★ <u>Tensoriality</u>

In view of future use, let us address the following question: given a map

$$\mathfrak{X}(M) \times \mathfrak{X}(M) \ldots \mathfrak{X}(M) \longmapsto \mathscr{C}^{\infty}(M)$$
$$(X_1 \quad X_2 \ldots X_k) \longmapsto T(X_1 \ldots X_n)$$

how can one ascertain its tensor character, i.e. whether it defines, in the specific case, a tensor (field) in $T^{(0,k)}(M)$?

Answer: check whether multilinearity <u>over $\mathscr{C}^{\infty}(M)$</u> holds that is, whether

$$T(\ldots, \alpha X_j^{(1)} + \beta X_j^{(2)}, \ldots) = \alpha\, T(\ldots X_j^{1} \ldots) + \beta\, T(\ldots X_j^{2} \ldots)$$

<u>with $\alpha, \beta \in \mathscr{C}^{\infty}(M)$.</u> (One has <u>pointwise</u> multilinearity)

‖ This also holds in general, for assessing tensoriality of prospective objects: check multilinearity over functions

<u>Examples</u>. Elements in $\mathfrak{X}(M)$ and $\Lambda^{k}(M)$ themselves are obviously tensors. A metric $g \in \mathcal{T}^{(0,2)}(M)$ has tensor character. We shall build up several objects which turn out to yield tensors, and others which will not give tensors.

<u>Remark</u>. It is quite common to call <u>tensor</u> fields simply <u>tensors</u>.