{A}$ (cf. Ch. 2, §9). (Notice that in general an $\mathscr{L}$-structure has more than one $\mathscr{L}'$-expansion.)

Let

$$\mathfrak{A} = \langle A, \langle R_i \rangle_{i \in I}, \langle c_j \rangle_{j \in J} \rangle,$$

$$\mathfrak{A}' = \langle A', \langle R_i' \rangle_{i \in I}, \langle c_j' \rangle_{j \in J} \rangle$$

be $\mathscr{L}$-structures. We say that $\mathfrak{A}$ is a *substructure* of $\mathfrak{A}'$ and write $\mathfrak{A} \subseteq \mathfrak{A}'$ if $A \subseteq A'$, for each $j \in J$, $c_j = c_j'$, and, for each $i \in I$, R_i is the restriction of R_i' to A, i.e. $R_i = R_i' \cap A^{\lambda(i)}$. If B is a non-empty subset of A which contains all the designated individuals c_j of $\mathfrak{A}$, we define the *restriction* $\mathfrak{A}|B$ of $\mathfrak{A}$ to B by

$$\mathfrak{A}|B = \langle B, \langle R_i \cap B^{\lambda(i)} \rangle_{i \in I}, \langle c_j \rangle_{j \in J} \rangle.$$

It is clear that for any subset B of A which contains all the designated individuals of $\mathfrak{A}$ we have $\mathfrak{A}|B \subseteq \mathfrak{A}$.

An *embedding* of $\mathfrak{A}$ into $\mathfrak{A}'$ is a one–one mapping f of A into A' such that

(i) $f(c_j) = c_j'$ for all $j \in J$;
(ii) $\langle a_1, \ldots, a_{\lambda(i)} \rangle \in R_i \Leftrightarrow \langle f(a_1), \ldots, f(a_{\lambda(i)}) \rangle \in R_i'$

for all $i \in I$ and all $a_1, \ldots, a_{\lambda(i)} \in A$.

If f is an embedding of $\mathfrak{A}$ into $\mathfrak{A}'$, it follows from (i) that $f[A]$ contains all the designated individuals of $\mathfrak{A}'$, so that we can form the restriction $\mathfrak{A}'|f[A]$. This is written $f[\mathfrak{A}]$ and is called the *image* of $\mathfrak{A}$ under f.

An *isomorphism* of $\mathfrak{A}$ onto $\mathfrak{A}'$ is an embedding of $\mathfrak{A}$ *onto* $\mathfrak{A}'$. If there is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{A}'$, we say that $\mathfrak{A}$ and $\mathfrak{A}'$ are *isomorphic* and write $\mathfrak{A} \cong \mathfrak{A}'$. Clearly, if f is an embedding of $\mathfrak{A}$ into $\mathfrak{A}'$, we have $\mathfrak{A} \cong f[\mathfrak{A}]$.

$\mathfrak{A}$ and $\mathfrak{A}'$ are said to be ($\mathscr{L}$-)*elementarily equivalent*, and we write $\mathfrak{A} \equiv \mathfrak{A}'$, if for any $\mathscr{L}$-sentence $\boldsymbol{\sigma}$ we have $\mathfrak{A} \models \boldsymbol{\sigma} \Leftrightarrow \mathfrak{A}' \models \boldsymbol{\sigma}$. Thus two $\mathscr{L}$-structures are elementarily equivalent if they cannot be distinguished by an $\mathscr{L}$-sentence.

1.1. PROBLEM. Let f be an isomorphism of $\mathfrak{A}$ onto $\mathfrak{A}'$. Show by induction on deg $\boldsymbol{\varphi}$ that for any formula $\boldsymbol{\varphi}$ all of whose free variables are among $\mathbf{v}_0,\ldots,\mathbf{v}_n$ and all $a_0,\ldots,a_n \in A$ we have

$$\mathfrak{A} \models \boldsymbol{\varphi}\,[a_0,\ldots,a_n] \Leftrightarrow \mathfrak{A}' \models \boldsymbol{\varphi}\,[f(a_0),\ldots,f(a_n)].$$

Infer that, *if* $\mathfrak{A} \cong \mathfrak{A}'$, *then* $\mathfrak{A} \equiv \mathfrak{A}'$. (We shall see later on that the converse is false.)

$\mathfrak{A}$ is said to be an ($\mathscr{L}$-)*elementary substructure* of $\mathfrak{A}'$, and $\mathfrak{A}'$ an ($\mathscr{L}$-)*elementary extension* of $\mathfrak{A}$ if $\mathfrak{A} \subseteq \mathfrak{A}'$ and for any $\mathscr{L}$-formula $\boldsymbol{\varphi}$ all of whose free variables are among $\mathbf{v}_0,\ldots,\mathbf{v}_n$ we have

$$\mathfrak{A} \models \boldsymbol{\varphi}\,[a_0,\ldots,a_n] \;\Leftrightarrow\; \mathfrak{A}' \models \boldsymbol{\varphi}\,[a_0,\ldots,a_n]$$

for all $a_0,\ldots,a_n \in A$. In this situation we write $\mathfrak{A} \prec \mathfrak{A}'$.

It is clear that $\mathfrak{A} \prec \mathfrak{A}' \Rightarrow \mathfrak{A} \equiv \mathfrak{A}'$. Our next problem shows that the converse is false.

1.2. PROBLEM. Let $\mathfrak{A} = \langle \omega - \{0\}, < \rangle$, $\mathfrak{A}' = \langle \omega, < \rangle$ where $<$ is the usual ordering of the natural numbers. Show that $\mathfrak{A} \cong \mathfrak{A}'$, $\mathfrak{A} \subseteq \mathfrak{A}'$, but not $\mathfrak{A} \prec \mathfrak{A}'$.

1.3. PROBLEM. Show that if $\mathfrak{A} \prec \mathfrak{A}'$, $\mathfrak{A}'' \prec \mathfrak{A}'$ and $\mathfrak{A} \subseteq \mathfrak{A}''$, then $\mathfrak{A} \prec \mathfrak{A}''$.

An embedding f of $\mathfrak{A}$ into $\mathfrak{A}'$ is called an ($\mathscr{L}$-)*elementary embedding* if for any $\mathscr{L}$-formula $\boldsymbol{\varphi}$ all of whose free variables are among $\mathbf{v}_0,\ldots,\mathbf{v}_n$ we have

$$\mathfrak{A} \models \boldsymbol{\varphi}\,[a_0,\ldots,a_n] \;\Leftrightarrow\; \mathfrak{A}' \models \boldsymbol{\varphi}\,[f(a_0),\ldots,f(a_n)]$$

for all $a_0,\ldots,a_n \in A$.

1.4. Problem. (i) Let f be an embedding of $\mathfrak{A}$ into $\mathfrak{A}'$. Show that f is an elementary embedding iff $f[\mathfrak{A}] \prec \mathfrak{A}'$.

(ii) Let $\mathfrak{A} \subseteq \mathfrak{A}'$. Show that $\mathfrak{A} \prec \mathfrak{A}'$ iff the natural injection of $\mathfrak{A}$ into $\mathfrak{A}'$ is an elementary embedding of $\mathfrak{A}$ into $\mathfrak{A}'$.

(iii) Let f be any mapping of A into A' such that, for all $a_0,\ldots,a_n \in A$ and all formulas $\boldsymbol{\varphi}$ with free variables among $\mathbf{v}_0,\ldots,\mathbf{v}_n$, $\mathfrak{A} \models \boldsymbol{\varphi}\,[a_0,\ldots,a_n] \Leftrightarrow$ $\Leftrightarrow \mathfrak{A}' \models \boldsymbol{\varphi}\,[f(a_0),\ldots,f(a_n)]$. Show that f is one–one and hence an elementary embedding of $\mathfrak{A}$ into $\mathfrak{A}'$.

$\mathfrak{A}$ is said to be *elementarily embeddable* in $\mathfrak{A}'$ if there is an elementary embedding of $\mathfrak{A}$ into $\mathfrak{A}'$. Clearly $\mathfrak{A}$ is elementarily embeddable in $\mathfrak{A}'$ iff $\mathfrak{A}$ is isomorphic to an elementary substructure of $\mathfrak{A}'$. Evidently, also, if $\mathfrak{A}$ is elementarily embeddable in $\mathfrak{A}'$, then $\mathfrak{A}$ is elementarily equivalent to $\mathfrak{A}'$.

We now prove some lemmas which will be very useful later.

The Löwenheim–Skolem Theorems

2.1. THEOREM. *Let* $\mathfrak{A}$ *be an infinite* $\mathscr{L}$*-structure, and let* $X \subseteq A$. *Then for any cardinal* α *satisfying* $\max\{|X|, \|\mathscr{L}\|\} \leqslant \alpha \leqslant \|\mathfrak{A}\|$ *there is an elementary substructure* $\mathfrak{B}$ *of* $\mathfrak{A}$ *such that* $\|\mathfrak{B}\| = \alpha$ *and* $X \subseteq B$.

2.2. DOWNWARD LÖWENHEIM–SKOLEM THEOREM. *Let* Σ *be a set of sentences of* $\mathscr{L}$ *with an infinite model of cardinality* $\alpha > |\Sigma|$. *Then* Σ *has a model of any cardinality* β *such that* $\max(|\Sigma|, \aleph_0) < \beta < \alpha$.

2.3. COROLLARY. *Any countable set of sentences with an infinite model has a countable model.* $\blacksquare$

2.4. LEMMA. *Let* $\mathfrak{A}$ *be an* $\mathscr{L}$*-structure. Then, for any cardinal* $\alpha \geqslant \|\mathfrak{A}\|$, $\mathfrak{A}$ *has an elementary extension of cardinality* α *iff* $\mathfrak{A}$ *is elementarily embeddable in a structure of cardinality* α.

2.5. COMPACTNESS THEOREM. *If each finite subset of a set* Σ *of sentences of* $\mathscr{L}$ *has a model, then* Σ *has a model.* ∎

2.6. THEOREM. *Let* $\mathfrak{A}$ *be an infinite* $\mathscr{L}$*-structure. Then* $\mathfrak{A}$ *has an elementary extension of any cardinality* $\alpha \geqslant \max(\|\mathfrak{A}\|, \|\mathscr{L}\|)$.

2.7. UPWARD LÖWENHEIM–SKOLEM THEOREM. *Let* Σ *be a set of* $\mathscr{L}$*-sentences with a model of cardinality* $\alpha \geqslant \aleph_0$. *Then* Σ *has a model of any cardinality* $\geqslant \max(\alpha, |\Sigma|)$.

2.11. PROBLEM. (i) Use the compactness theorem to show that if a set of sentences Σ has arbitrarily large finite models, then it has an infinite model. (Show that each finite subset of the set $\Sigma \cup \{\sigma_n : n \in \omega\}$ has a model, where σ_n is a sentence which asserts that there are at least n distinct individuals.)

1) $\sigma_n \equiv \exists x_1 - x_n \,.\, x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge \cdots \wedge x_{n-1} \neq x_n$

2) Ogni sottoinsieme finito di $\Sigma \cup \{\sigma_n\}_{n<\omega} = \Sigma' \cup \{\sigma_{i_1}, \ldots, \sigma_{i_k}\}$ ha un modello (basta prendere un modello di Σ che abbia almeno $i_j = \max\{i_1, \ldots, i_k\}$ elementi)

3) per il teor. di compattezza $\Sigma \cup \{\sigma_n\}_{n<\omega}$ ha un modello. $\mathfrak{A}$

4) $|\mathfrak{A}| \geq \aleph_0$: infatti se $|\mathfrak{A}| = m$ avremmo che $\mathfrak{A} \not\models \sigma_{m+1}$

5) $\mathfrak{A}$ è anche un modello per Σ