Fourier analysis of discrete-time signals

(Lathi Chapt. 10 and these slides)

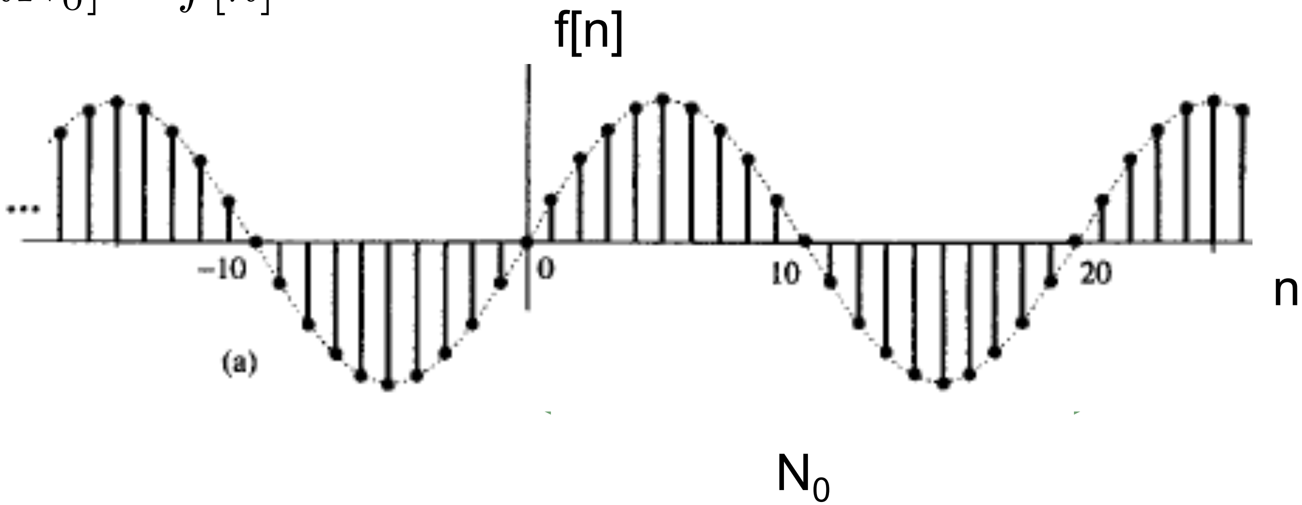
Towards the discrete-time Fourier transform

- How we will get there?
  - Periodic discrete-time signal representation by Discrete-time Fourier series
  - Extension to non-periodic DT signals using the “periodization trick”
  - Derivation of the Discrete Time Fourier Transform (DTFT)
  - Discrete Fourier Transform

Discrete-time periodic signals

- A periodic DT signal of period N_0 is called N_0-periodic signal

$$f[n + kN_0] = f[n]$$



- For the frequency it is customary to use a different notation: the frequency of a DT sinusoid with period N_0 is

$$\Omega_0 = \frac{2\pi}{N_0}$$

Fourier series representation of DT periodic signals

- DT N_0-periodic signals can be represented by DTFS with fundamental frequency $\Omega_0 = \frac{2\pi}{N_0}$ and its multiples

- The exponential DT exponential basis functions are

$$e^{0k}, e^{\pm j\Omega_0 k}, e^{\pm j2\Omega_0 k}, \ldots, e^{\pm jn\Omega_0 k} \quad \text{Discrete time}$$

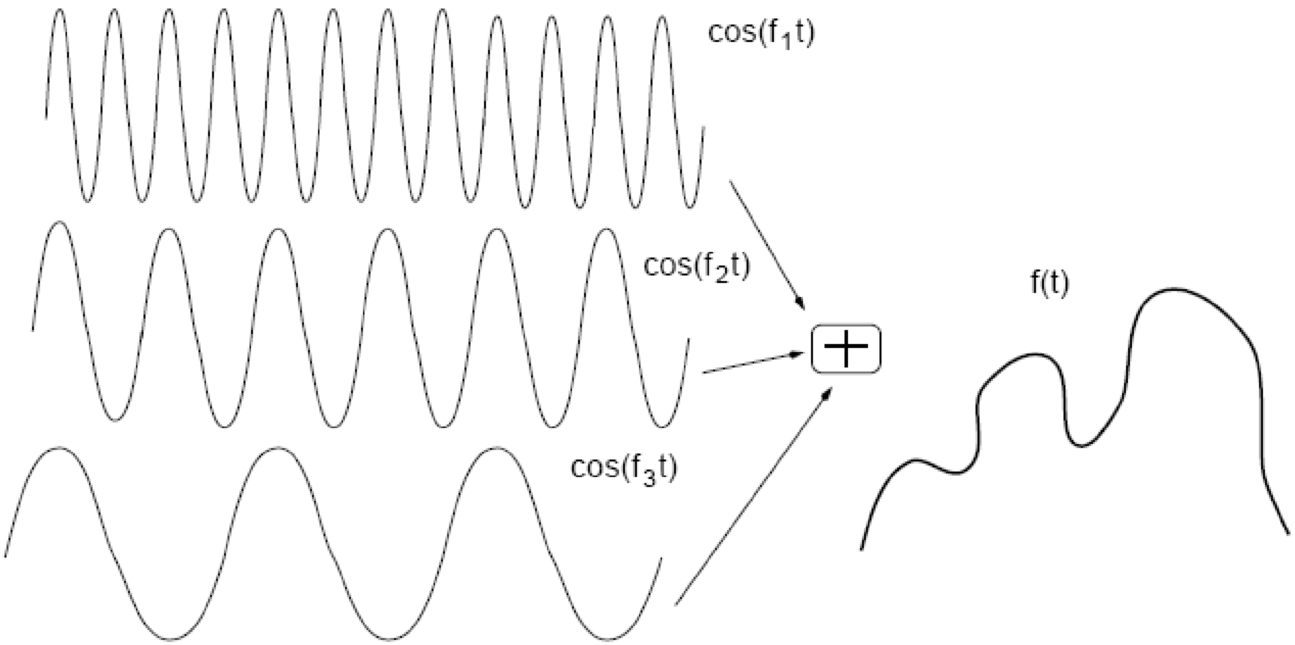
$$e^{0k}, e^{\pm j\omega_0 t}, e^{\pm j2\omega_0 t}, \ldots, e^{\pm jn\omega_0 t} \quad \text{Continuous time}$$

- Important difference with respect to the continuous case: only a finite number of exponentials are different!
  - This is because the DT exponential series is periodic of period 2π

$$e^{\pm j(\Omega \pm 2\pi)k} = e^{\pm j\Omega k}$$

Increasing the frequency: continuous time

- Consider a continuous time sinusoid with increasing frequency: the number of oscillations per unit time increases with frequency



Increasing the frequency: discrete time

- Discrete-time sinusoid

$$s[n] = \sin(\Omega_0 n)$$

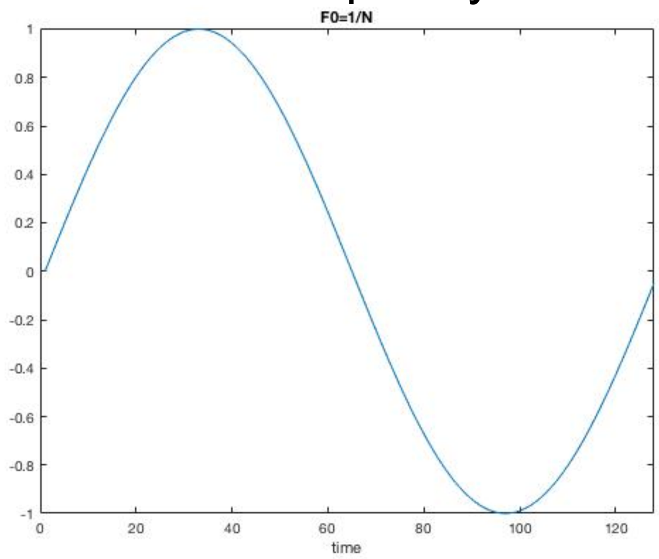
- Changing the frequency by 2pi leaves the signal unchanged

$$s[n] = \sin((\Omega_0 + 2\pi)n) = \sin(\Omega_0 n + 2\pi n) = \sin(\Omega_0 n)$$

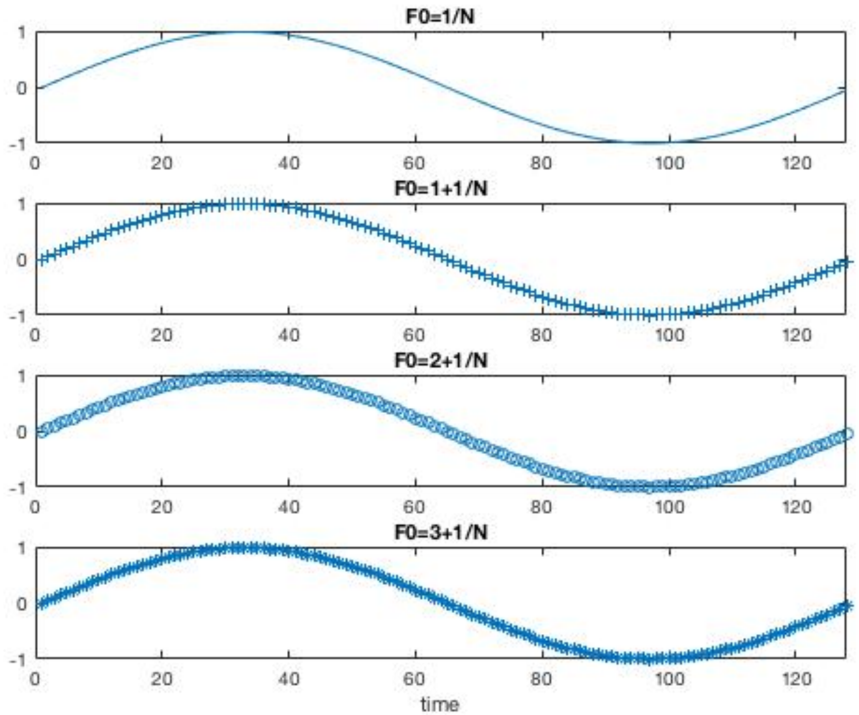
- Thus when the frequency increases from zero, the number of oscillations per unit time increase until the frequency reaches pi, then decreases again towards the value that it had in zero.
- The values of the signal are the same, for each time index n, as those for frequencies differing of 2pi

Discrete time sinusoids

N=128, sinusoid with minimum frequency



Changing Omega0 by 2pi or, equivalently, F0 by 1

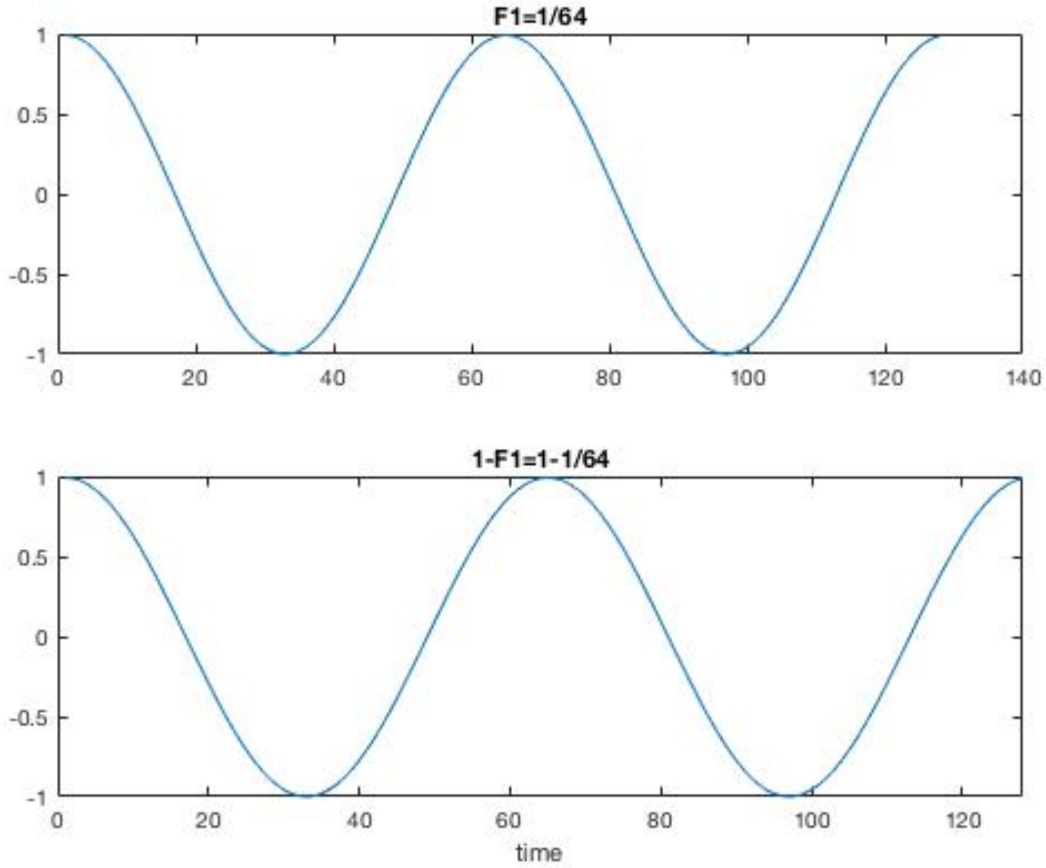


$$N = 128, T_s = 1 \to F_s = 1, T_0 = N = 128, F_0 = 1/T_0 = 1/N$$

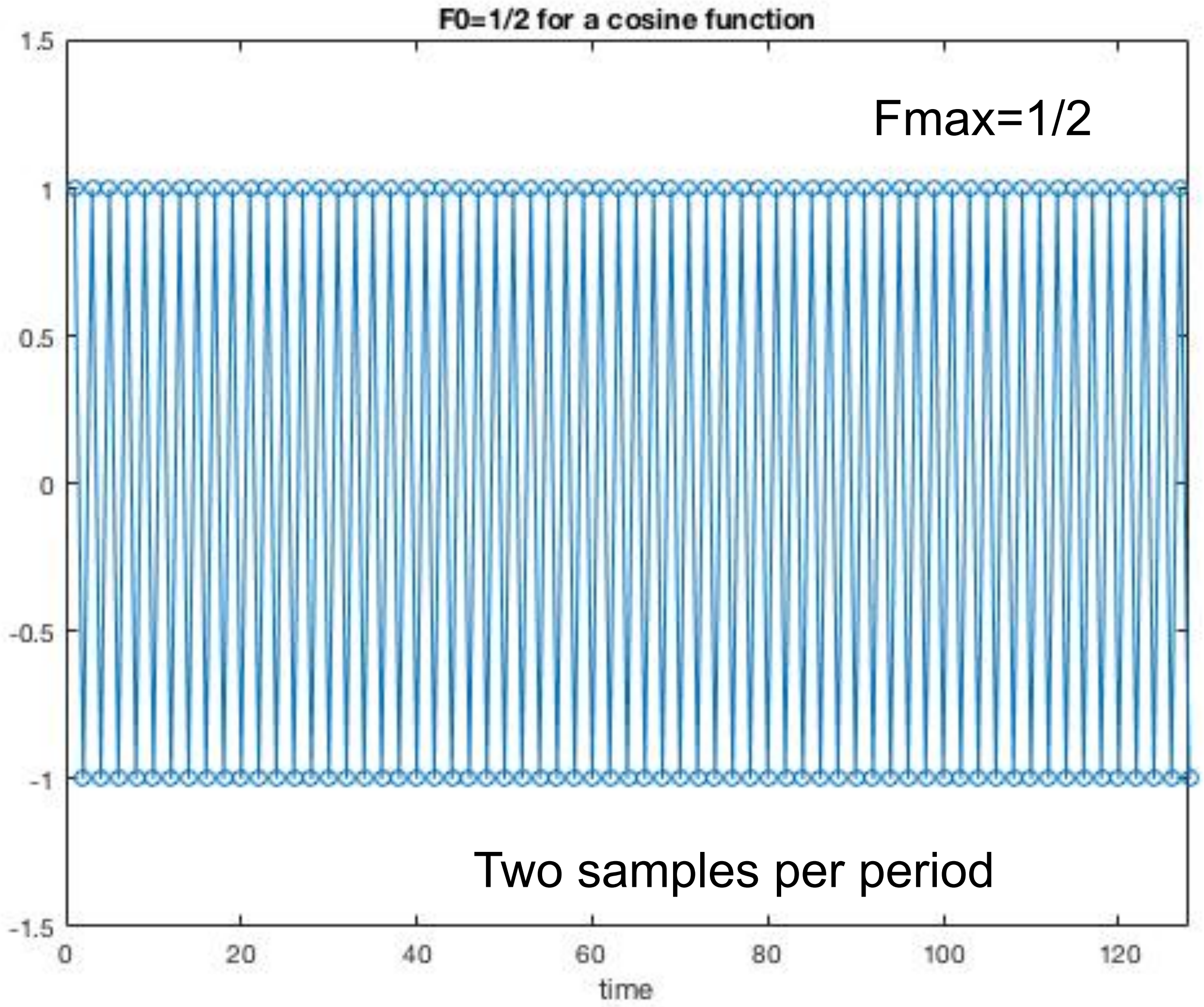
Range of possible frequencies F (multiples of F0)

$$0, 1/N, 2/N, \ldots, (N-1)/N, 1$$

Discrete time sinusoids: conjugate symmetry



Discrete time sinusoids: maximum frequency



DT cosine

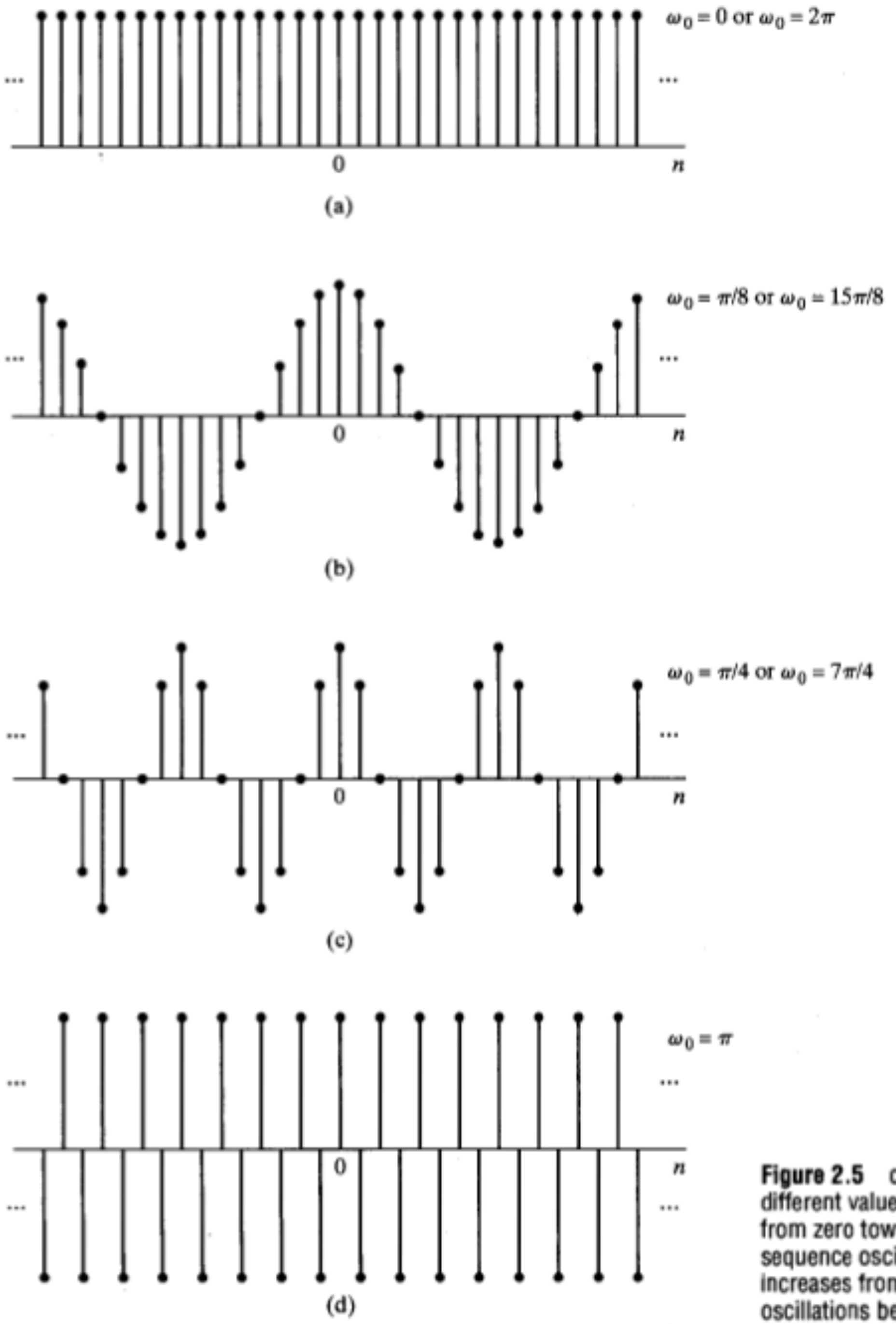


Figure 2.5 $\cos \omega_0 n$ for several different values of ω_0. As ω_0 increases from zero toward π (parts a–d), the sequence oscillates more rapidly. As ω_0 increases from π to 2π (parts d–a), the oscillations become slower.

Warning

- Discrete time sinusoids are periodic only if

$$\Omega N = 2\pi k \rightarrow \frac{2\pi}{T} N = 2\pi k \rightarrow T = \frac{N}{k}$$

k integer

- That is the period is a fraction of the number of samples

- Not all discrete sinusoids are periodic

Periodicity of the DT FS

- The r-th harmonic is IDENTICAL to the (r+N_0)-th harmonic
- Let the r-th harmonic be

$$g_r[n] = e^{jr\Omega_0 n}$$

Index of the harmonic

Independent variable

$$g_r(t) = e^{jr\omega_0 t}$$

- Thus

$$g_{r+N_0} = e^{j(r+N_0)\Omega_0 k} = e^{j(r\Omega_0 k+2\pi k)} = e^{jr\Omega_0 k} = g_r$$

$$\Omega_0 = \frac{2\pi}{N_0}$$

Periodicity of the DT FS

- Follows

$$g_r = g_{r+N_0} = g_{r+2N_0} = \cdots = g_{r+mN_0} \qquad m, \text{ integer}$$

- Thus, the first harmonic is identical to the N_0+1 harmonic, the second to the N_0+2 etc
- There are only N_0 distinct harmonics ranging on the period 0-2 pi
  - Since they are separated by $\Omega_0 = \frac{2\pi}{N_0}$
- If the signal is real, Hermitian symmetry holds and only $N_0/2$ different sinusoids are possible

Periodicity of the DT FS

Thus, the first harmonic is identical to the (N_0+1)st harmonic, the second harmonic is identical to the (N_0+2)nd harmonic, and so on. In other words, there are only N_0 independent harmonics, and they range over an interval 2π (because the harmonics are separated by $\Omega_0 = \frac{2\pi}{N_0}$. We may choose these N_0 independent harmonics as $e^{jr\Omega_0 k}$ over $0 \le r \le N_0 - 1$, or over $-1 \le r \le N_0 - 2$, or over $1 \le r \le N_0$, or over any other suitable choice for that matter. Every one of these sets will have the same harmonics, although in different order. Let us take the first choice $(0 \le r \le N_0 - 1)$.

Let us take the first choice $(0 \le r \le N_0 - 1)$.

Fourier series for an N_0-periodic signal $f[k]$ consists of only these N_0 harmonics, and can be expressed as

$$f[k] = \sum_{r=0}^{N_0-1} \mathcal{D}_r e^{jr\Omega_0 k} \qquad \Omega_0 = \frac{2\pi}{N_0}$$

Periodicity of the DTFS coefficients

To compute coefficients $\mathcal{D}_r$ in the Fourier series (10.4), we multiply both sides of (10.4) by $e^{-jm\Omega_0 k}$ and sum over k from $k = 0$ to $(N_0 - 1)$.

$$\sum_{k=0}^{N_0-1} f[k]e^{-jm\Omega_0 k} = \sum_{k=0}^{N_0-1} \sum_{r=0}^{N_0-1} \mathcal{D}_r e^{j(r-m)\Omega_0 k} \tag{10.5}$$

The right-hand sum, after interchanging the order of summation, results in

$$\sum_{r=0}^{N_0-1} \mathcal{D}_r \left[\sum_{k=0}^{N_0-1} e^{j(r-m)\Omega_0 k} \right] \tag{10.6}$$

The inner sum, according to Eq. (5.43), is zero for all values of $r \neq m$. It is nonzero with a value N_0 only when $r = m$. This fact means the outside sum has only one term $\mathcal{D}_m N_0$ (corresponding to $r = m$). Therefore, the right-hand side of Eq. (10.5) is equal to $\mathcal{D}_m N_0$, and

$$\sum_{k=0}^{N_0-1} f[k]e^{-jm\Omega_0 k} = \mathcal{D}_m N_0$$

Periodicity of the DTFS coefficients

- Then

$$\mathcal{D}_m = \frac{1}{N_0} \sum_{k=0}^{N_0-1} f[k] e^{-jm\Omega_0 k}$$

- In summary

synthesis

$$f[k] = \sum_{r=0}^{N_0-1} \mathcal{D}_r e^{jr\Omega_0 k}$$

analysis

$$\mathcal{D}_r = \frac{1}{N_0} \sum_{k=0}^{N_0-1} f[k] e^{-jr\Omega_0 k} \qquad \Omega_0 = \frac{2\pi}{N_0}$$

Same number of samples

$$D_r = \frac{1}{N_0} \sum_{k=0}^{N_0-1} f[k] e^{-j\frac{2\pi}{N_0}kr}$$

Frequency resolution

$$f[k] = \sum_{r=0}^{N_0-1} D_r e^{j\frac{2\pi}{N_0}rk}$$

Example

$$f[k] = \sin[0.1\pi k]$$

$$\Omega_0 = \frac{2\pi}{N_0} = 0.1\pi \rightarrow$$

$$N_0 = \frac{2\pi}{\Omega_0} = \frac{2\pi}{0.1\pi} = 20$$

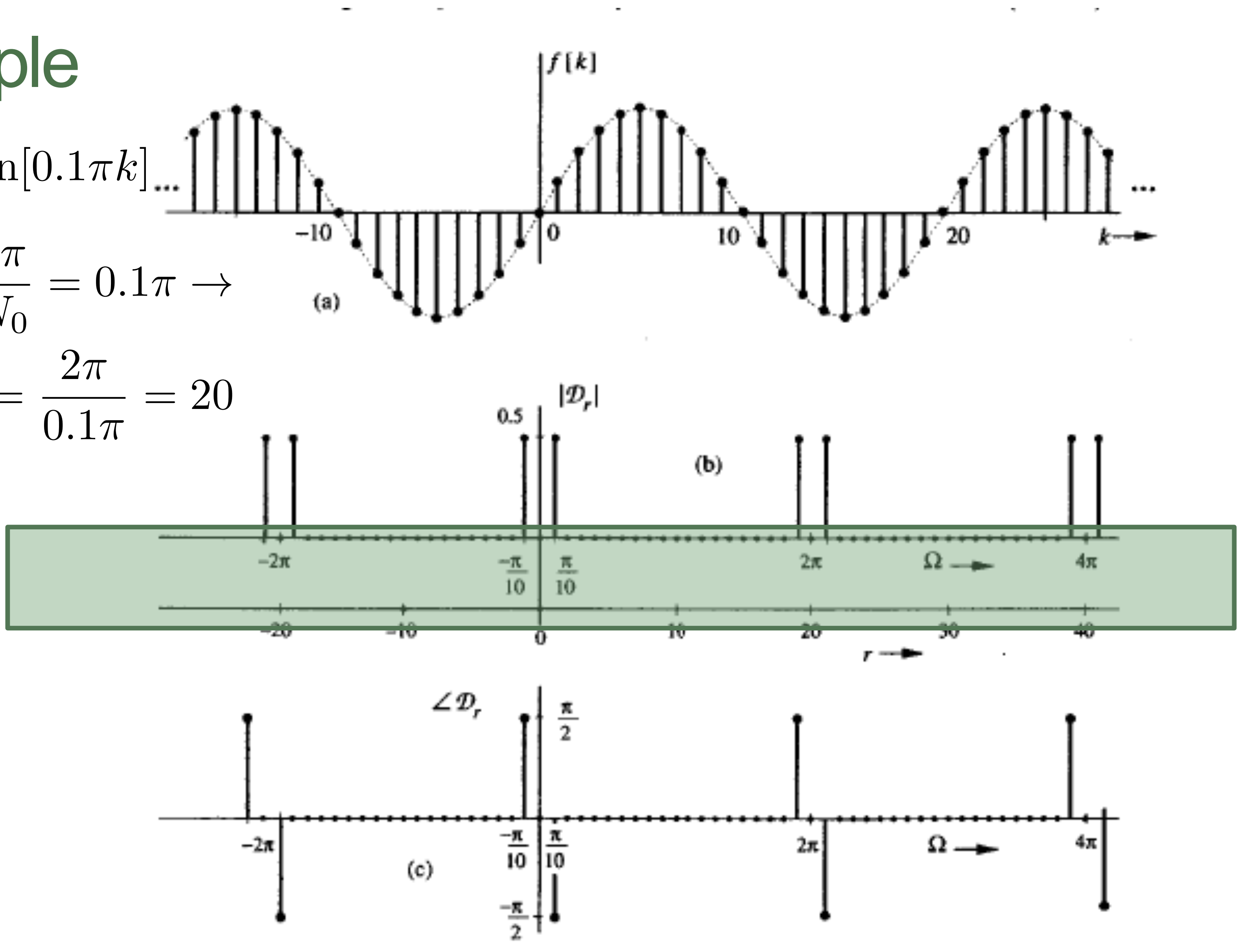


Fig. 10.1 Discrete-time sinusoid sin $0.1\pi k$ and its Fourier spectra.

Example

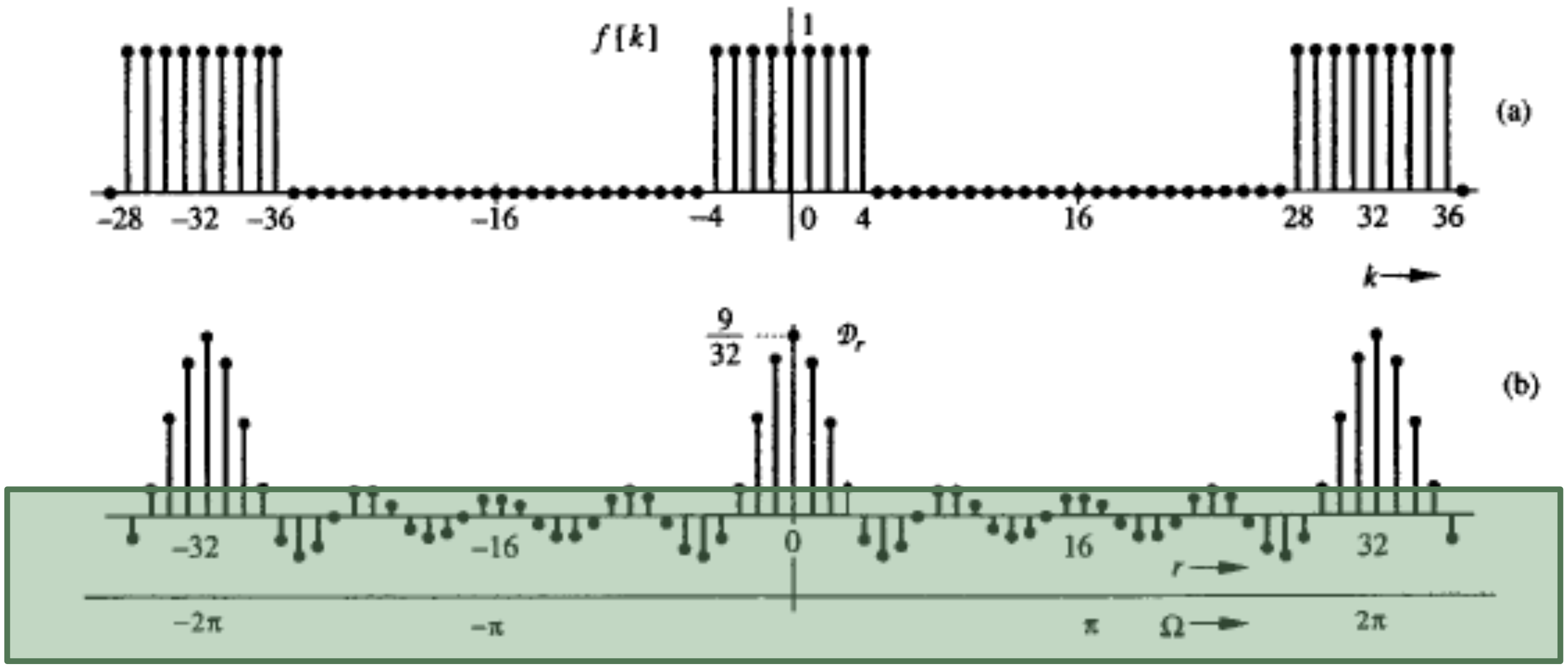


Fig. 10.2 Periodic sampled gate pulse and its Fourier spectrum.

Be careful: the spectrum is periodic with period 2π, BUT it can be represented as a function of the *harmonic index*, that ranges between 1 and N_0-1

A-periodic signals

- Same “trick”

$$\lim_{N_0 \to \infty} f_{N_0}[k] = f[k]$$

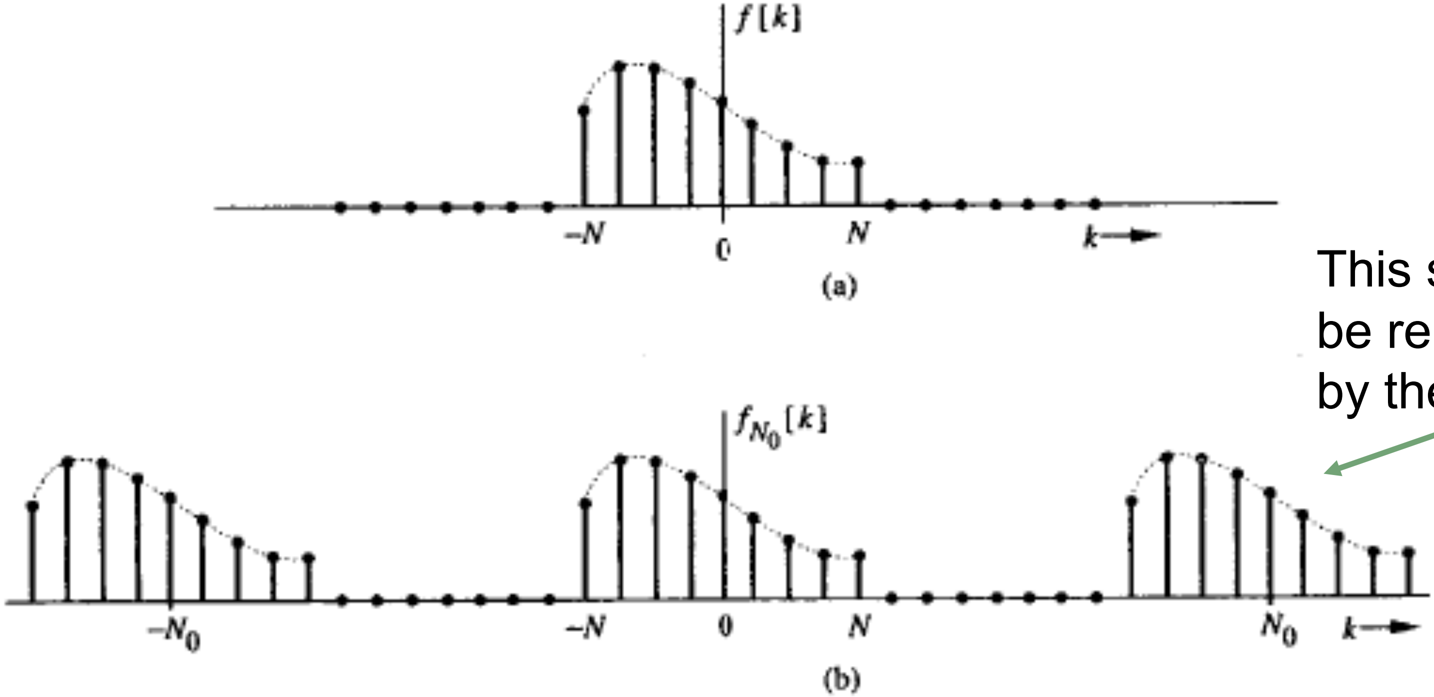


Fig. 10.3 Generation of a periodic signal by periodic extension of a signal $f[k]$.

This signal can be represented by the DTFS

A-periodic signals

- Using the same arguments and steps as we did for going from continuous-time periodic signals to continuous-time a-periodic signals, we can
  - 1) derive the expression of the Fourier-representation of DT non-periodic signals (Discrete Time Fourier Transform, DTFT)
  - 2) link the periodic and a–periodic cases

- Of note:
  - Discrete-time signals ➔ Periodic spectra
  - Periodic signals ➔ Discrete spectra
  - And respective combinations

A-periodic signals

- We define the function

$$F(\Omega) = \sum_{k=-\infty}^{\infty} f[k]e^{-j\Omega k}$$

Continuous frequency variable

- We can sow that

$$\mathcal{D}_r = \frac{1}{N_0}F(r\Omega_0)$$

⟹ **F(Ω) is the envelop of D_r**

Coefficient of the r-th harmonic component of the periodized signal

DTFT

- Increasing N_0 the frequency resolution increases and the frequency spacing decreases and in the limit N0->infty we can define the DTFT for a non-periodic signal as

$$F(\Omega) = \sum_{k=-\infty}^{\infty} f[k] e^{-j\Omega k}$$

$$f[k] = \frac{1}{2\pi} \int_{2\pi} F(\Omega) e^{jk\Omega} d\Omega$$

$$F(\Omega + 2\pi) = \sum_{k=-\infty}^{\infty} f[k] e^{-j(\Omega+2\pi)k} = \sum_{k=-\infty}^{\infty} f[k] e^{-j\Omega k} e^{-j2\pi k} = F(\Omega)$$

F(Ω) is 2π-periodic

A-periodic signals: rect

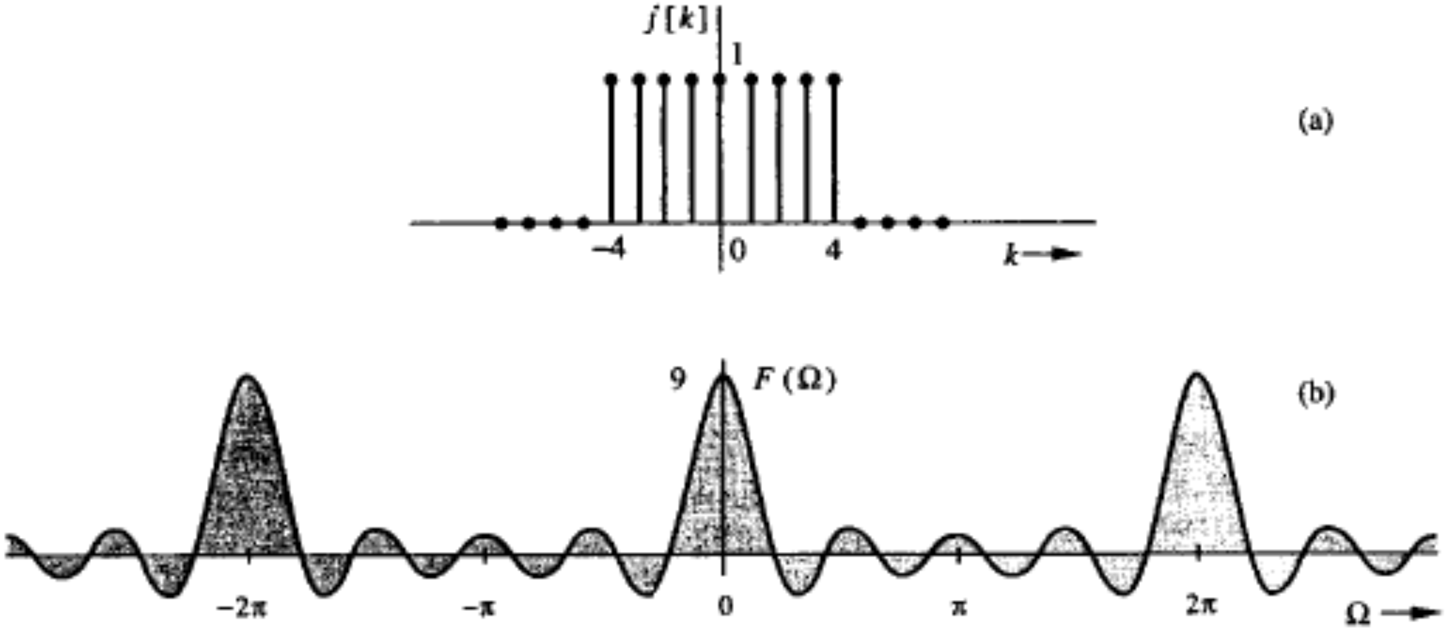


Fig. 10.6 Discrete-time gate pulse and its Fourier spectrum.

DTFT is continuous and 2pi periodic

Discrete Fourier Transform

- DTFS “corresponds” to DFT
- Now we can say that our signal has “period” N_0

$$f[k] = \sum^{N_0-1} \mathcal{D}_r e^{jr\Omega_0 k}$$

$$\mathcal{D}_r = \frac{1}{N_0} \sum_{k=0}^{N_0-1} f[k] e^{-jr\Omega_0 k} \qquad \Omega_0 = \frac{2\pi}{N_0}$$

- Reformulating

$N_0 \rightarrow N$

$D_r \rightarrow F[k]$

$f[k] \rightarrow f[n]$

$\Omega_0 \rightarrow \frac{2\pi}{N}$

$$f[n] = \sum_{k=0}^{N-1} F[k] e^{j2\pi \frac{nk}{N}}$$ synthesis

$$F[k] = \frac{1}{N} \sum_{n=0}^{N-1} f[n] e^{-j2\pi \frac{nk}{N}}$$ analysis

2π periodic *or* N-periodic

Discrete Fourier Transform

- The DFT can be considered as a generalization of the CTFT to *discrete series* of a *finite number of samples*
- It is the FT of a discrete (sampled) function of one variable
- The 1/N factor is put either in the analysis formula or in the synthesis one, or the 1/sqrt(N) is put in front of both.
- Calculating the DFT takes about N^2 calculations
  - The FFT algorithm is used

Properties of DFT

1. **Linearity:** If $f[k] \Longleftrightarrow F_r$ and $g[k] \Longleftrightarrow G_r$, then

$$a_1 f[k] + a_2 g[k] \Longleftrightarrow a_1 F_r + a_2 G_r$$

2. **Conjugate Symmetry:** For real $f[k]$

$$F_{N_0-r} = F_r^* \qquad (10.71)$$

There is a conjugate symmetry about $N_0/2$, which enables us to determine roughly half the values of F_r from the other half of the values, when $f[k]$ is real. For instance, in a 7-point DFT, $F_6 = F_1^*$, $F_5 = F_2^*$ and $F_4 = F_3^*$. In an 8-point DFT, $F_7 = F_1^*$, $F_6 = F_2^*$, $F_5 = F_3^*$, and so on.

3. **Time Shifting (Circular Shifting):**

$$f[k-n] \Longleftrightarrow F_r e^{-jr\Omega_0 n} \qquad (10.72)$$

4. **Frequency Shifting:**

$$f[k]e^{jk\Omega_0 m} \Longleftrightarrow F_{r-m} \qquad (10.73)$$

DFT

- About N^2 multiplications are needed to calculate the DFT
- The transform F[k] has the same number of components of f[n], that is N
- The DFT always exists for signals that do not go to infinity at any point
- Using the Eulero's formula $e^{j\theta} = \cos\theta + j\sin\theta.$

$$F[k] = \frac{1}{N}\sum_{n=0}^{N-1} f[n] e^{-j2\pi\frac{nk}{N}} =$$

$$\frac{1}{N}\sum_{n=0}^{N-1} f[n]\left(cos(2\pi\frac{nk}{N}) - j\sin(2\pi\frac{nk}{N})\right)$$

frequency component *k*

discrete trigonometric functions

Properties of DFT

5. **Circular (or Periodic) Convolution:**

$$f[k] \circledast g[k] \Longleftrightarrow F_r G_r \tag{10.74a}$$

and

$$f[k]g[k] \Longleftrightarrow \frac{1}{N_0} F_r \circledast G_r \tag{10.74b}$$

where the circular (or periodic) convolution of two N_0-point periodic sequences $f[k]$ and $g[k]$ is defined as

$$f[k] \circledast g[k] = \sum_{n=0}^{N_0-1} f[n]g[k-n] = \sum_{n=0}^{N_0-1} g[n]f[k-n] \tag{10.75}$$

Note that the circular convolution differs from the regular (linear) convolution by the fact that the summation is over one period (starting at any point). In the linear convolution, the summation is from $-\infty$ to ∞. The result of a periodic convolution is also an N_0-periodic sequence.

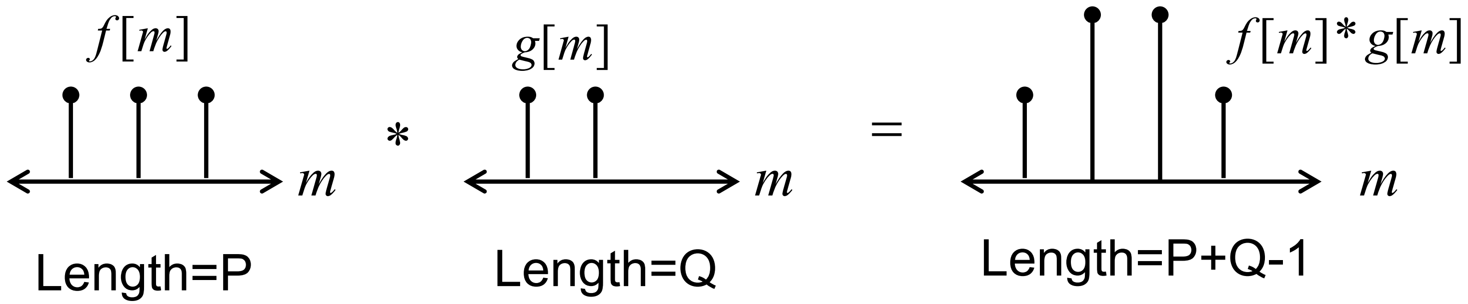
Circular convolution

- Finite length signals (N_0 samples) → circular or periodic convolution
  - the summation is over 1 period
  - the result is a N_0 period sequence

$$c[k] = f[k] \otimes g[k] = \sum_{n=0}^{N_0-1} f[n]g[k-n]$$

- The circular convolution is equivalent to the linear convolution of the zero-padded equal length sequences

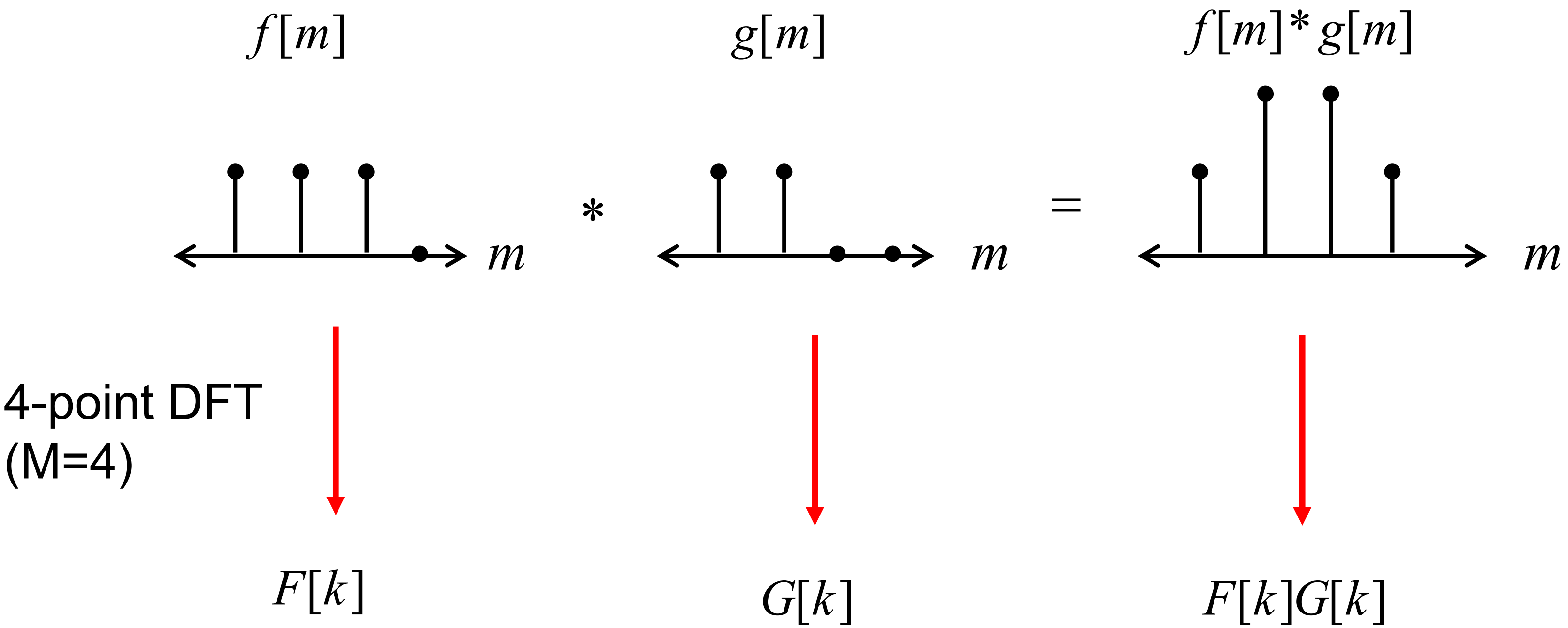
$$f[m] * g[m] \Leftrightarrow F[k]G[k]$$



For the convolution property to hold, M (the number of points used for calculating the DFT) must be *greater than or equal* to P+Q-1.

Circular convolution

- Zero padding $f[m]*g[m] \Leftrightarrow F[k]G[k]$



In words

- Given 2 sequences of length N and M, let y[k] be their linear convolution

$$y[k] = f[k] * h[k] = \sum_{n=-\infty}^{+\infty} f[n]h[k-n]$$

- y[k] is also equal to the circular convolution of the two suitably zero padded sequences making them consist of the same number of samples

$$c[k] = f[k] \otimes h[k] = \sum_{n=0}^{N_0 - 1} f[n]h[k-n]$$

$$N_0 = N_f + N_h - 1: \text{ length of the zero-padded seq}$$

- In this way, the linear convolution between two sequences having a different length (filtering) can be computed by the DFT (which rests on the circular convolution)
  - The procedure is the following
    - Pad f[n] with N_h-1 zeros and h[n] with N_f-1 zeros
    - Find Y[r] as the product of F[r] and H[r] (which are the DFTs of the corresponding zero-padded signals)
    - Find the inverse DFT of Y[r]

- **Allows to perform linear filtering using DFT**

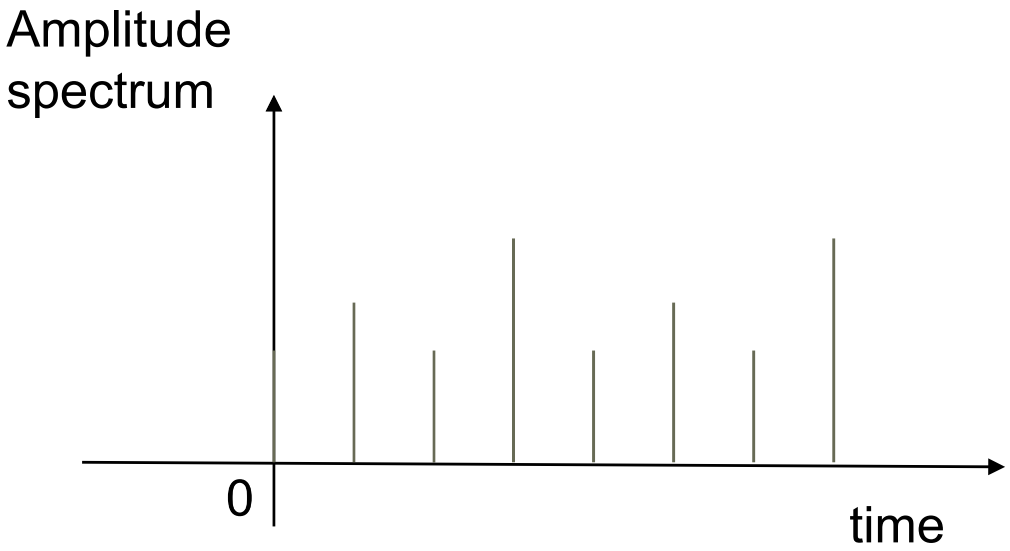
In practice..

- In order to calculate the DFT we start with k=0, calculate F(0) as in the formula below, then we change to u=1 etc

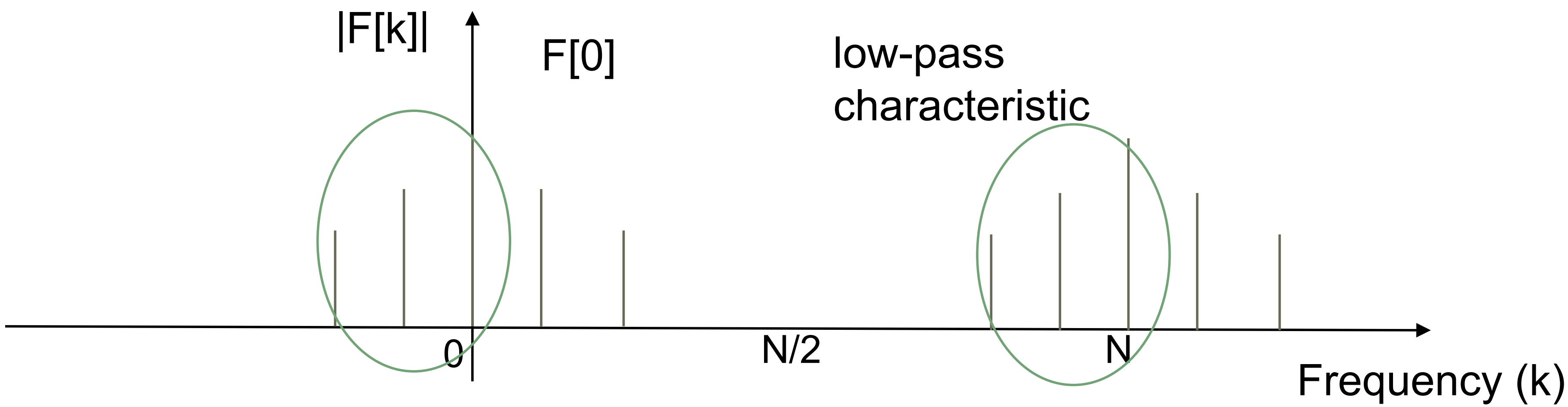
$$F[0] = \frac{1}{N}\sum_{n=0}^{N-1} f[n] e^{-j2\pi 0 n/N} = \frac{1}{N}\sum_{n=0}^{N-1} f[n] = \bar{f}$$

- F[0] is the mean value of the function f[n]
  - This is also the case for the CTFT
- The transformed function F[k] has the same number of terms as f[n] and always exists
- The transform is always reversible by construction so that we can always recover f given F

Highlights on DFT properties



The DFT of a **real** signal is **symmetric (Hermitian symmetry)**
The DFT of a **real symmetric** signal (even like the cosine) is **real and symmetric**
The DFT is **N-periodic**
Hence
The DFT of a real symmetric signal only needs to be specified in [0, N/2]



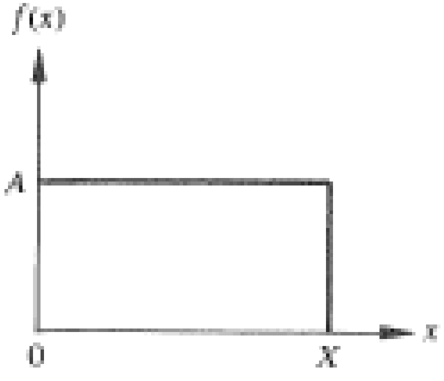
Visualization of the basic repetition

- To show a full period, we need to translate the origin of the transform at **u=N/2** (or at **(N/2,N/2)** in 2D)

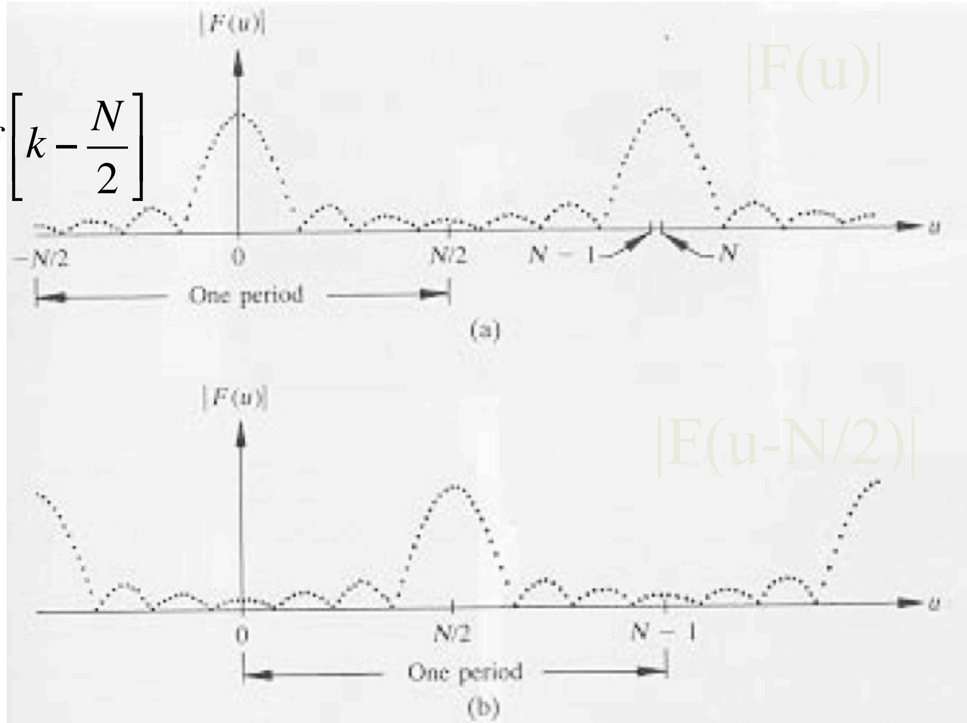
$$f[n]e^{2\pi u_0 n} \rightarrow f[k-u_0]$$

$$u_0 = \frac{N}{2}$$

$$f[n]e^{2\pi\frac{N}{2}n} = f[n]e^{\pi N n} = (-1)^n f[n] \rightarrow f\left[k-\frac{N}{2}\right]$$

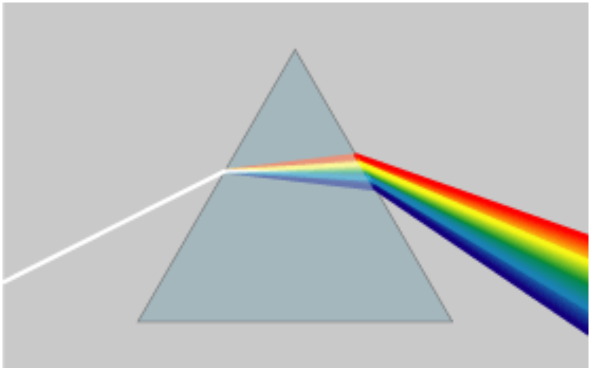


(a)



Going back to the intuition

- The FT decomposed the signal over its harmonic components and thus represents it as a sum of linearly independent complex exponential functions
- Thus, it can be interpreted as a "mathematical prism"



DFT

- Each term of the DFT, namely each value of F[k], results of the contributions of all the samples in the signal (f[n] for n=1,..,N)
- The samples of f[n] are multiplied by trigonometric functions of different frequencies
- The domain over which F[k] lives is called *frequency domain*
- Each term of the summation which gives F[k] is called *frequency component* of *harmonic component*

DFT is a complex number

- F[k] in general are complex numbers

$$F[k] = \mathrm{Re}\{F[k]\} + j\,\mathrm{Im}\{F[k]\}$$

$$F[k] = |F[k]| \exp\{j \mathrm{R} F[k]\}$$

$$\left\{ \begin{array}{l} |F[k]| = \sqrt{\mathrm{Re}\{F[k]\}^2 + \mathrm{Im}\{F[k]\}^2} \\ \mathrm{R}\, F[k] = \tan^{-1}\left\{ -\dfrac{\mathrm{Im}\{F[k]\}}{\mathrm{Re}\{F[k]\}} \right\} \end{array} \right\}$$

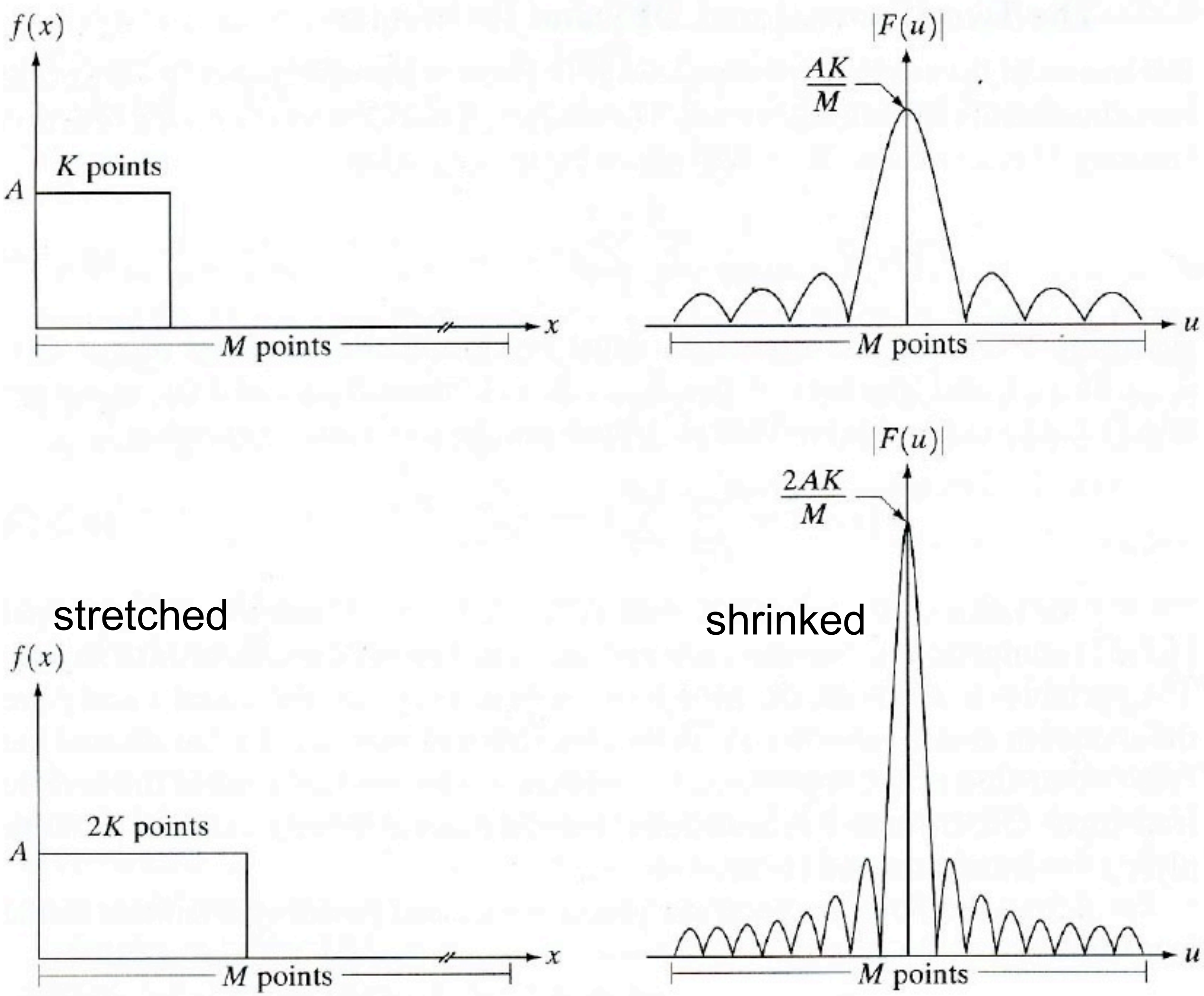
magnitude or spectrum

phase or angle

$$P[k] = |F[k]|^2$$

power spectrum

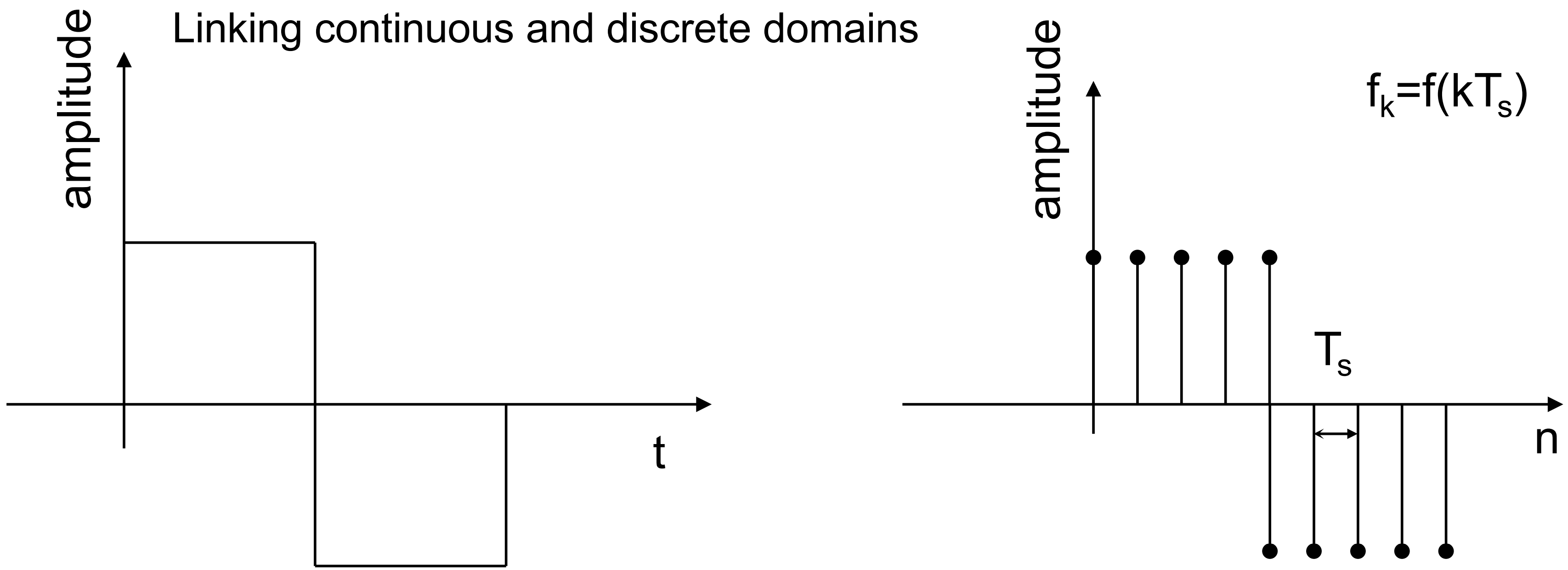
Stretching vs shrinking



a b
c d

FIGURE 4.2 (a) A discrete function of M points, and (b) its Fourier spectrum. (c) A discrete function with twice the number of nonzero points, and (d) its Fourier spectrum.

Periodization vs discretization



- DT (discrete time) signals can be seen as *sampled* versions of CT (continuous time) signals
- Both CT and DT signals can be of finite duration or periodic
- There is a duality between *periodicity* and *discretization*
  - Periodic signals have discrete frequency (sampled) transform
  - Discrete time signals have periodic transform
  - DT periodic signals have discrete (sampled) periodic transforms

Increasing the resolution by Zero Padding

- Consider the analysis formula

$$F[k] = \frac{1}{N}\sum_{n=0}^{N-1} f[n] e^{-\frac{2\pi jkn}{N}}$$

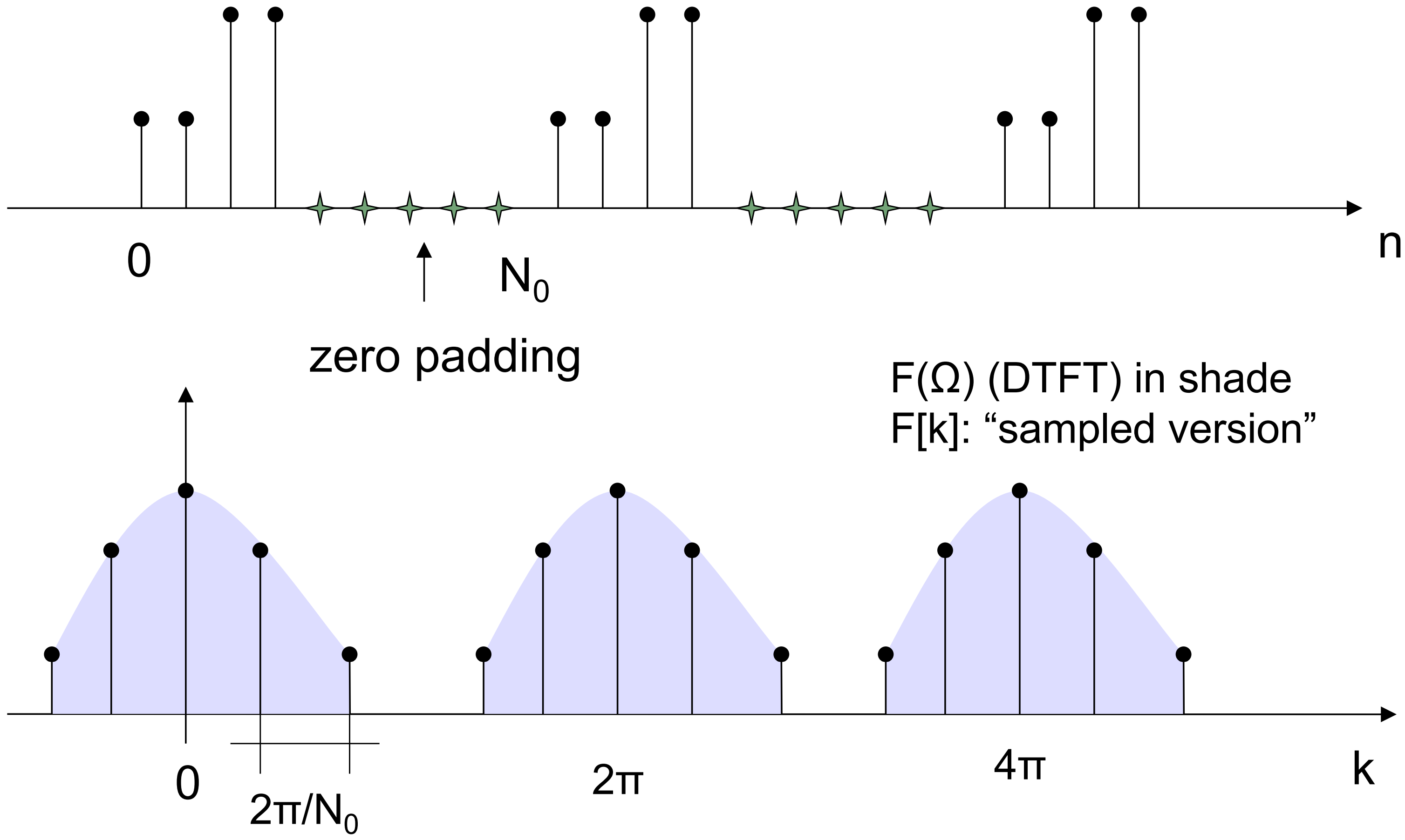
- If f[n] consists of N samples than F[k] consists of N samples as well, it is discrete (k is an integer) and it is periodic (because the signal f[n] is discrete time, namely n is an integer)
- The value of each F[k], for all k, is given by a weighted sum of the values of f[n], for n=1,.., N-1
- Key point: if we artificially increase the length of the signal adding M zeros on the right, we get a signal $f_1[m]$ for which m=1,…,N+M-1. Since

$$f_1[m] = \begin{cases} f[m] & for \quad 0 \le m < N \\ 0 & for \quad N \le m < N+M \end{cases}$$

Increasing the resolution through ZP

- Then the value of each F[k] is obtained by a weighted sum of the “real” values of f[n] for 0≤k≤N-1, which are the only ones different from zero, but they happen at different “normalized frequencies” since the frequency axis has been rescaled. In consequence, F[k] is more “densely sampled” and thus features a higher resolution.

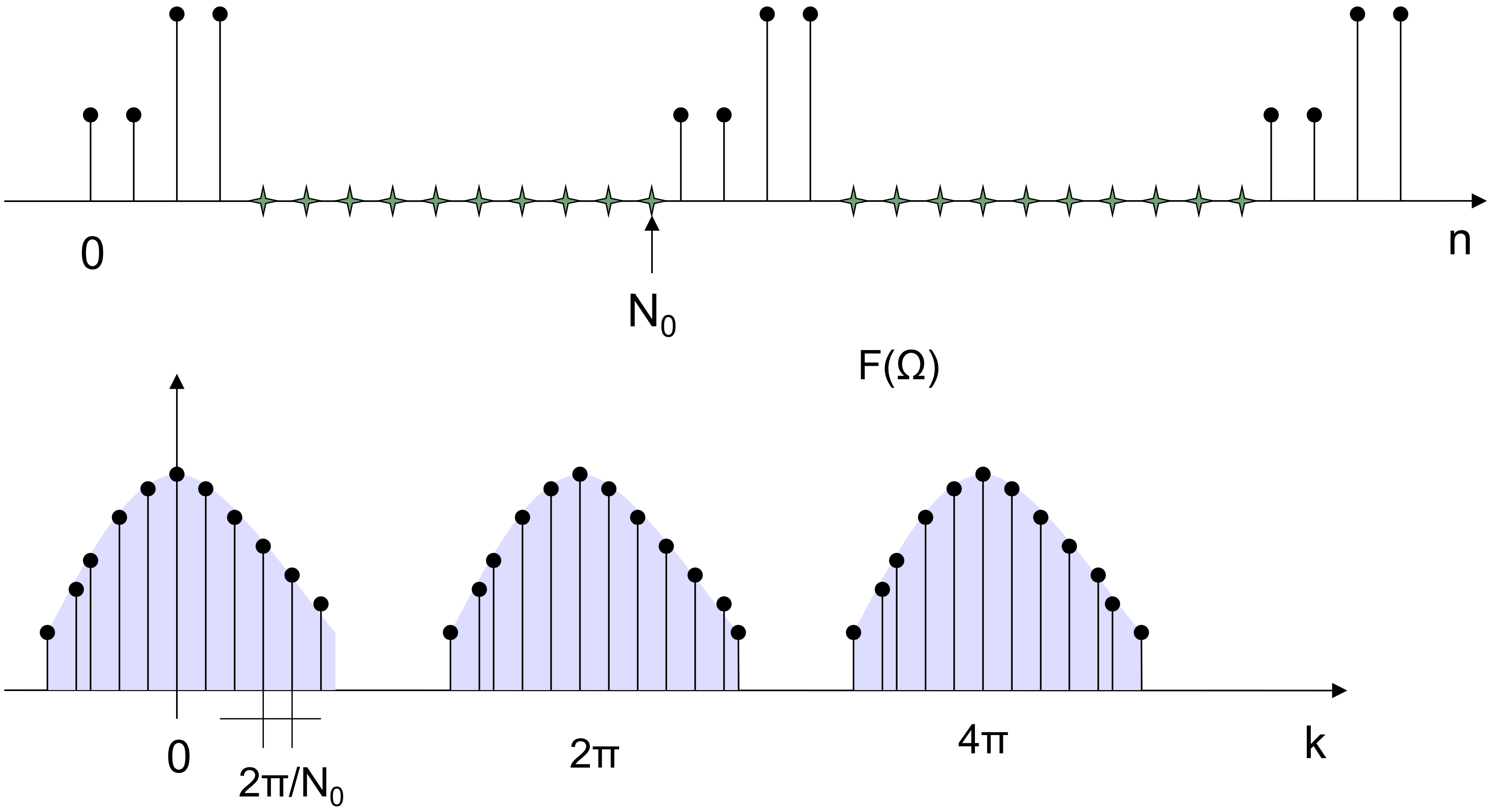
Increasing the resolution by Zero Padding



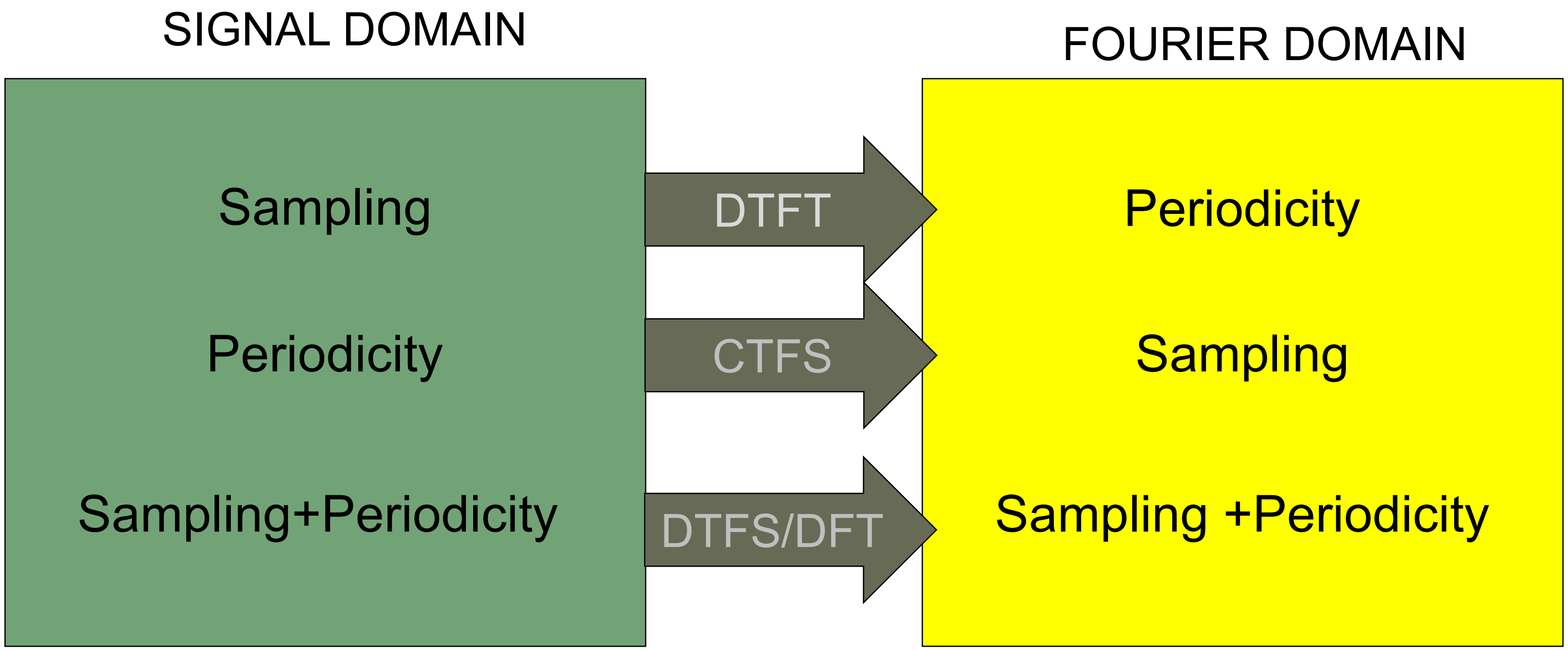
Zero padding

Increasing the number of zeros augments the "resolution" of the transform since the samples of the DFT get "closer"

zero padding



Summary of dualities



Discrete Cosine Transform (DCT)

Applies to digital (sampled) finite length signals AND uses only **cosines**.

The DCT coefficients are all **real numbers**

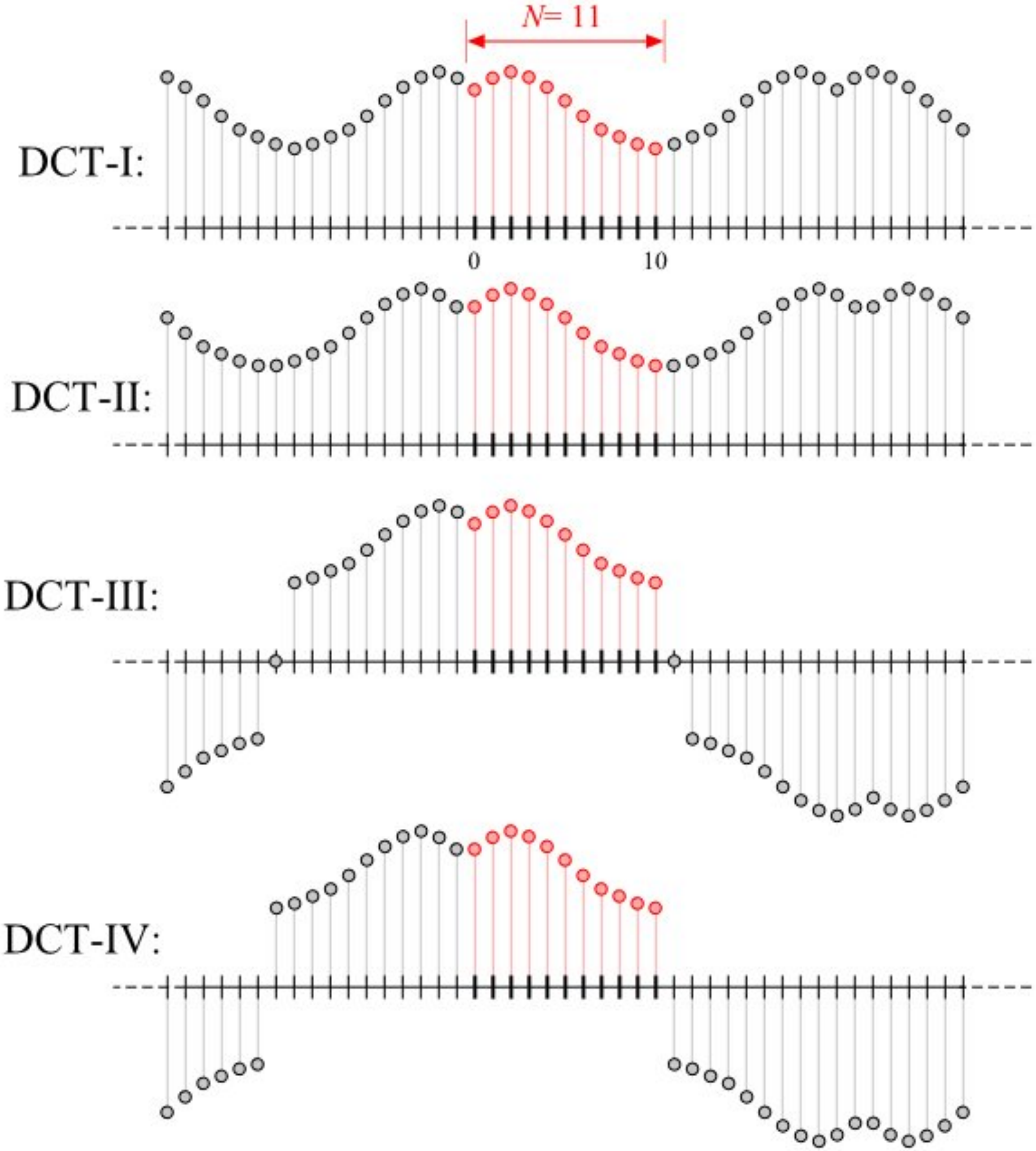
Discrete *Cosine* Transform (DCT)

- Operate on finite discrete sequences (as DFT)
- A **discrete cosine transform** (**DCT**) expresses a sequence of finitely many data points in terms of a sum of cosine functions oscillating at different frequencies
- DCT is a Fourier-related transform similar to the DFT but using only real numbers
- DCT is equivalent to DFT of roughly twice the length, operating on real data with even symmetry (since the Fourier transform of a real and even function is real and even), where in some variants the input and/or output data are shifted by half a sample
- There are eight standard DCT variants, out of which four are common
- Strong connection with the Karunen-Loeven transform
    - VERY important for signal compression

DCT

- DCT implies different boundary conditions than the DFT or other related transforms
- A DCT, like a cosine transform, implies an *even periodic* extension of the original function
- Tricky part
  - First, one has to specify whether the function is even or odd at *both* the left and right boundaries of the domain
  - Second, one has to specify around *what point* the function is even or odd
    - In particular, consider a sequence *abcd* of four equally spaced data points, and say that we specify an even *left* boundary. There are two sensible possibilities: either the data is <u>even about the sample *a*</u>, in which case the even extension is ***dcbabcd***, or the data is <u>even about the point *halfway* between *a* and the previous point</u>, in which case the even extension is ***dcbaabcd*** (*a* is repeated).

Symmetries



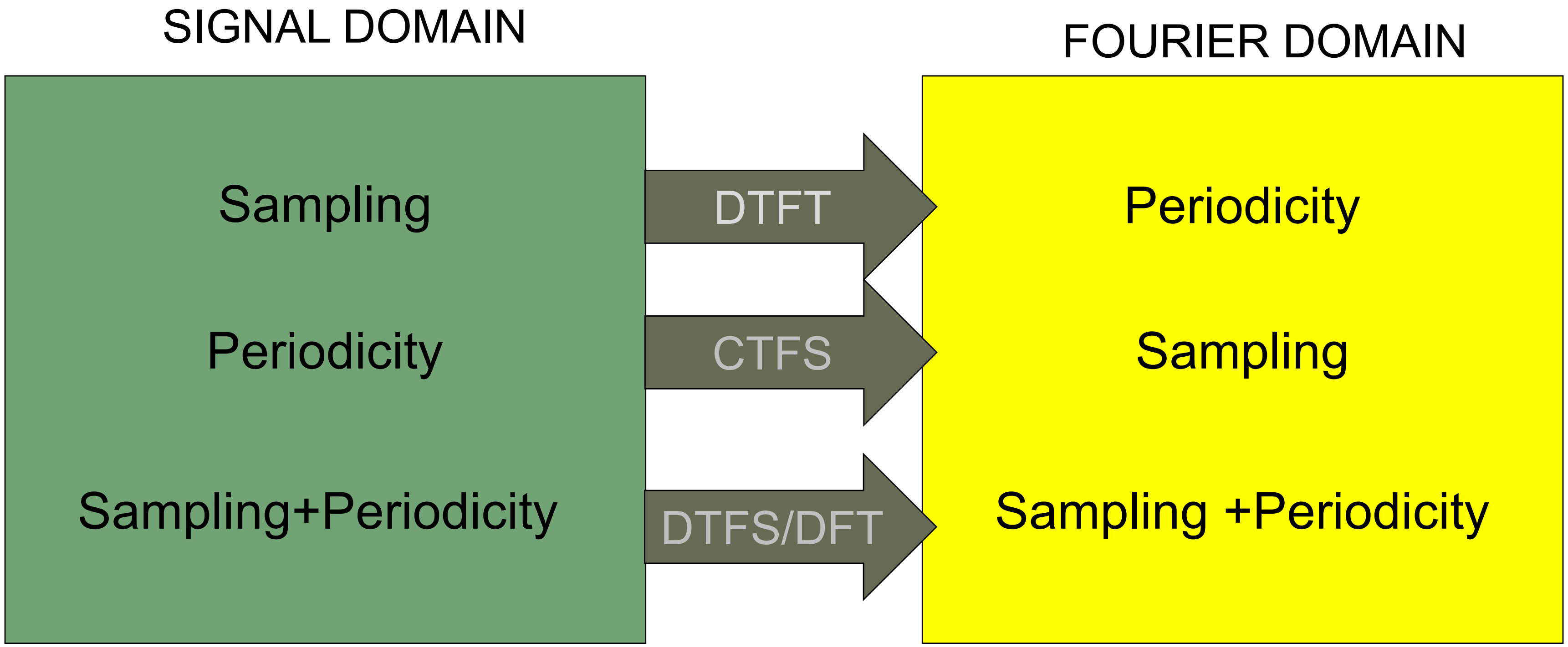
DCT

$$X_k = \sum_{n=0}^{N_0-1} x_n \cos\left[\frac{\pi}{N_0}\left(n+\frac{1}{2}\right)k\right] \qquad k = 0,\ldots., N_0 - 1$$

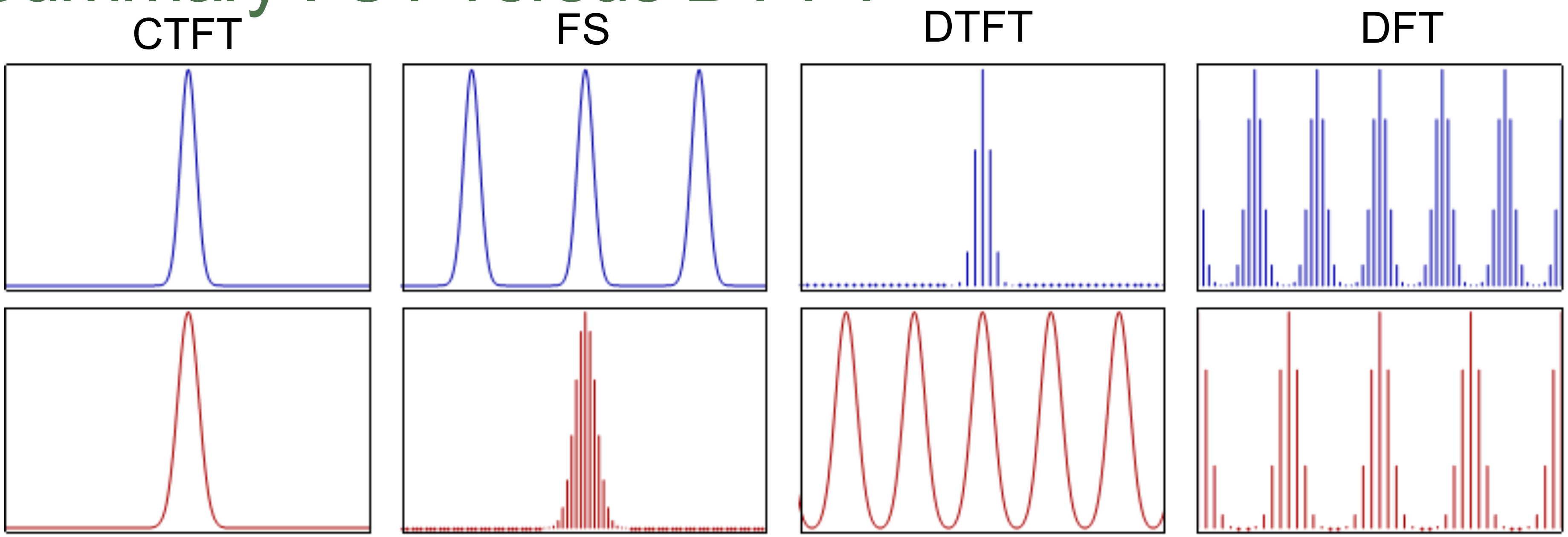
$$x_n = \frac{2}{N_0}\left\{\frac{1}{2}X_0 + \sum_{k=0}^{N_0-1} X_k \cos\left[\frac{\pi k}{N_0}\left(k+\frac{1}{2}\right)\right]\right\}$$

- *Warning*: the normalization factor in front of these transform definitions is merely a convention and differs between treatments.
    - Some authors multiply the transforms by $(2/N_0)^{1/2}$ so that the inverse does not require any additional multiplicative factor.
        - Combined with appropriate factors of √2 (see above), this can be used to make the transform matrix orthogonal.

Summary of dualities



Summary : CT versus DT FT



Relationship between the (continuous) Fourier transform and the discrete Fourier transform.
Left column: A continuous function (top) and its Fourier transform (bottom).
Center-left column: Periodic summation of the original function (top). Fourier transform (bottom) is zero except at discrete points. The inverse transform is a sum of sinusoids called Fourier series.
Center-right column: Original function is discretized (multiplied by a Dirac comb) (top). Its Fourier transform (bottom) is a periodic summation (DTFT) of the original transform.
Right column: The DFT (bottom) computes discrete samples of the continuous DTFT. The inverse DFT (top) is a periodic summation of the original samples. The FFT algorithm computes one cycle of the DFT and its inverse is one cycle of the DFT inverse.