# Fourier Transform

# Contents

- Signals as functions (1D, 2D)
  - Tools
- Continuos Time Fourier Transform (CTFT)
- Discrete Time Fourier Transform (DTFT)
- Discrete Fourier Transform (DFT)
- Discrete Cosine Transform (DCT)
- Sampling theorem

# Signals as functions

1. Continuous functions of real independent variables
   - 1D: $f=f(x)$
   - 2D: $f=f(x,y)$ $x,y$
   - Real world signals (audio, ECG, images)
2. Real valued functions of discrete variables
   - 1D: $f=f[k]$
   - 2D: $f=f[i,j]$
   - *Sampled* signals
3. Discrete functions of discrete variables
   - 1D: $y=y[k]$
   - 2D: $y=y[i,j]$
   - *Sampled and quantized* signals
   - For ease of notations, we will use the same notations for 2 and 3

# Fourier Transform

- Different formulations for the different classes of signals
  - Summary table: Fourier transforms with various combinations of continuous/ discrete time and frequency variables.
  - Notations:
    - CTFT: continuous time FT: t is real and f real (f=ω) (CT, CF)
    - DTFT: Discrete Time FT: t is discrete (t=n), f is real (f=ω) (DT, CF)
    - CTFS: CT Fourier Series (summation synthesis): t is real AND the function is periodic, f is discrete (f=k), (CT, DF)
    - DTFS: DT Fourier Series (summation synthesis): t=n AND the function is periodic, f discrete (f=k), (DT, DF)
    - P: periodical signals
    - T: sampling period
    - $\omega_s$: sampling frequency ($\omega_s=2\pi/T$)
    - For DTFT: $T=1 \rightarrow \omega_s=2\pi$
- *This is a hint for those who are interested in a more exhaustive theoretical approach*

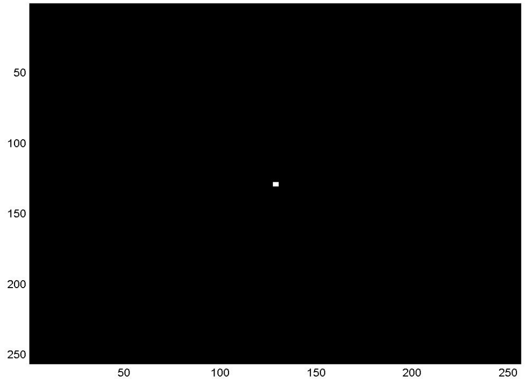
# Images as functions

- Gray scale images: 2D functions
  - Domain of the functions: set of $(x,y)$ values for which $f(x,y)$ is defined : 2D lattice $[i,j]$ defining the pixel locations
  - Set of values taken by the function : gray levels

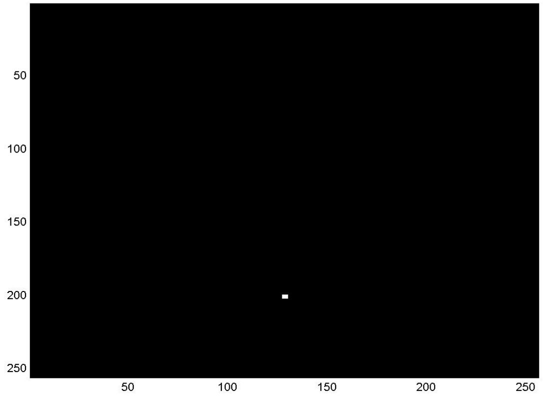
- Digital images can be seen as functions defined over a discrete domain $\{i,j: 0<i<I, 0<j<J\}$
  - $I,J$: number of rows (columns) of the matrix corresponding to the image
  - $f=f[i,j]$: gray level in position $[i,j]$

# Example 1: δ function

$$\delta[i,j] = \begin{cases} 1 & i = j = 0 \\ 0 & i, j \neq 0; i \neq j \end{cases}$$



$$\delta[i, j - J] = \begin{cases} 1 & i = 0; j = J \\ 0 & otherwise \end{cases}$$



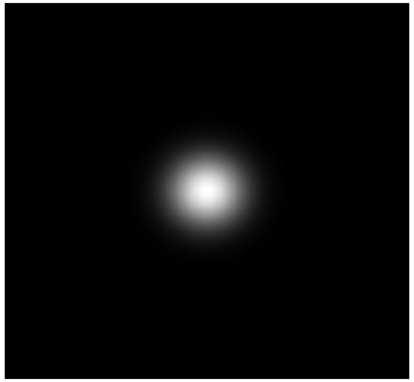
# Example 2: Gaussian

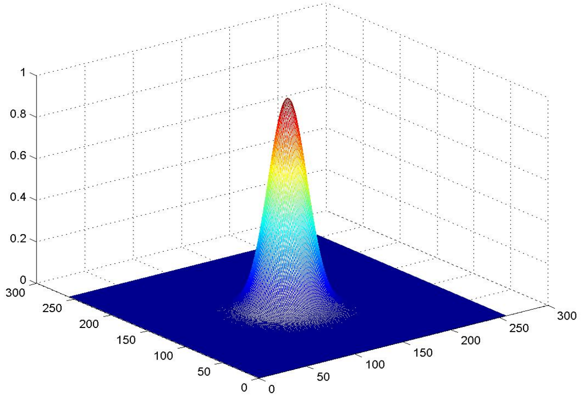
Continuous function

$$f(x,y)=\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{x^2+y^2}{2\sigma^2}}$$

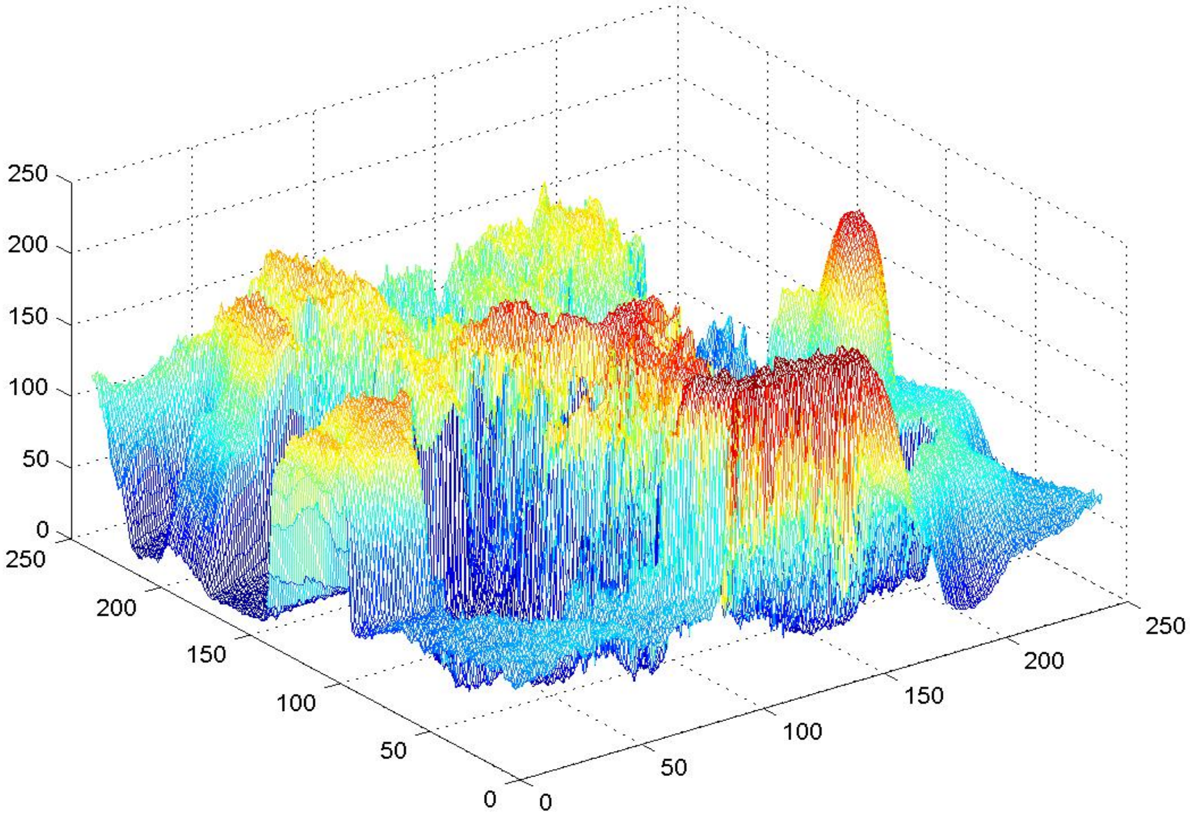
Discrete version

$$f[i,j]=\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{i^2+j^2}{2\sigma^2}}$$





# Example 3: Natural image



# Example 3: Natural image

# Mathematical Background: Complex Numbers

- A complex number **x** is of the form:

$$x = a + jb, \text{ where } j = \sqrt{-1}$$

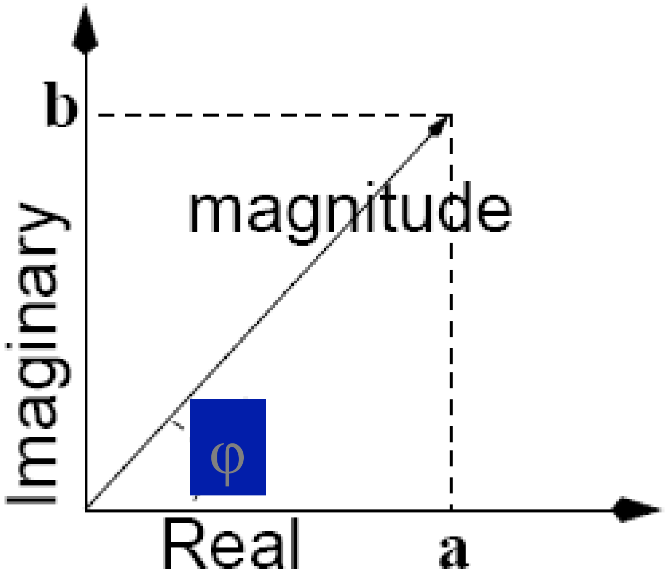
a: **real part**, b: **imaginary** part

- Addition $$(a + jb) + (c + jd) = (a + c) + j(b + d)$$

- Multiplication $$(a + jb) . (c + jd) = (ac - bd) + j(ad + bc)$$

# Mathematical Background: Complex Numbers (cont’ d)

- Magnitude-Phase (i.e.,vector) representation



**Magnitude:**

**Phase:**

$$|x| = \sqrt{a^2 + b^2}$$

$$\phi(x) = \tan^{-1}(b/a)$$

Phase – Magnitude notation:

$$x = |x|e^{j\phi(x)}$$

# Mathematical Background:
# Complex Numbers (cont' d)

- Multiplication using magnitude-phase representation

$$xy = |x|e^{j\phi(x)} \,.\, |y|e^{j\phi(y)} = |x|\ |y|\ e^{j(\phi(x)+\phi(y))}$$

- Complex conjugate

$$x^* = a - jb$$

- Properties

$$|x| = |x^*|$$
$$\phi(x) = -\,\phi(x^*)$$
$$xx^* = |x|^2$$

# Mathematical Background: Complex Numbers (cont’ d)

- Euler’ s formula

$$e^{\pm j\theta} = cos(\theta) \pm jsin(\theta)$$

$$|e^{\pm j\theta}| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1$$

- Properties

$$\phi(e^{\pm j\theta}) = \tan^{-1}(\pm \frac{sin(\theta)}{cos(\theta)}) = \tan^{-1}(\pm tan(\theta)) = \pm\theta$$

$$sin(\theta) = \frac{1}{2i_{j}}(e^{j\theta} - e^{-j\theta})$$

$$cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$$

# Mathematical Background:
# Sine and Cosine Functions

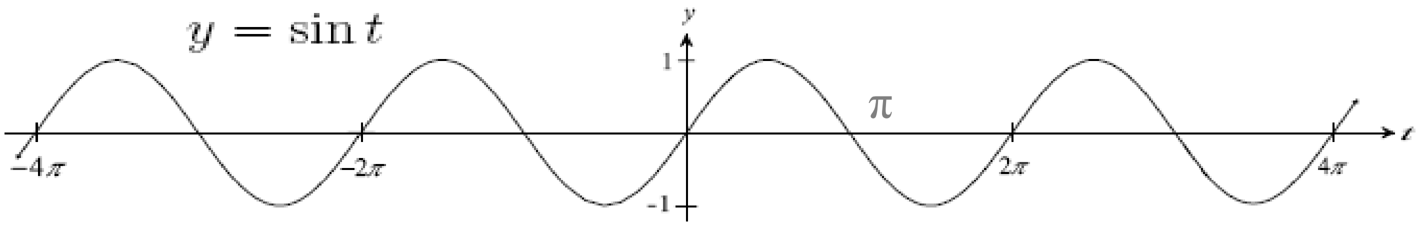
- Periodic functions
- General form of sine and cosine functions:

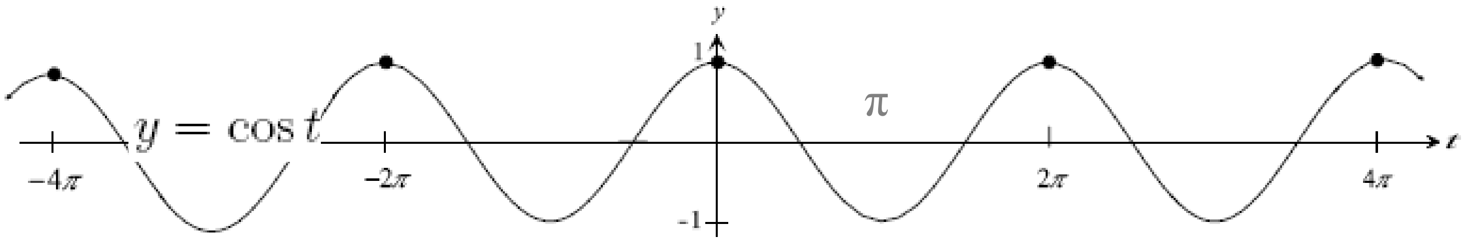
$$y\left(t\right) = A\sin[a(t+b)] \qquad y\left(t\right) = A\cos[a(t+b)]$$

| | |
|---|---|
| $\|A\|$ | amplitude |
| $\frac{2\pi}{\|a\|}$ | period |
| $b$ | phase shift |

# Mathematical Background: Sine and Cosine Functions

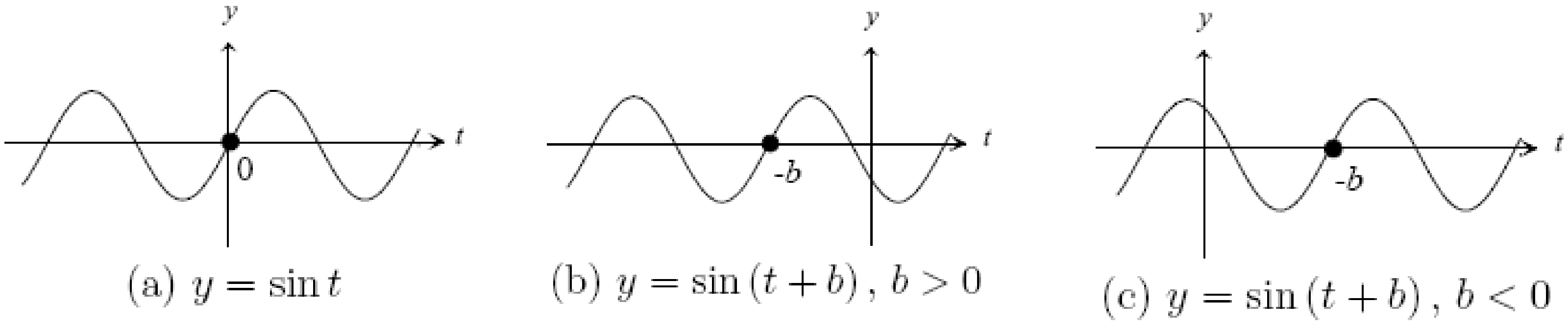
Special case: A=1, b=0, α=1





# Mathematical Background:
# Sine and Cosine Functions (cont' d)

- Shifting or translating the sine function by a const b



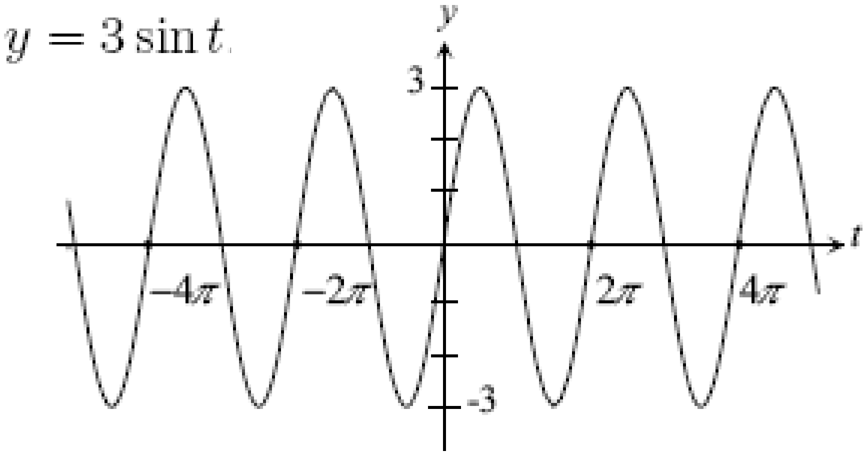
(a) $y = \sin t$ (b) $y = \sin(t + b)$, $b > 0$ (c) $y = \sin(t + b)$, $b < 0$

Note: cosine is a shifted sine function:

$$\cos(t) = \sin(t + \frac{\pi}{2})$$

# Mathematical Background:
# Sine and Cosine Functions (cont’ d)

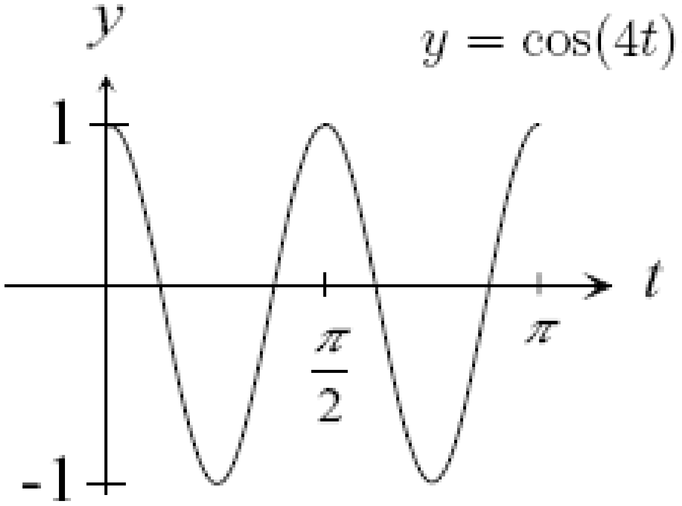
- Changing the amplitude A



# Mathematical Background:
# Sine and Cosine Functions (cont' d)

- Changing the period T=2π/|α|

  consider A=1, b=0: y=cos(αt)



α =4

period 2π/4=π/2

shorter period
higher frequency
(i.e., oscillates faster)

**Frequency** is defined as f=1/T

Alternative notation: sin(αt)=sin(2πt/T)=sin(2πft)

# Fourier Series Theorem

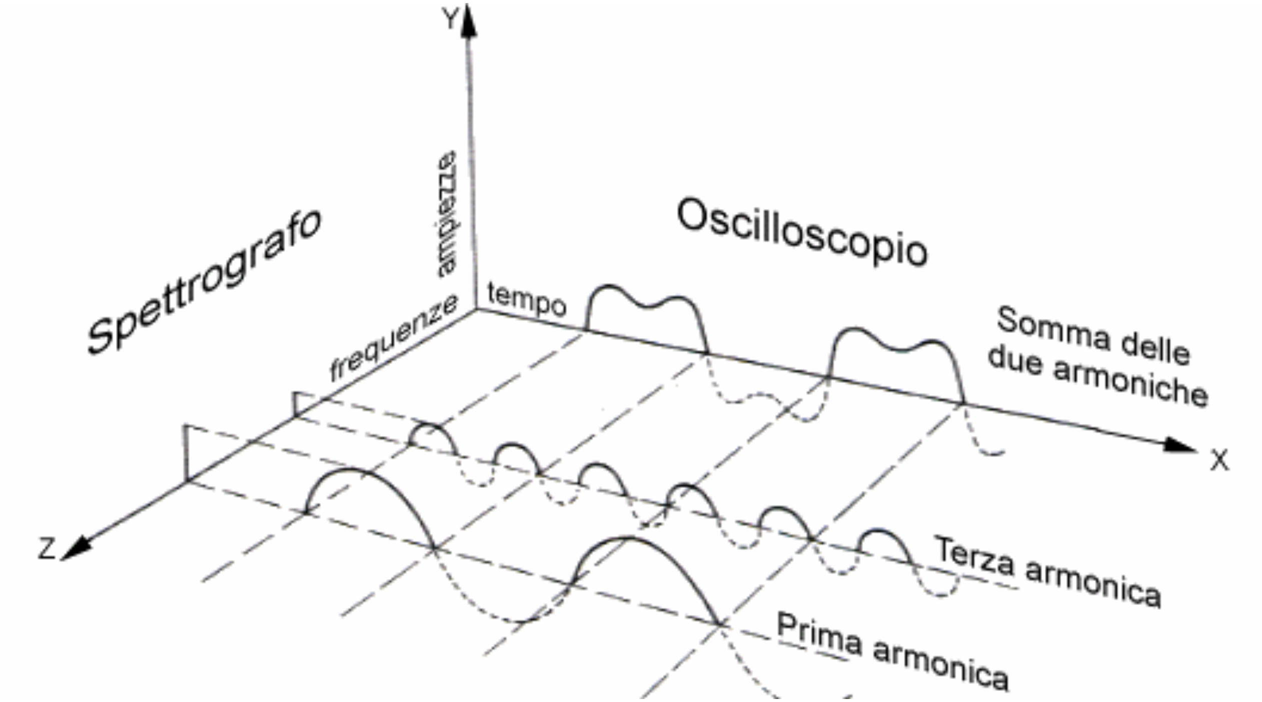
- Any periodic function can be expressed as a weighted sum (infinite) of sine and cosine functions of varying frequency:

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n cos(nf_0 t) + \sum_{n=1}^{\infty} b_n sin(nf_0 t)$$

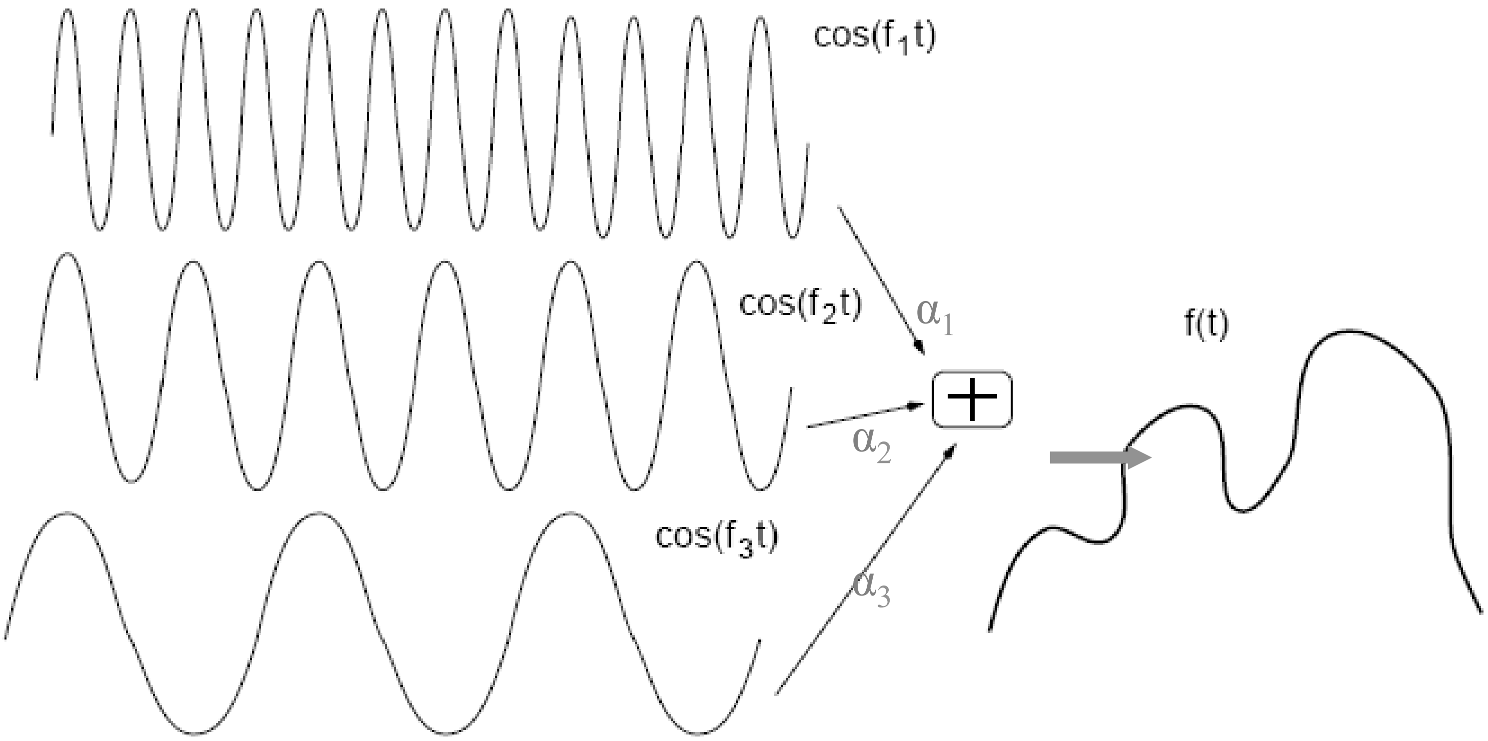
$f_0$ is called the “fundamental frequency”

$$a_n = 1/\pi \int_{-\pi}^{\pi} f(x) \cos(nf_0 t) \, dt \qquad b_n = 1/\pi \int_{-\pi}^{\pi} f(x) \sin(nf_0 t) \, dt$$

# Illustrazione



# Fourier Series (cont’ d)



# Continuous Fourier Transform (FT)

- Transforms a signal (i.e., function) from the **spatial** domain to the **frequency** domain.

Forward FT: $$F(f(x)) = F(u) = \int_{-\infty}^{\infty} f(x)e^{-j2\pi ux}dx$$

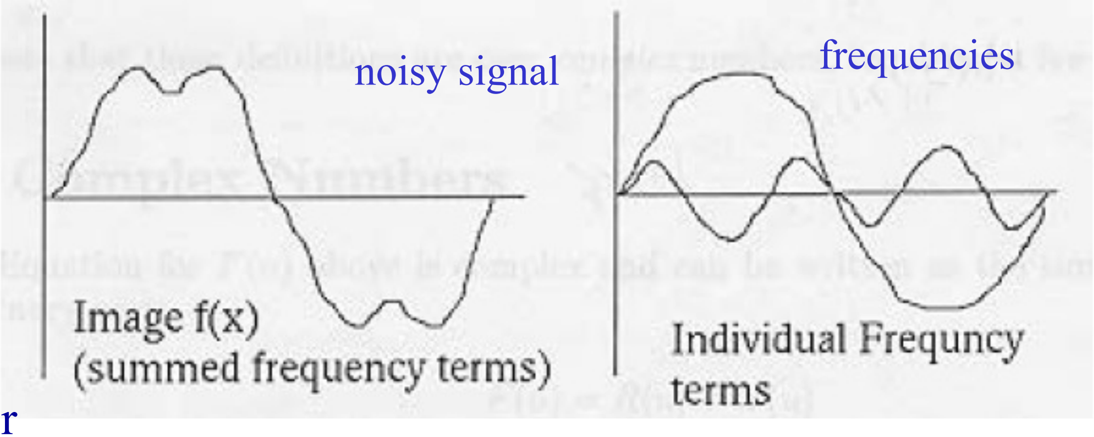
Inverse FT: (IFT) $$F^{-1}(F(u)) = f(x) = \int_{-\infty}^{\infty} F(u)e^{j2\pi ux}du$$

where $e^{\pm j\theta} = cos(\theta) \pm jsin(\theta)$

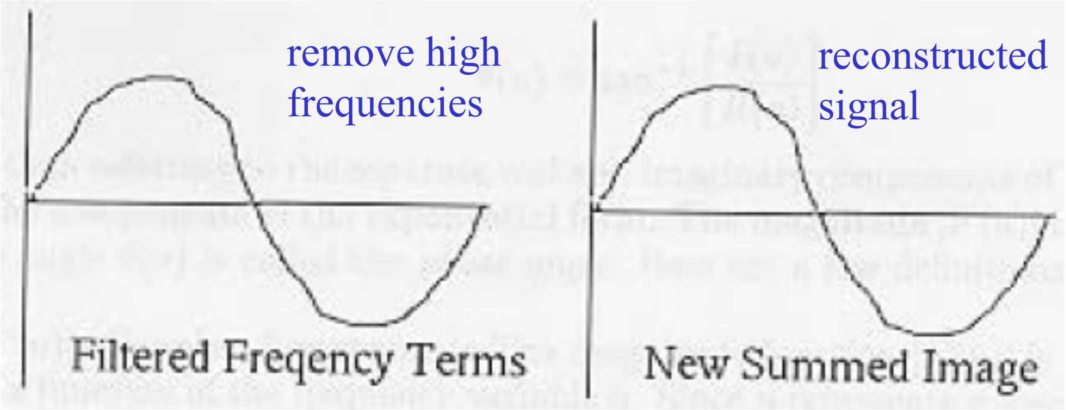
# Why is FT Useful?

- Easier to remove undesirable frequencies.
- Faster perform certain operations in the **frequency** domain than in the **spatial** domain.

# Example: Removing undesirable frequencies



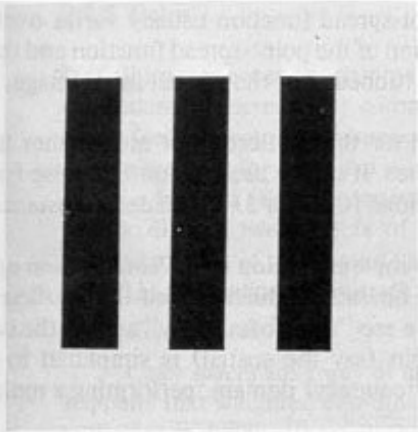
To remove certain frequencies, set their corresponding F(u) coefficients to zero!



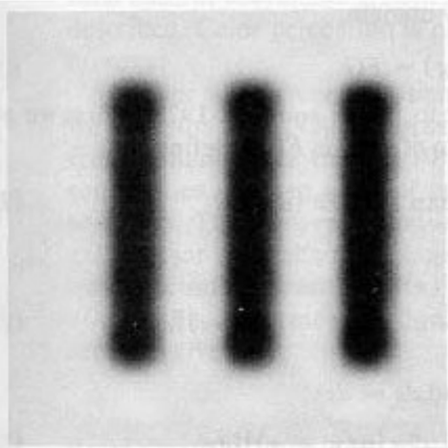
# How do frequencies show up in an image?

- Low frequencies correspond to slowly varying information (e.g., continuous surface).
- High frequencies correspond to quickly varying information (e.g., edges)

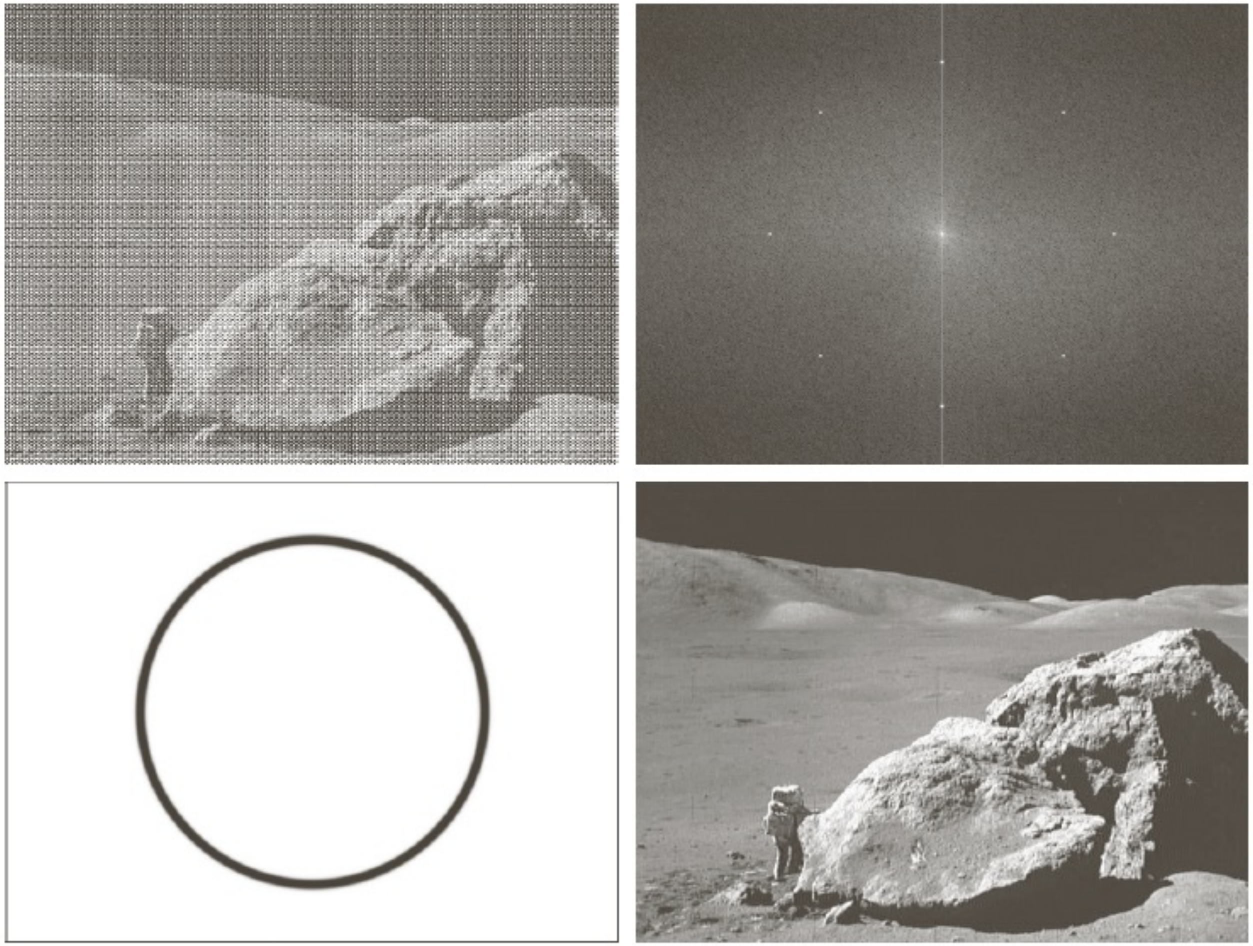
Original Image



Low-passed



# Example of noise reduction using FT



# Frequency Filtering Steps

1. Take the FT of f(x):

$$F(f(x))$$

2. Remove undesired frequencies:

$$D(F(f(x)))$$

3. Convert back to a signal:

$$\hat{f}(x) = F^{-1}(D(F(f(x))))$$

We' ll talk more about this later .....

# Definitions

- **F(u)** is a complex function:

$$F(u) = R(u) + jI(u)$$

- Magnitude of FT (spectrum):

$$|F(u)| = \sqrt{R^2(u) + I^2(u)}$$

- Phase of FT: $\phi(F(u)) = \tan^{-1}(\frac{I(u)}{R(u)})$

- Magnitude-Phase representation:

$$F(u) = |F(u)|e^{j\phi(u)}$$

$$R^2(u) + I^2(u)$$

- Energy of f(x): **P(u)**=|F(u)|$^2$

# Continuous Time Fourier Transform (CTFT)

Time is a real variable (t)

Frequency is a real variable (ω)

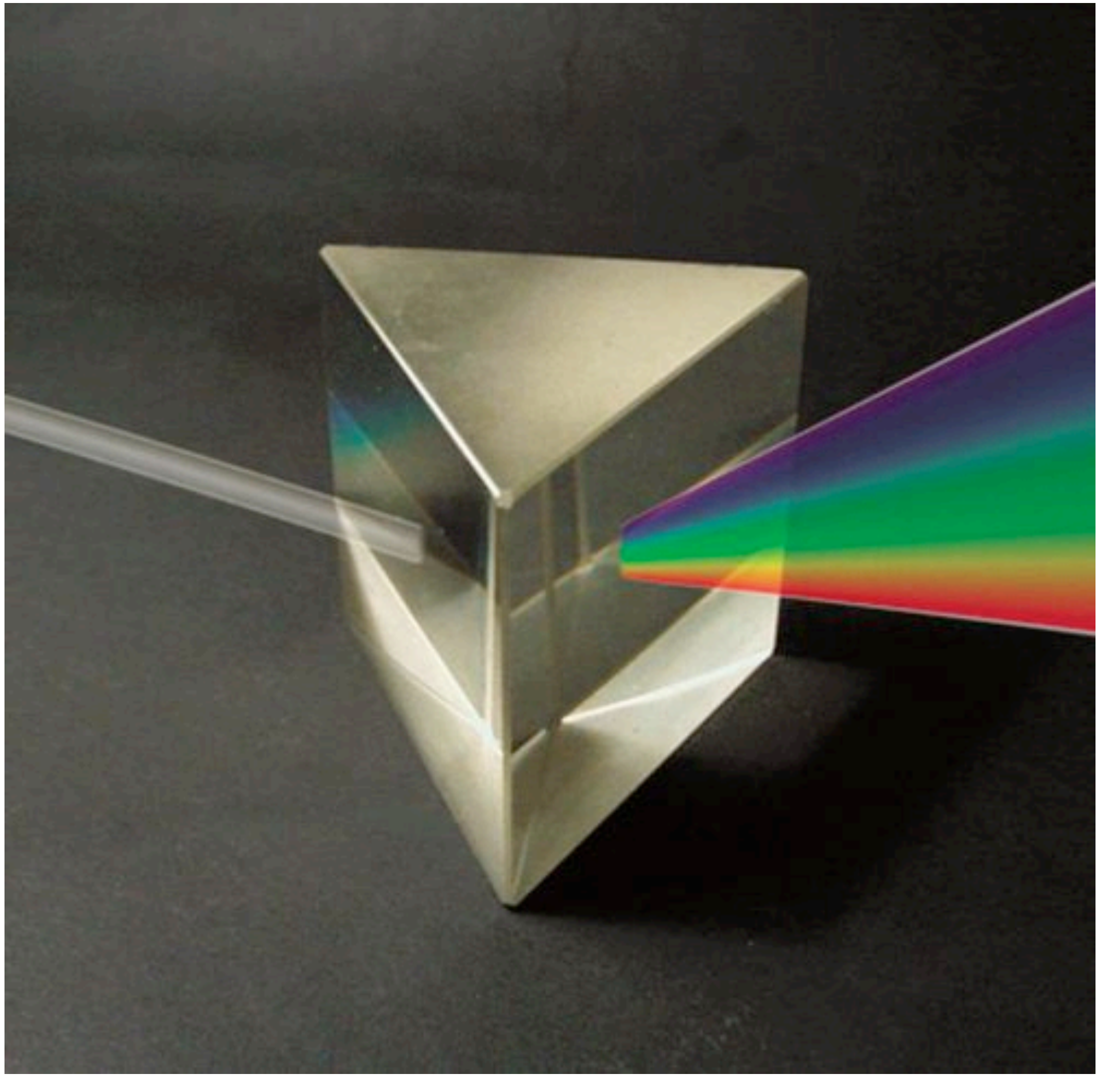
Signals : 1D

# The idea

A signal can be interpreted as en electromagnetic wave. This consists of lights of different “color”, or frequency, that can be split apart usign an optic prism. Each component is a “monochromatic” light with sinusoidal shape.

Following this analogy, each signal can be decomposed into its “sinusoidal” components which represent its “colors”.

Of course these components in general do not correspond to visible monochromatic light. However, they give an idea of how fast are the changes of the signal.



# CTFT: Concept

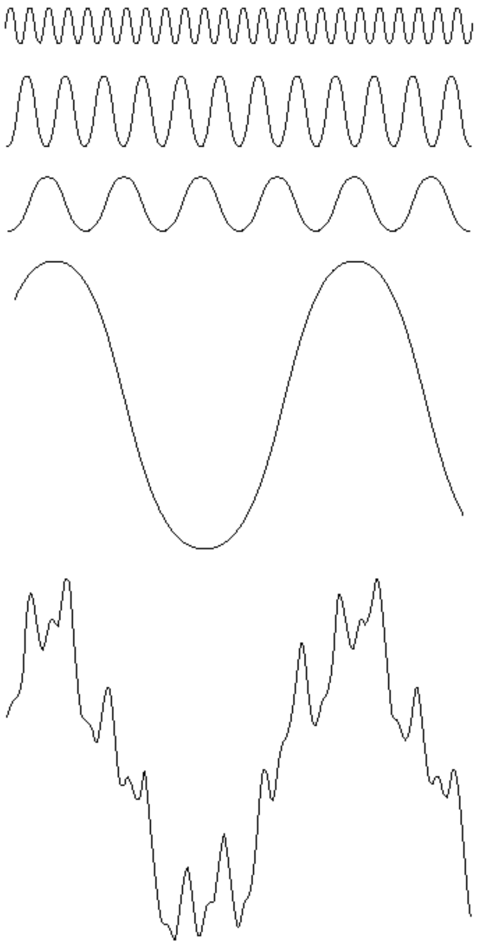


**FIGURE 4.1** The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

- A signal can be represented as a weighted sum of sinusoids.

- Fourier Transform is a change of basis, where the basis functions consist of sins and cosines (complex exponentials).

[Gonzalez Chapter 4]

# Continuous Time Fourier Transform (CTFT)

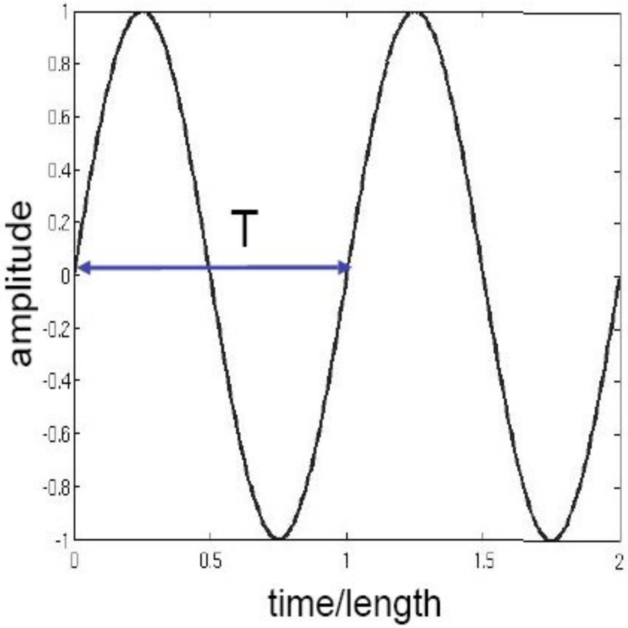
- Define frequency

  = 1/T

  *cycles per unit time*

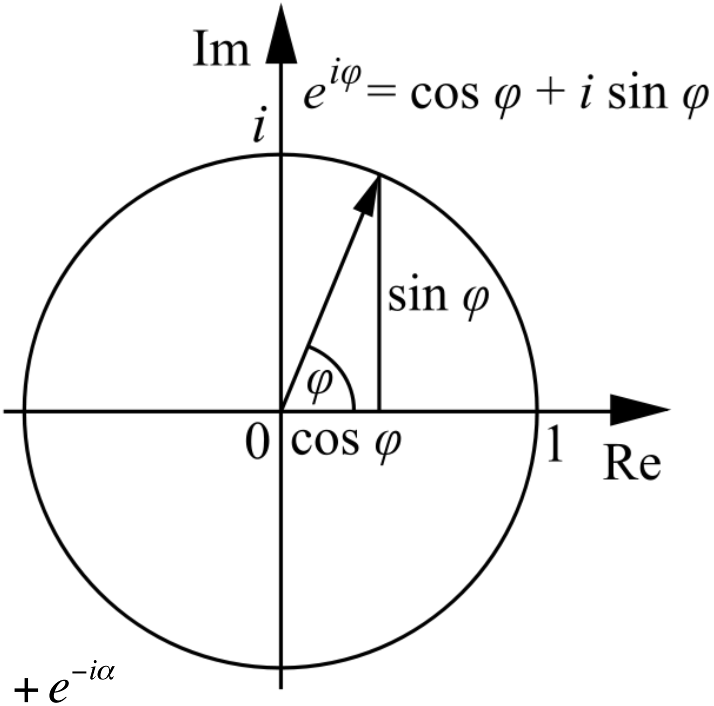
  *cycles per unit distance*

- Here f = 1 T=1



# Fourier Transform

- Cosine/sine signals are easy to define and interpret.
- Analysis and manipulation of sinusoidal signals is greatly simplified by dealing with related signals called complex exponential signals.
- A complex number has real and imaginary parts: $z = x + j\,y$
- *The Eulero formula* links complex exponential signals and trigonometric functions



$$r\,e^{j\alpha} = r(\cos\alpha + j\sin\alpha) \quad \Longleftrightarrow \quad \begin{aligned} \cos\alpha &= \frac{e^{i\alpha}+e^{-i\alpha}}{2} \\ \sin\alpha &= \frac{e^{i\alpha}-e^{-i\alpha}}{2i} \end{aligned}$$

# CTFT

- Continuous Time Fourier Transform
- Continuous time *a-periodic* signal
- Both time (space) and frequency are continuous variables
  - NON normalized frequency ω is used
- Fourier integral can be regarded as a Fourier series with fundamental frequency approaching zero
- Fourier spectra are continuous
  - A signal is represented as a sum of sinusoids (or exponentials) of all frequencies over a continuous frequency interval

Fourier *integral*

$$F(\omega) = \int_t f(t) e^{-j\omega t} dt$$

analysis

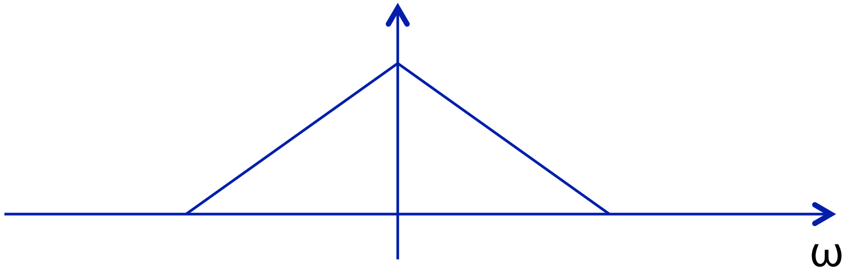
$$f(t) = \frac{1}{2\pi} \int_\omega F(\omega) e^{j\omega t} d\omega$$

synthesis

# CTFT of real signals

- Real signals: each signal sample is a real number
- Property: the CTFT is symmetric

$$\hat{f}(-\omega) = f(\omega)$$



$$f(t) \rightarrow \hat{f}(\omega)$$

$$f(-t) \rightarrow \hat{f}(-\omega) = \hat{f}^{*}(\omega)$$

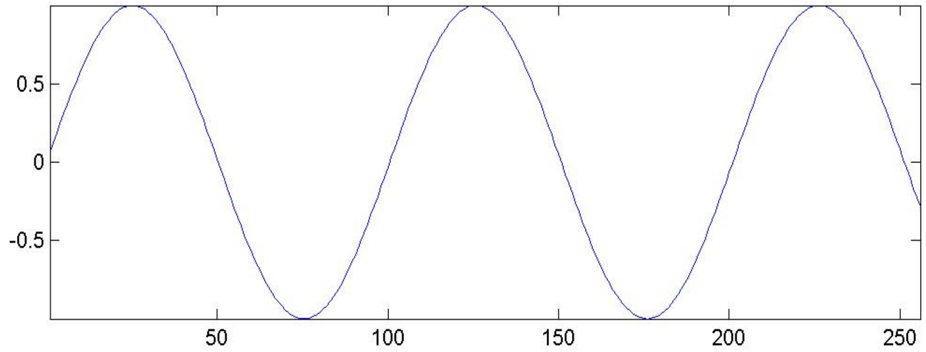
$$\Pr oof$$

$$\Im\{f(-t)\} = \int_{-\infty}^{+\infty} f(-t)e^{-j\omega t}dt = \int_{-\infty}^{+\infty} f(t')e^{j\omega t'}dt' = \hat{f}(-\omega)$$

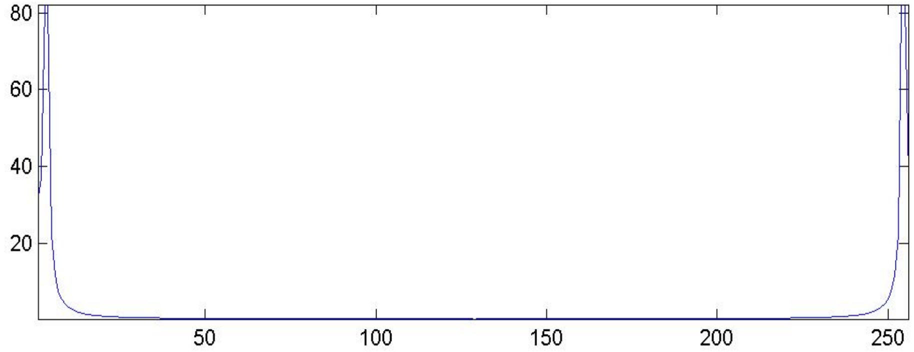
# Sinusoids

- Frequency domain characterization of signals

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt$$



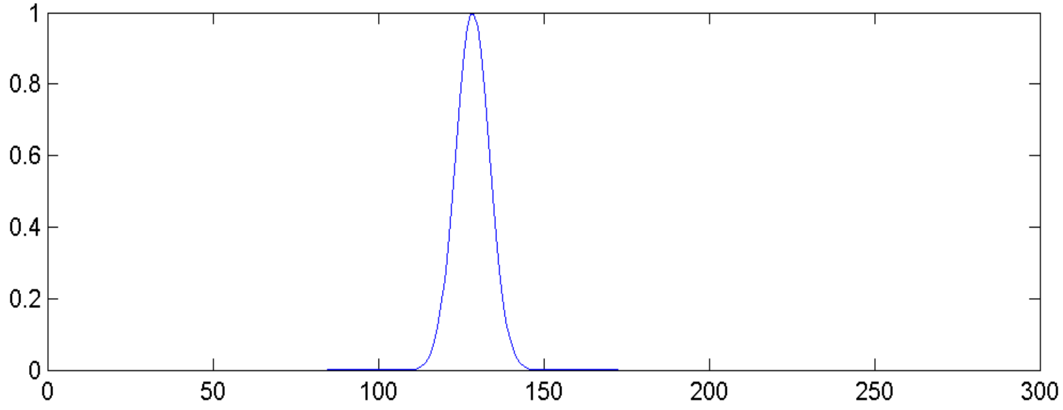
Signal domain



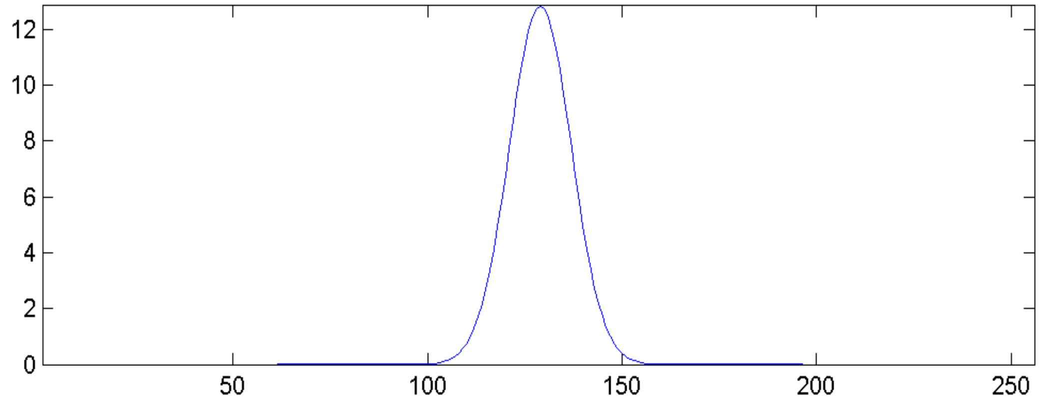
Frequency domain (spectrum, absolute value of the transform)

# Gaussian

Time domain

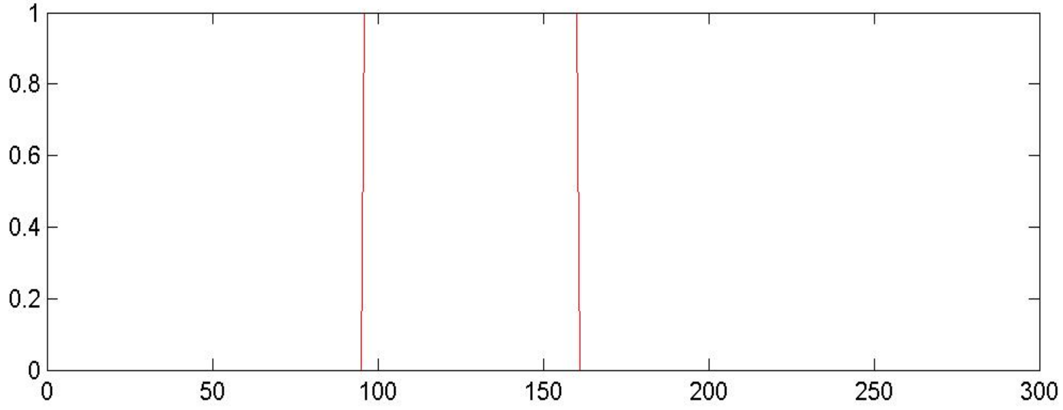


Frequency domain

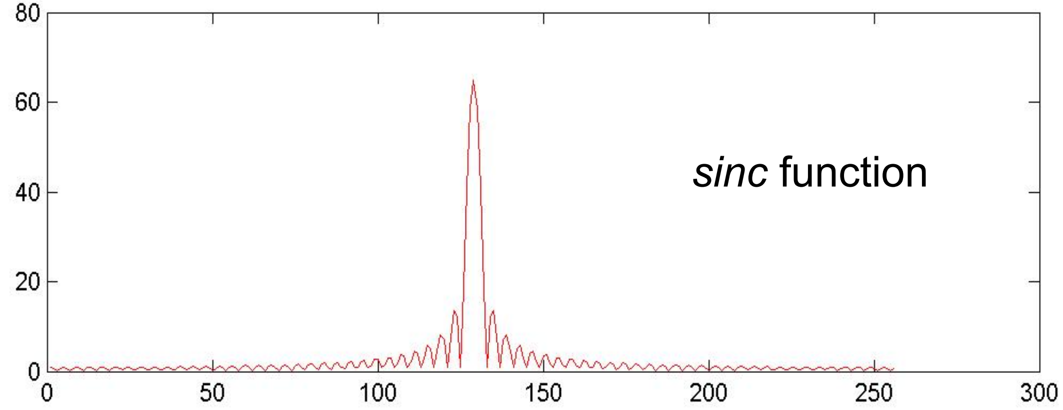


# rect

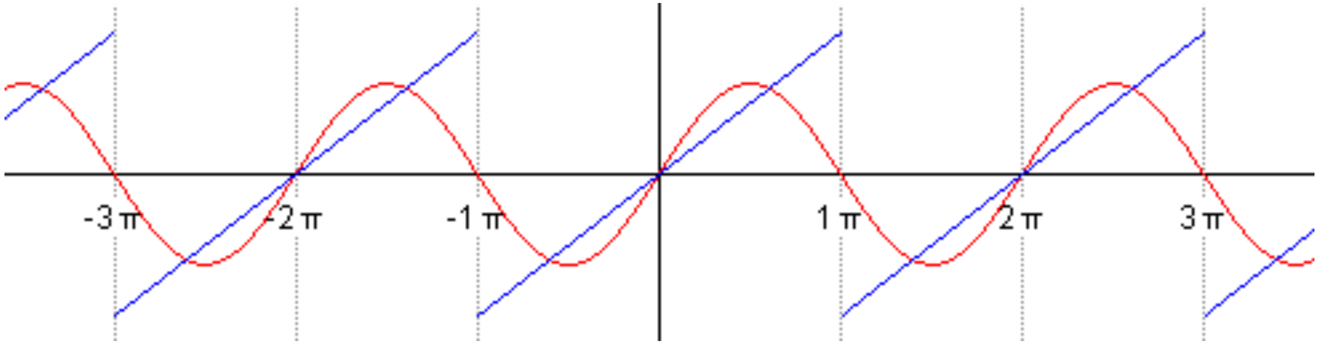
Time domain



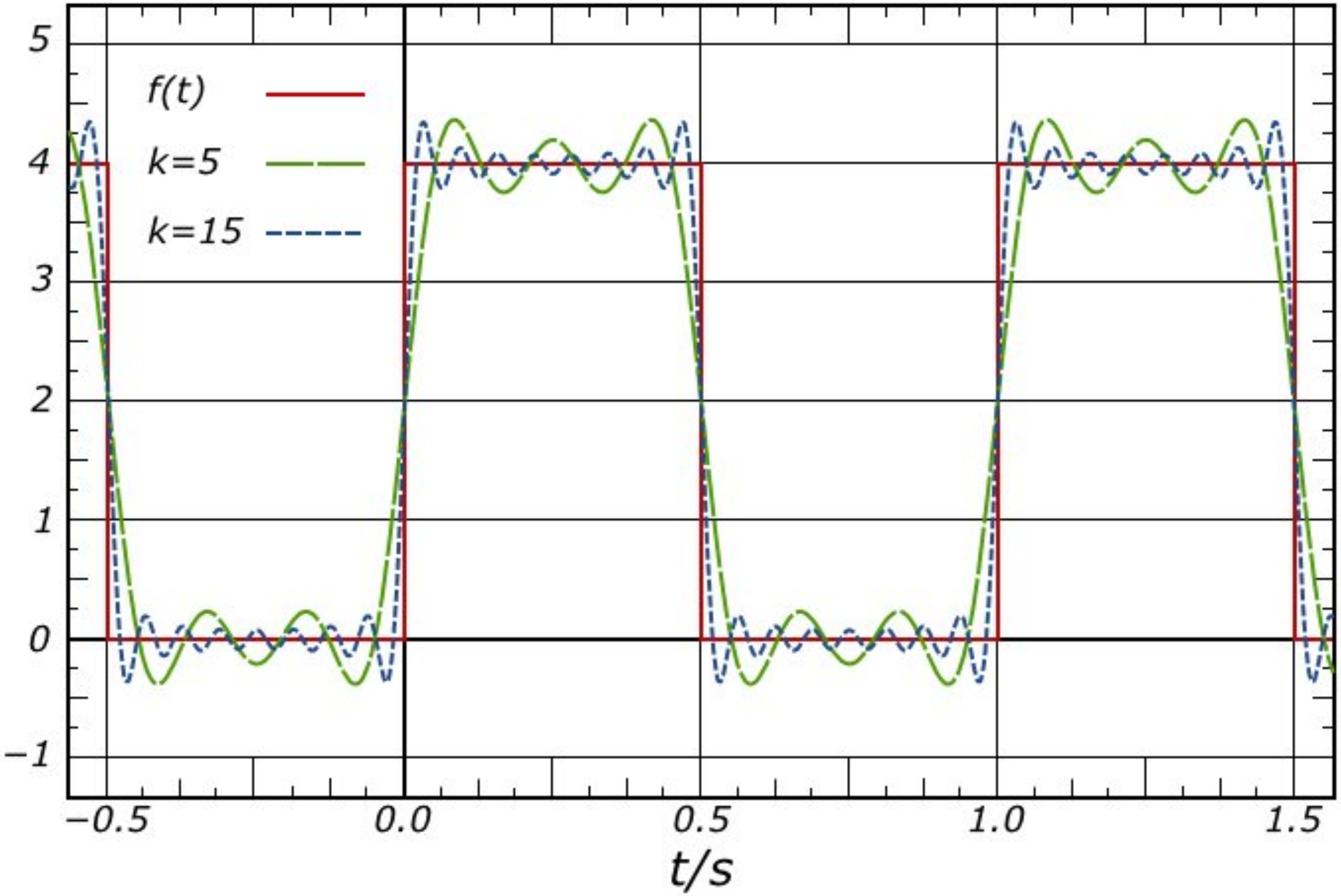
Frequency domain


*sinc* function

# Example



# Example



# Properties

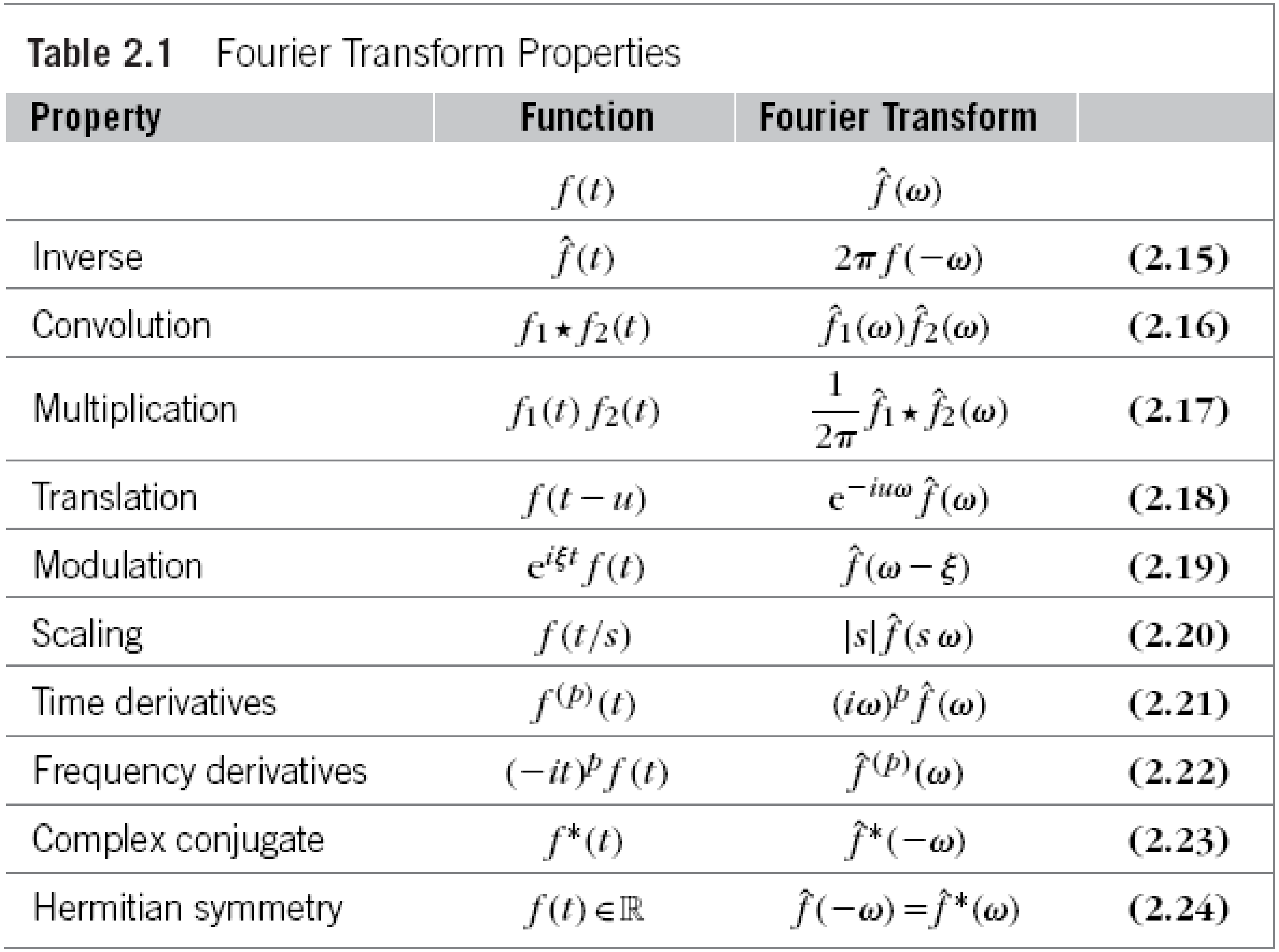


**Table 2.1** Fourier Transform Properties

| Property | Function | Fourier Transform | |
|---|---|---|---|
| | $f(t)$ | $\hat{f}(\omega)$ | |
| Inverse | $\hat{f}(t)$ | $2\pi f(-\omega)$ | (2.15) |
| Convolution | $f_1 \star f_2(t)$ | $\hat{f}_1(\omega)\hat{f}_2(\omega)$ | (2.16) |
| Multiplication | $f_1(t)\, f_2(t)$ | $\frac{1}{2\pi}\hat{f}_1 \star \hat{f}_2(\omega)$ | (2.17) |
| Translation | $f(t-u)$ | $e^{-iu\omega}\hat{f}(\omega)$ | (2.18) |
| Modulation | $e^{i\xi t} f(t)$ | $\hat{f}(\omega-\xi)$ | (2.19) |
| Scaling | $f(t/s)$ | $\|s\|\hat{f}(s\,\omega)$ | (2.20) |
| Time derivatives | $f^{(p)}(t)$ | $(i\omega)^p\hat{f}(\omega)$ | (2.21) |
| Frequency derivatives | $(-it)^p f(t)$ | $\hat{f}^{(p)}(\omega)$ | (2.22) |
| Complex conjugate | $f^*(t)$ | $\hat{f}^*(-\omega)$ | (2.23) |
| Hermitian symmetry | $f(t)\in\mathbb{R}$ | $\hat{f}(-\omega)=\hat{f}^*(\omega)$ | (2.24) |

# CTFT

- Change of variables for simplified notations: ω=2πu

$$F(2\pi u) = F(u) = \int_{-\infty}^{\infty} f(x)e^{-j2\pi ux}dx =$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)e^{j2\pi ux}d\left(2\pi u\right) = \int_{-\infty}^{\infty} F(u)e^{j2\pi ux}du$$

- More compact notations (same as in GW)

$$F(u) = \int_{-\infty}^{\infty} f(x)e^{-j2\pi ux}dx$$

$$f\left(x\right) = \int_{-\infty}^{\infty} F(u)e^{j2\pi ux}du$$

# Images vs Signals

## 1D

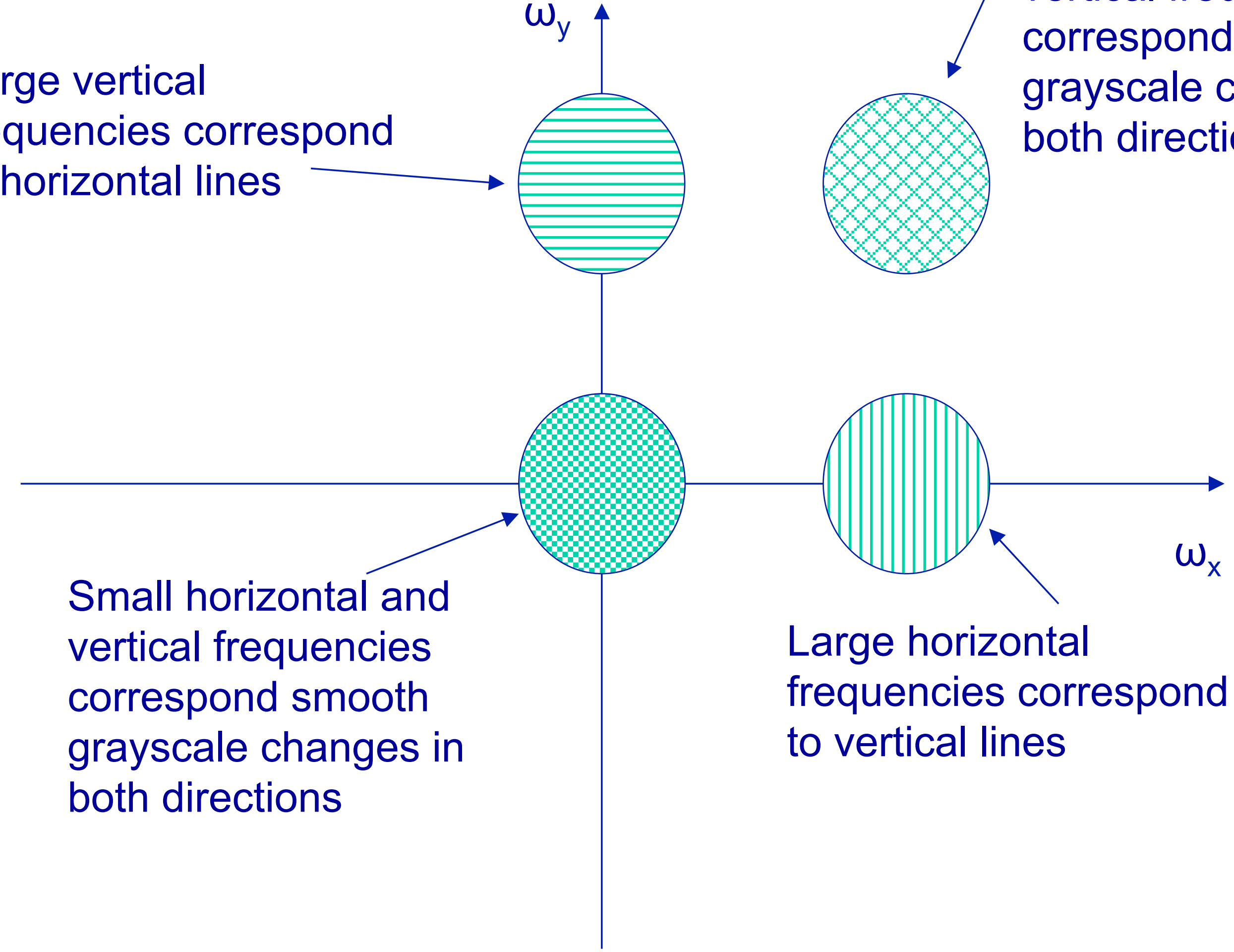
- Signals
- Frequency
  - Temporal
  - Spatial
- Time (space) frequency characterization of signals
- Reference space for
  - Filtering
  - Changing the sampling rate
  - Signal analysis
  - ….

## 2D

- Images
- Frequency
  - Spatial
- Space/frequency characterization of 2D signals
- Reference space for
  - Filtering
  - Up/Down sampling
  - Image analysis
  - Feature extraction
  - Compression
  - ….

# 2D Continuous FT

# 2D Frequency domain



Large vertical frequencies correspond to horizontal lines

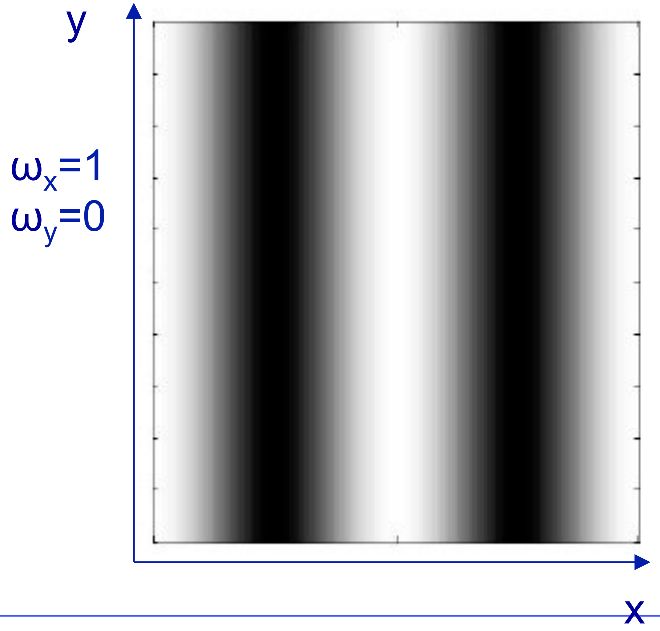
Large horizontal and vertical frequencies correspond sharp grayscale changes in both directions

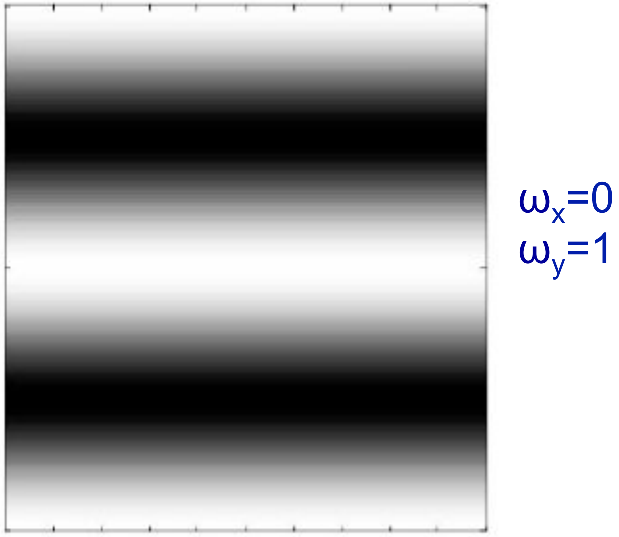
Small horizontal and vertical frequencies correspond smooth grayscale changes in both directions

Large horizontal frequencies correspond to vertical lines

# 2D spatial frequencies

- 2D spatial frequencies characterize the image spatial changes in the horizontal (x) and vertical (y) directions
  - Smooth variations -> low frequencies
  - Sharp variations -> high frequencies





# 2D Continuous Fourier Transform

- 2D Continuous Fourier Transform (notation 2)

$$\hat{f}(u,v)=\int_{-\infty}^{+\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy$$

$$f(x,y)=\int_{-\infty}^{+\infty} \hat{f}(u,v)e^{j2\pi(ux+vy)}dudv=$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|f(x,y)\right|^{2}dxdy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\hat{f}(u,v)\right|^{2}dudv$$

Plancherel’ s equality

# Delta

- Sampling property of the 2D-delta function (Dirac’s delta)

$$\int_{-\infty}^{\infty} \delta(x - x_0, y - y_0) f(x, y) dxdy = f(x_0, y_0)$$

- Transform of the delta function

$$F\left\{\delta(x, y)\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x, y) e^{-j2\pi(ux+vy)} dxdy = 1$$

$$F\left\{\delta(x - x_0, y - y_0)\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x - x_0, y - y_0) e^{-j2\pi(ux+vy)} dxdy = e^{-j2\pi(ux_0+vy_0)}$$

shifting property

# Constant functions

- Inverse transform of the impulse function

$$F^{-1}\left\{\delta(u,v)\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(u,v)e^{j2\pi(ux+vy)}dudv = e^{j2\pi(0x+v0)} = 1$$

- Fourier Transform of the constant (=1 for all x and y)

$$k(x,y) = 1 \qquad \forall x, y$$

$$F\left(u,v\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-j2\pi(ux+vy)}dxdy = \delta(u,v)$$

# Trigonometric functions

- Cosine function oscillating along the x axis
  - Constant along the y axis

$$s(x,y)=\cos(2\pi fx)$$

$$F\left\{\cos(2\pi fx)\right\}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cos(2\pi fx)e^{-j2\pi(ux+vy)}dxdy=$$

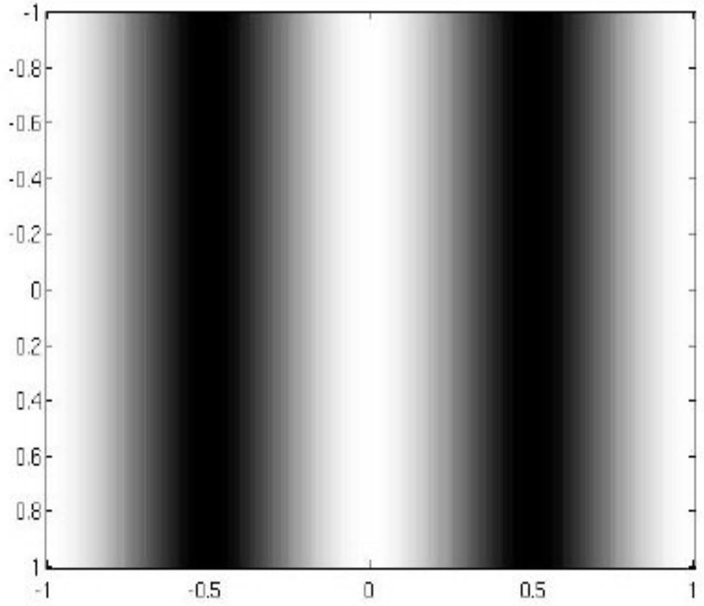
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{e^{j2\pi(fx)}+e^{-j2\pi(fx)}}{2}\right]e^{-j2\pi(ux+vy)}dxdy$$

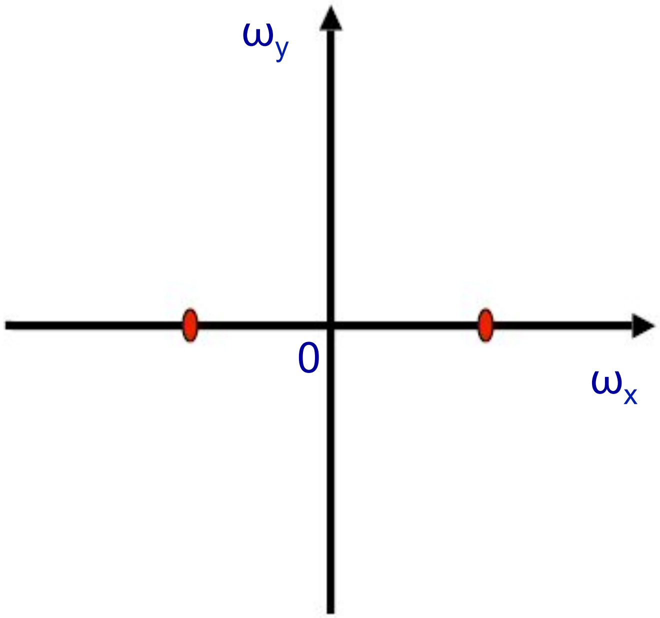
$$=\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[e^{-j2\pi(u-f)x}+e^{-j2\pi(u+f)x}\right]e^{-j2\pi vy}dxdy=$$

$$=\frac{1}{2}\int_{-\infty}^{\infty}e^{-j2\pi vy}dy\int_{-\infty}^{\infty}\left[e^{-j2\pi(u-f)x}+e^{-j2\pi(u+f)x}\right]dx=\frac{1}{2}1\int_{-\infty}^{\infty}\left[e^{-j2\pi(u-f)x}+e^{-j2\pi(u+f)x}\right]dx=$$

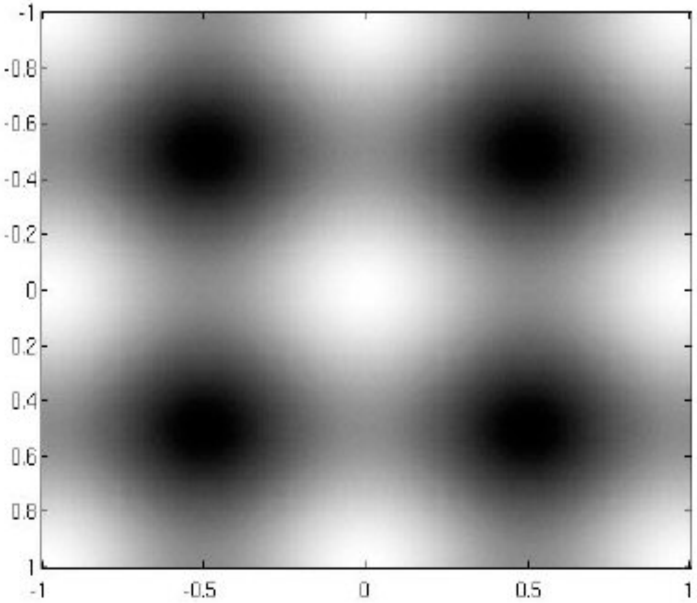
$$\frac{1}{2}\left[\delta(u-f)+\delta(u+f)\right]$$

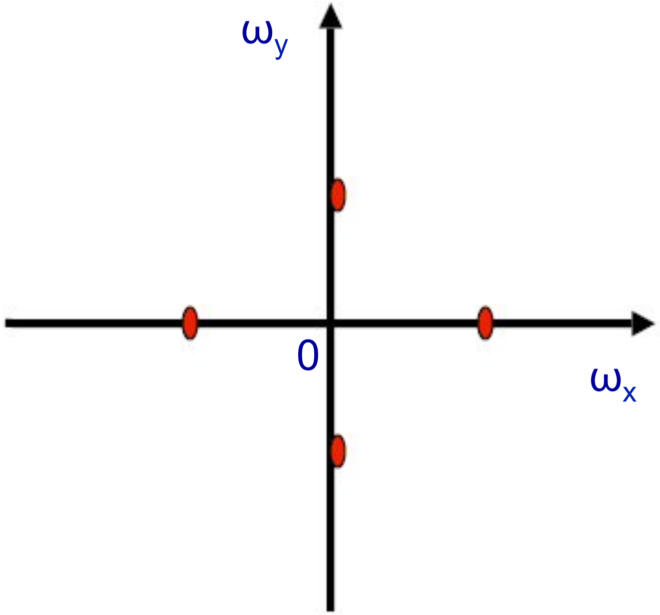
# Vertical grating



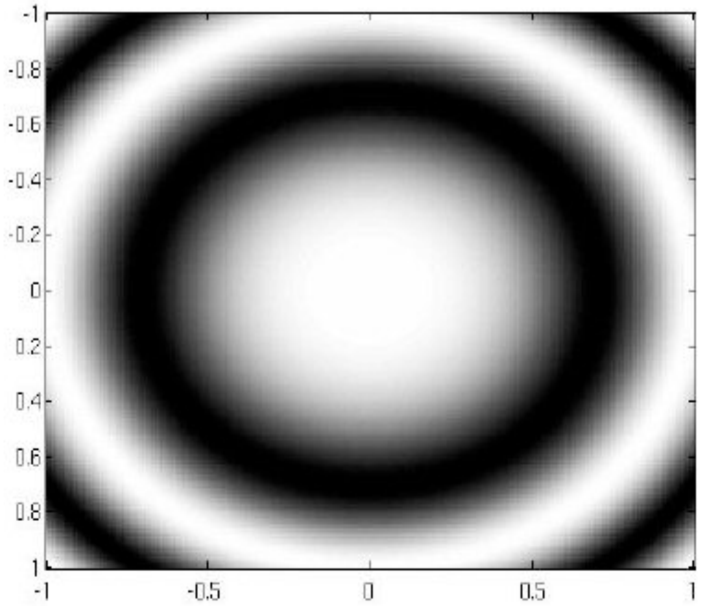


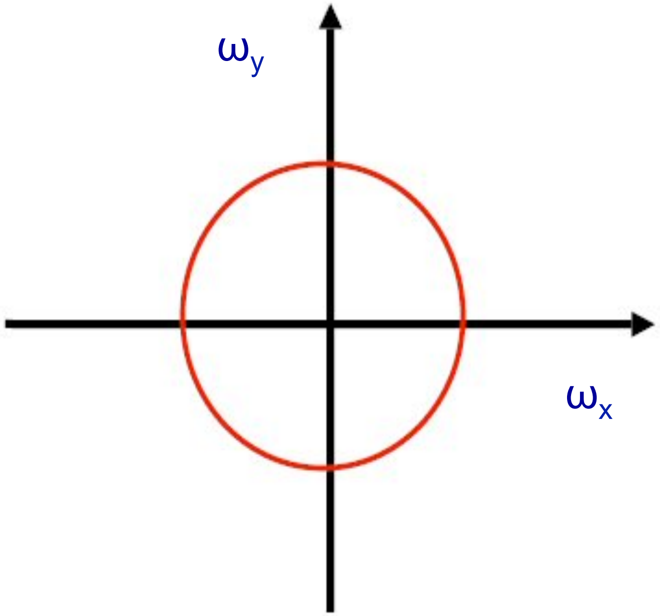
# Double grating



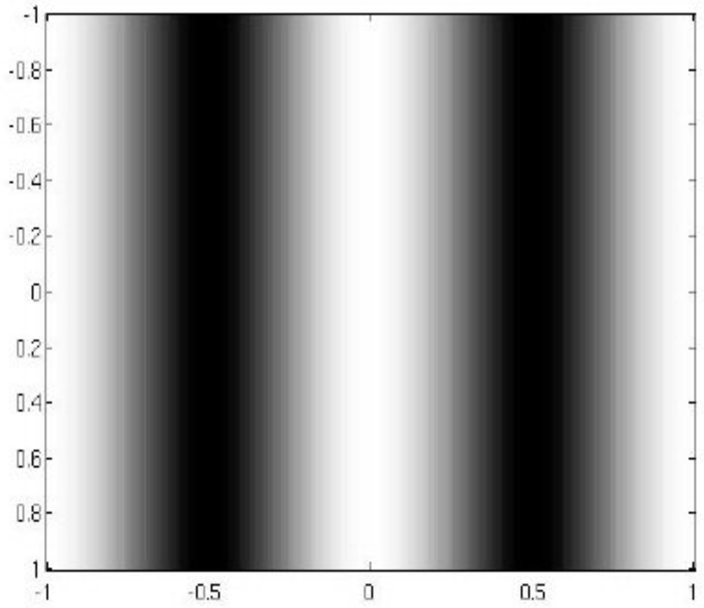


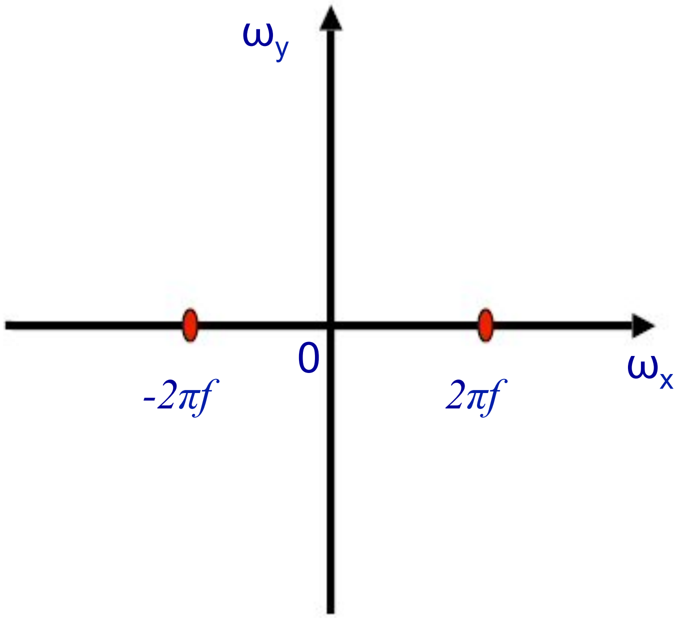
# Smooth rings



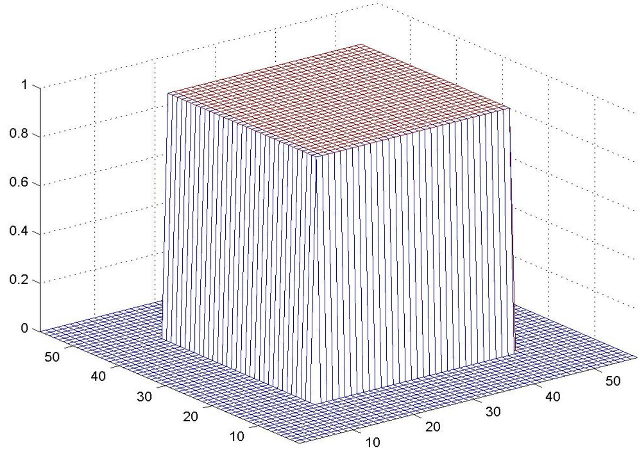


# Vertical grating

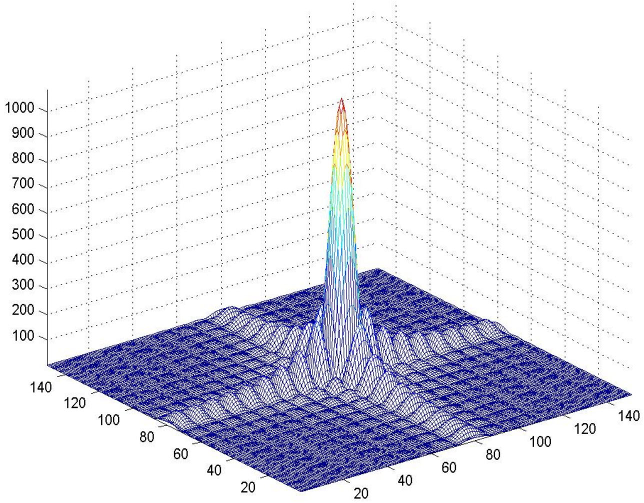


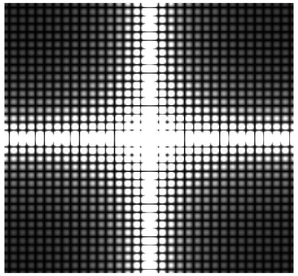


# 2D box



2D sinc





# CTFT properties

| | |
|---|---|
| ■ Linearity | $af(x,y)+bg(x,y) \Leftrightarrow aF(u,v)+bG(u,v)$ |
| ■ Shifting | $f(x-x_0,y-x_0) \Leftrightarrow e^{-j2\pi(ux_0+vy_0)}F(u,v)$ |
| ■ Modulation | $e^{j2\pi(u_0x+v_0y)}f(x,y) \Leftrightarrow F(u-u_0,v-v_0)$ |
| ■ Convolution | $f(x,y)*g(x,y) \Leftrightarrow F(u,v)G(u,v)$ |
| ■ Multiplication | $f(x,y)g(x,y) \Leftrightarrow F(u,v)*G(u,v)$ |
| ■ Separability | $f(x,y)=f(x)f(y) \Leftrightarrow F(u,v)=F(u)F(v)$ |

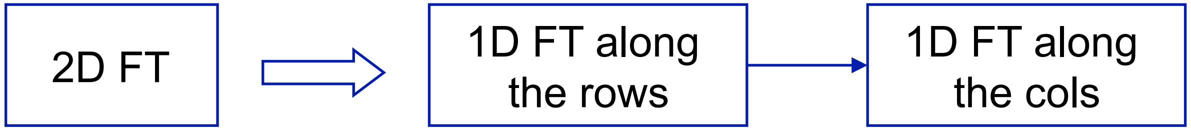
# Separability

1. Separability of the 2D Fourier transform
   - 2D Fourier Transforms can be implemented as a sequence of 1D Fourier Transform operations performed *independently* along the two axis

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi ux}e^{-j2\pi vy}dxdy = \int_{-\infty}^{\infty} e^{-j2\pi vy}dy\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi ux}dx =$$

$$= \int_{-\infty}^{\infty} F(u,y)e^{-j2\pi vy}dy = F\left(u,v\right)$$



2D FT

1D FT along the rows

1D FT along the cols

# Separability

- Separable functions can be written as $f\left(x,y\right)=f\left(x\right)g\left(y\right)$

2. The FT of a separable function is the product of the FTs of the two functions

$$F(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)e^{-j2\pi(ux+vy)}dxdy=$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h(x)g\left(y\right)e^{-j2\pi ux}e^{-j2\pi vy}dxdy=\int_{-\infty}^{\infty}g\left(y\right)e^{-j2\pi vy}dy\int_{-\infty}^{\infty}h(x)e^{-j2\pi ux}dx=$$

$$=H\left(u\right)G\left(v\right)$$

$$f\left(x,y\right)=h\left(x\right)g\left(y\right)\Rightarrow F\left(u,v\right)=H\left(u\right)G\left(v\right)$$

# Discrete Time Fourier Transform (DTFT)

Applies to Discrete time (sampled) signals and time series

1D

# Fourier Transform: 2D Discrete Signals

- Fourier Transform of a 2D discrete signal is defined as

$$F(u,v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f[m,n] e^{-j2\pi(um+vn)}$$

where $$\frac{-1}{2} \le u,v < \frac{1}{2}$$

$\omega_x$=2πu
$\omega_y$=2πv
F($\omega_x$, $\omega_y$) is 2π-periodic
f(x,y) is real -> F($\omega_x$, $\omega_y$) is symmetric so it can be completely specified in [0-π]. Accordingly, F(u,v) is periodic with period 1 and can be specified in [-1/2, 1/2]

- Inverse Fourier Transform

$$f[m,n] = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} F(u,v) e^{j2\pi(um+vn)} dudv$$

# Properties

- Periodicity: 2D Fourier Transform of a *discrete* a-periodic signal is *periodic*
  - *The period is 1 for the unitary frequency notations and 2π for normalized frequency notations*.
  - Proof (referring to the firsts case)

$$F(u+k,v+l)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}f[m,n]e^{-j2\pi\left((u+k)m+(v+l)n\right)}$$

Arbitrary integers

$$=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}f[m,n]e^{-j2\pi(um+vn)}\overset{1}{e^{-j2\pi km}}\,\overset{1}{e^{-j2\pi ln}}$$

$$=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}f[m,n]e^{-j2\pi(um+vn)}$$

$$=F(u,v)$$

# Fourier Transform: Properties

- *Periodicity:* Fourier Transform of a discrete signal is periodic with period 1.

$$F(u+k, v+l) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f[m,n] e^{-j2\pi\left((u+k)m+(v+l)n\right)}$$

Arbitrary integers

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f[m,n] e^{-j2\pi\left(um+vn\right)} \cancelto{1}{e^{-j2\pi km}} \cancelto{1}{e^{-j2\pi ln}}$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f[m,n] e^{-j2\pi(um+vn)}$$

$$= F(u,v)$$

# Fourier Transform: Properties

- Linearity, shifting, modulation, convolution, multiplication, separability, energy conservation properties also exist for the 2D Fourier Transform of discrete signals.

# DTFT Properties

- Linearity $af[m,n]+bg[m,n] \Leftrightarrow aF(u,v)+bG(u,v)$
- Shifting $f[m-m_0,n-n_0] \Leftrightarrow e^{-j2\pi(um_0+vn_0)}F(u,v)$
- Modulation $e^{j2\pi(u_0m+v_0n)}f[m,n] \Leftrightarrow F(u-u_0,v-v_0)$
- Convolution $f[m,n]*g[m,n] \Leftrightarrow F(u,v)G(u,v)$
- Multiplication $f[m,n]g[m,n] \Leftrightarrow F(u,v)*G(u,v)$
- Separable functions $f[m,n]=f[m]f[n] \Leftrightarrow F(u,v)=F(u)F(v)$
- Energy conservation $\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\left|f[m,n]\right|^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|F(u,v)\right|^2 dudv$

# Fourier Transform: Properties

- Define *Kronecker delta function*

$$\delta[m,n] = \begin{cases} 1, & \text{for } m = 0 \text{ and } n = 0 \\ 0, & \text{otherwise} \end{cases}$$

- Fourier Transform of the Kronecker delta function

$$F(u,v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left[ \delta[m,n] e^{-j2\pi(um+vn)} \right] = e^{-j2\pi(u0+v0)} = 1$$

# DTFT Properties

- Fourier Transform of 1

$$f(m,n)=1 \Leftrightarrow F(u,v)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\left[1e^{-j2\pi(um+vn)}\right]=\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\delta(u-k,v-l)$$

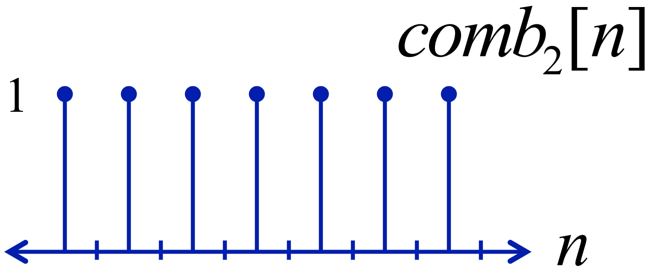
*To prove: Take the inverse Fourier Transform of the Dirac delta function and use the fact that the Fourier Transform has to be periodic with period 1.*

# Impulse Train

- Define a *comb* function (impulse train) as follows

$$comb_{M,N}[m,n] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \delta[m-kM, n-lN]$$

where *M* and *N* are integers



# Impulse Train

$$comb_{M,N}[m,n] \triangleq \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\delta[m-kM,n-lN]$$

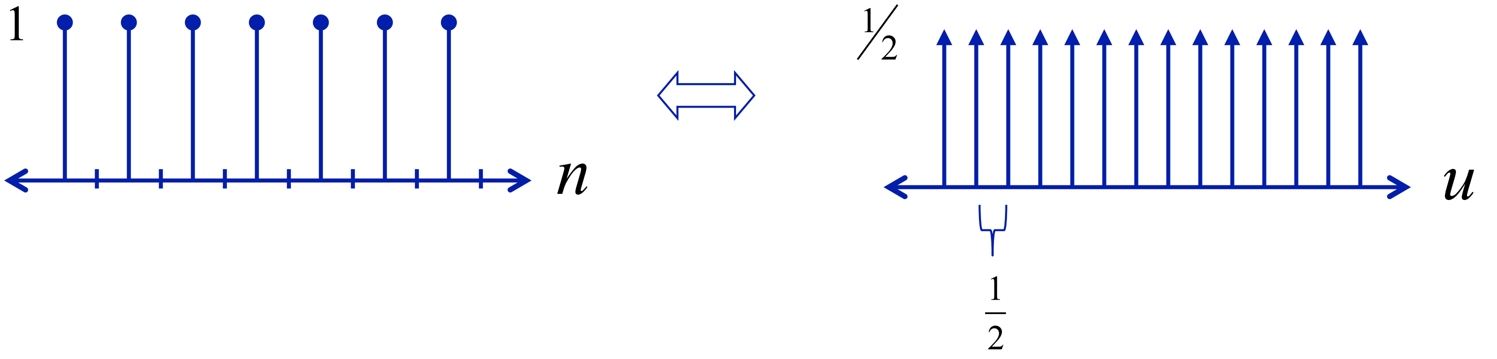
$$comb_{M,N}(x,y) \triangleq \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\delta\left(x-kM,y-lN\right)$$

- Fourier Transform of an impulse train is also an impulse train:

$$\underbrace{\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\delta\left[m-kM,n-lN\right]}_{comb_{M,N}[m,n]} \Leftrightarrow \frac{1}{MN}\underbrace{\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\delta\left(u-\frac{k}{M},v-\frac{l}{N}\right)}_{comb_{\frac{1}{M},\frac{1}{N}}(u,v)}$$

# Impulse Train

$$comb_2[n] \qquad \Leftrightarrow \qquad \frac{1}{2}\,comb_{\frac{1}{2}}(u)$$



# Impulse Train

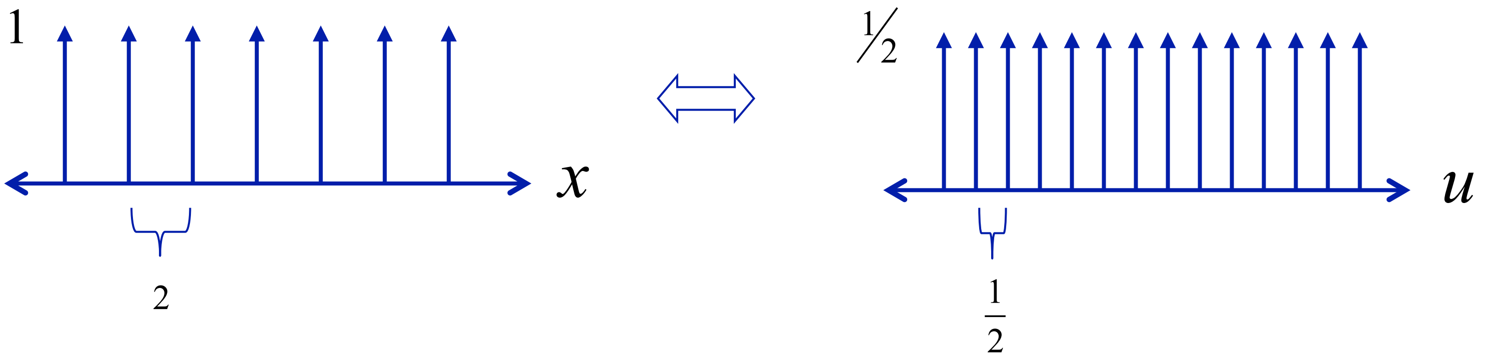
$$comb_{M,N}(x,y)=\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\delta\left(x-kM,y-lN\right)$$

- In the case of continuous signals:

$$\underbrace{\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\delta\left(x-kM,y-lN\right)}_{comb_{M,N}(x,y)}\Leftrightarrow\frac{1}{MN}\underbrace{\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\delta\left(u-\frac{k}{M},v-\frac{l}{N}\right)}_{comb_{\frac{1}{M},\frac{1}{N}}(u,v)}$$

# Impulse Train

$$comb_2(x) \qquad \Longleftrightarrow \qquad \frac{1}{2} comb_{\frac{1}{2}}(u)$$



# 2D DTFT: constant

- Fourier Transform of 1

$$f[k,l] = 1, \forall k,l$$

$$F[u,v] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \left[ 1e^{-j2\pi(uk+vl)} \right] =$$

$$= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \delta(u-k, v-l)$$

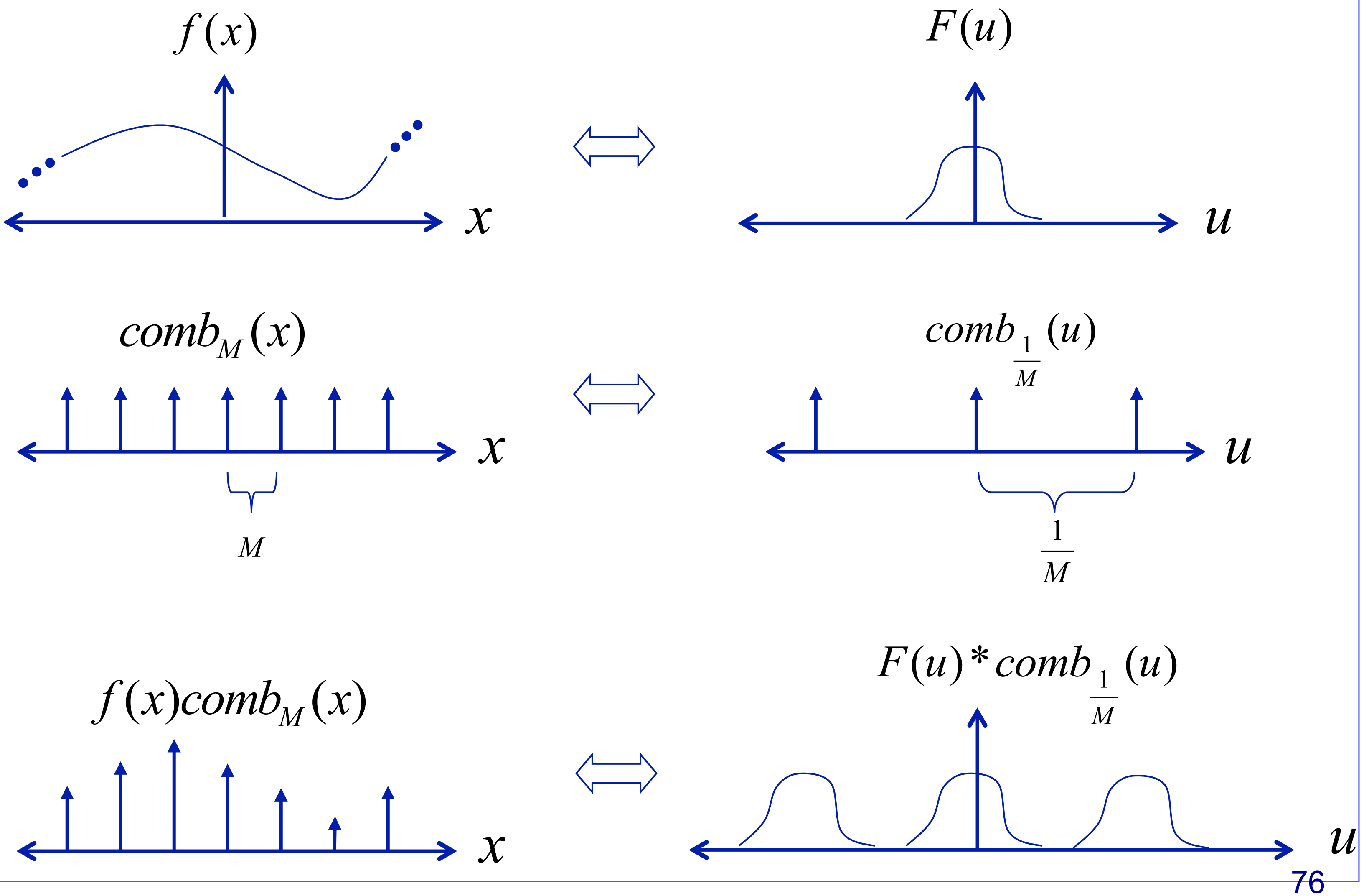
periodic with period 1
along u and v

*To prove: Take the inverse Fourier Transform of the Dirac delta function and use the fact that the Fourier Transform has to be periodic with period 1.*

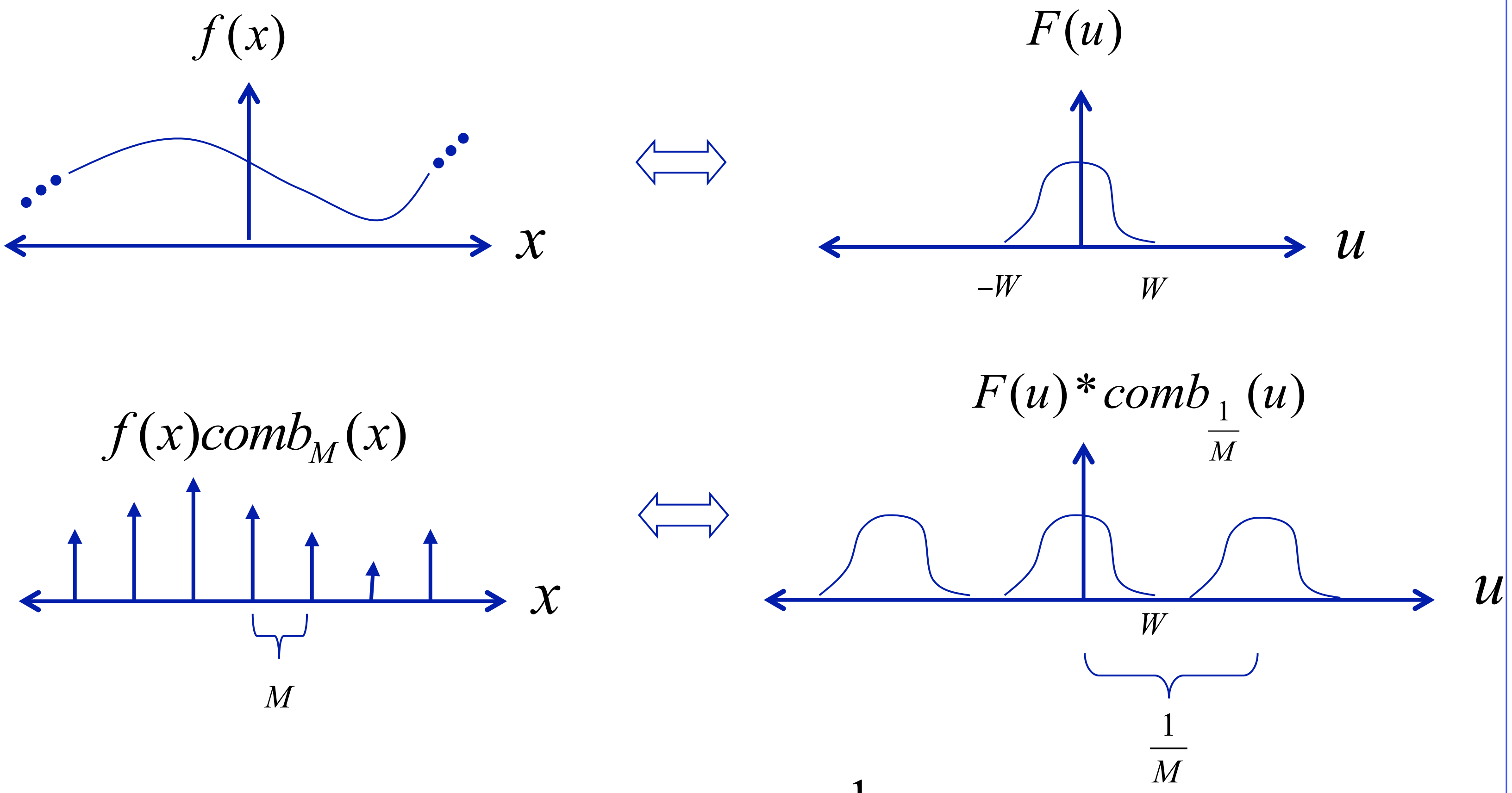
# Consequences

Sampling (Nyquist) theorem

# Sampling

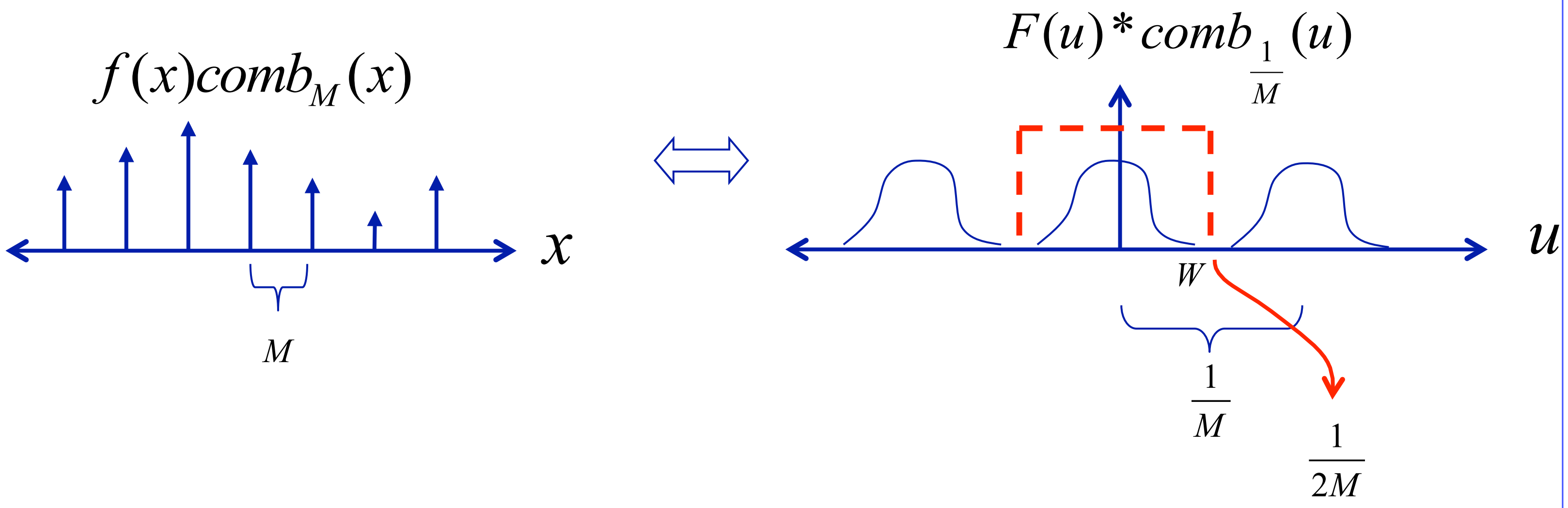

$f(x)$
$F(u)$
$x$
$u$
$comb_M(x)$
$comb_{\frac{1}{M}}(u)$
$M$
$\frac{1}{M}$
$x$
$u$
$f(x)comb_M(x)$
$F(u) * comb_{\frac{1}{M}}(u)$
$x$
$u$

# Sampling



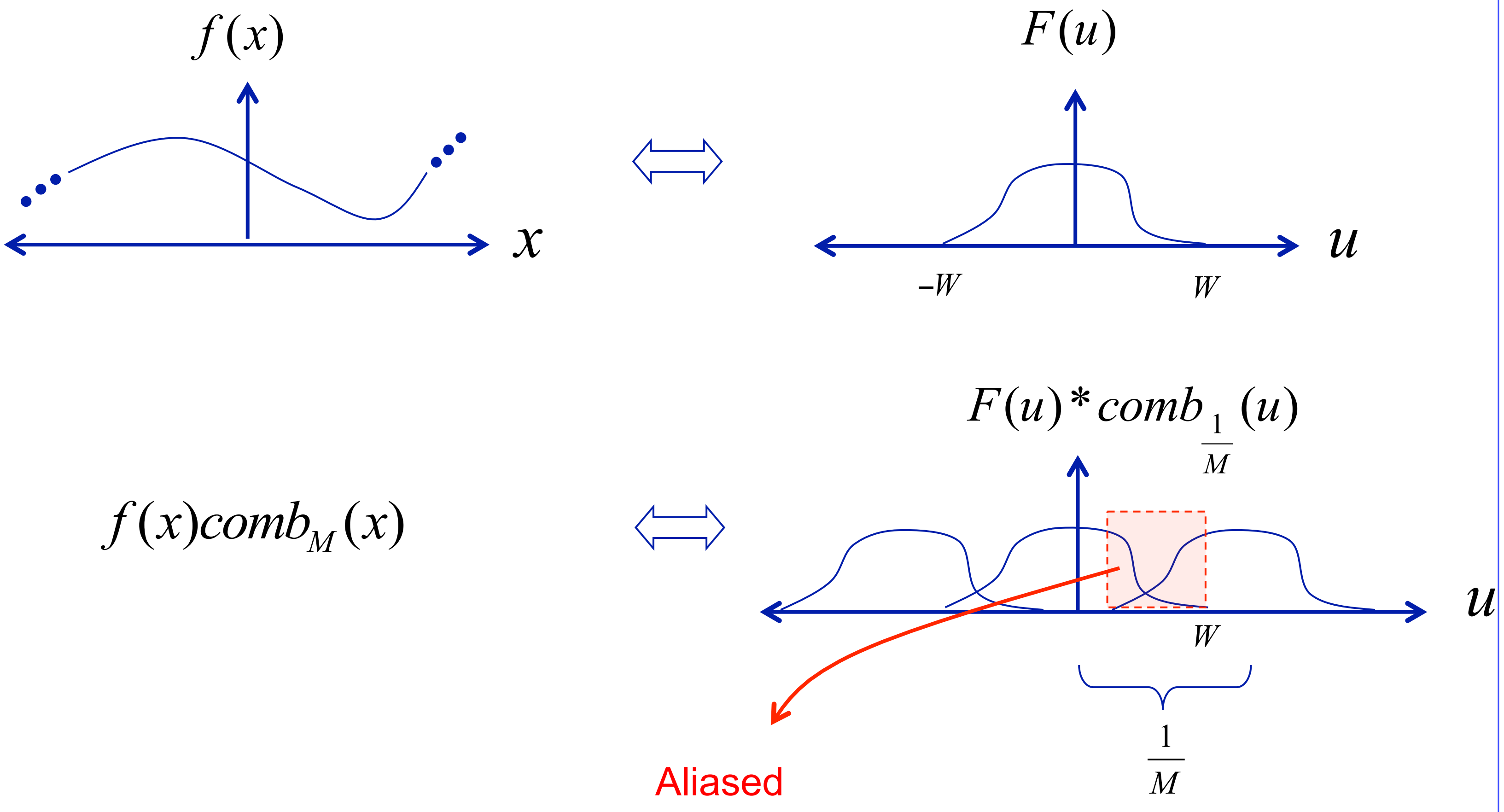
*Nyquist theorem: No aliasing if* $\frac{1}{M} > 2W$

# Sampling

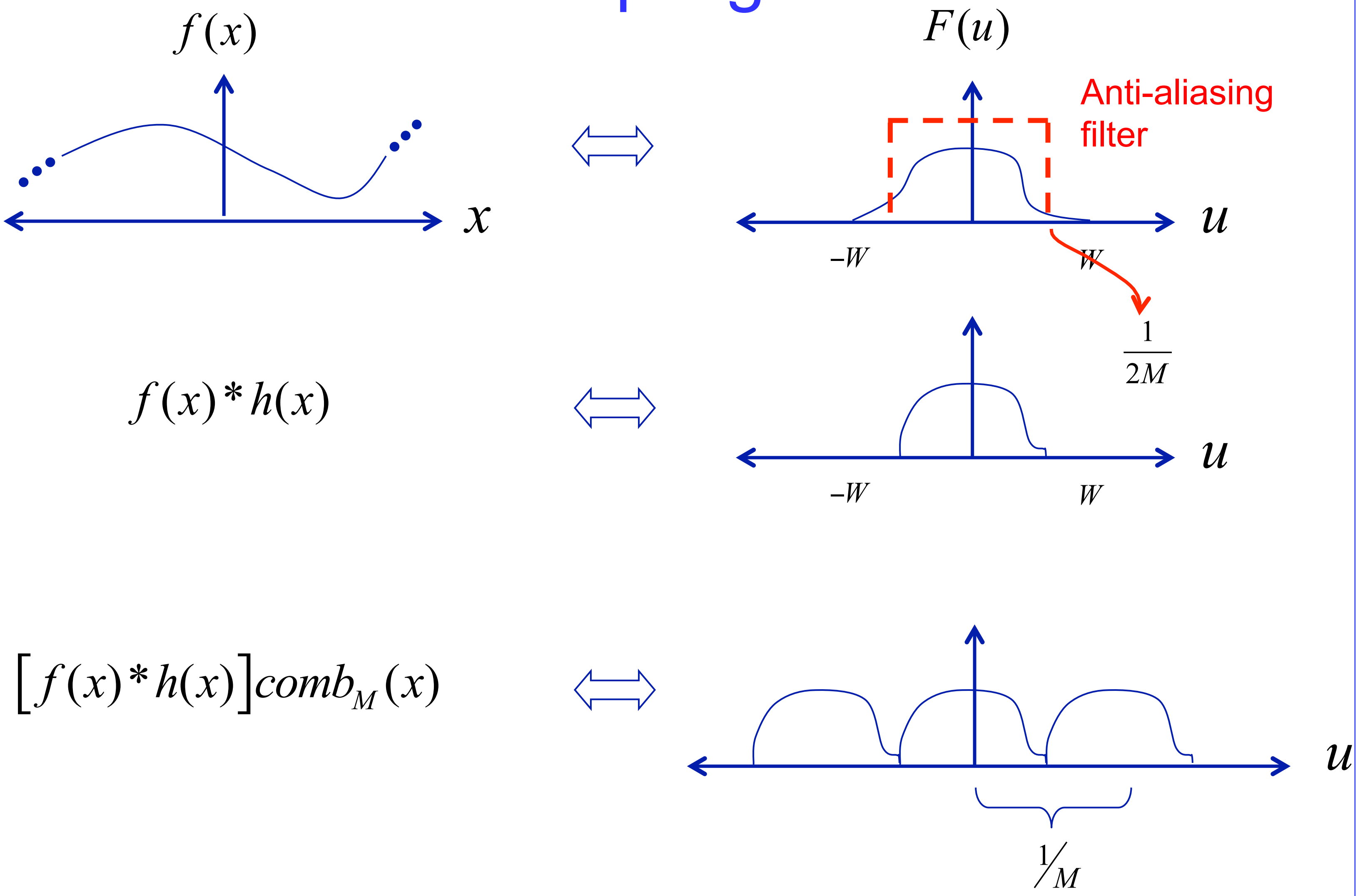


*If there is no aliasing, the original signal can be recovered from its samples by low-pass filtering.*

# Sampling


$f(x)$
$x$
$\Leftrightarrow$
$F(u)$
$u$
$-W$
$W$
$f(x)comb_M(x)$
$\Leftrightarrow$
$F(u) * comb_{\frac{1}{M}}(u)$
$u$
$W$
$\frac{1}{M}$
Aliased

# Sampling



$f(x) \Leftrightarrow F(u)$

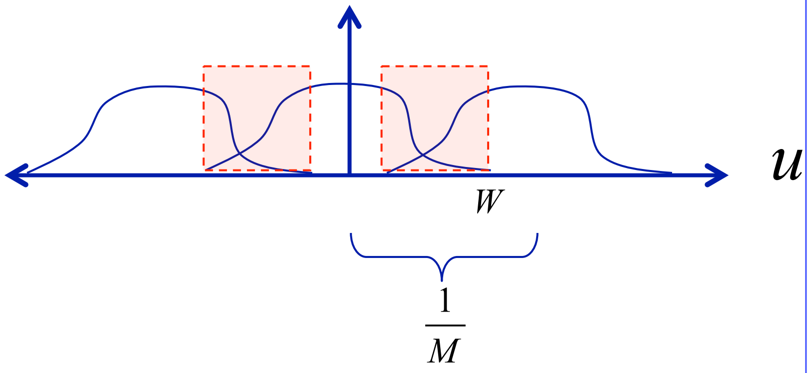
$f(x) * h(x) \Leftrightarrow$

$\left[ f(x) * h(x) \right] comb_M(x) \Leftrightarrow$

# Sampling

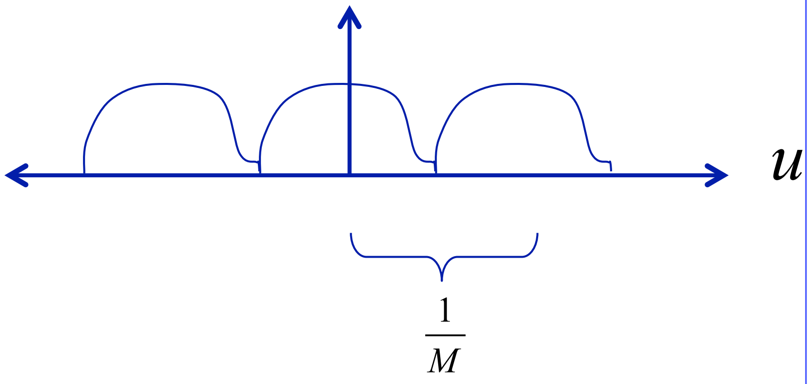
- Without anti-aliasing filter:

$$f(x)comb_M(x) \Leftrightarrow$$

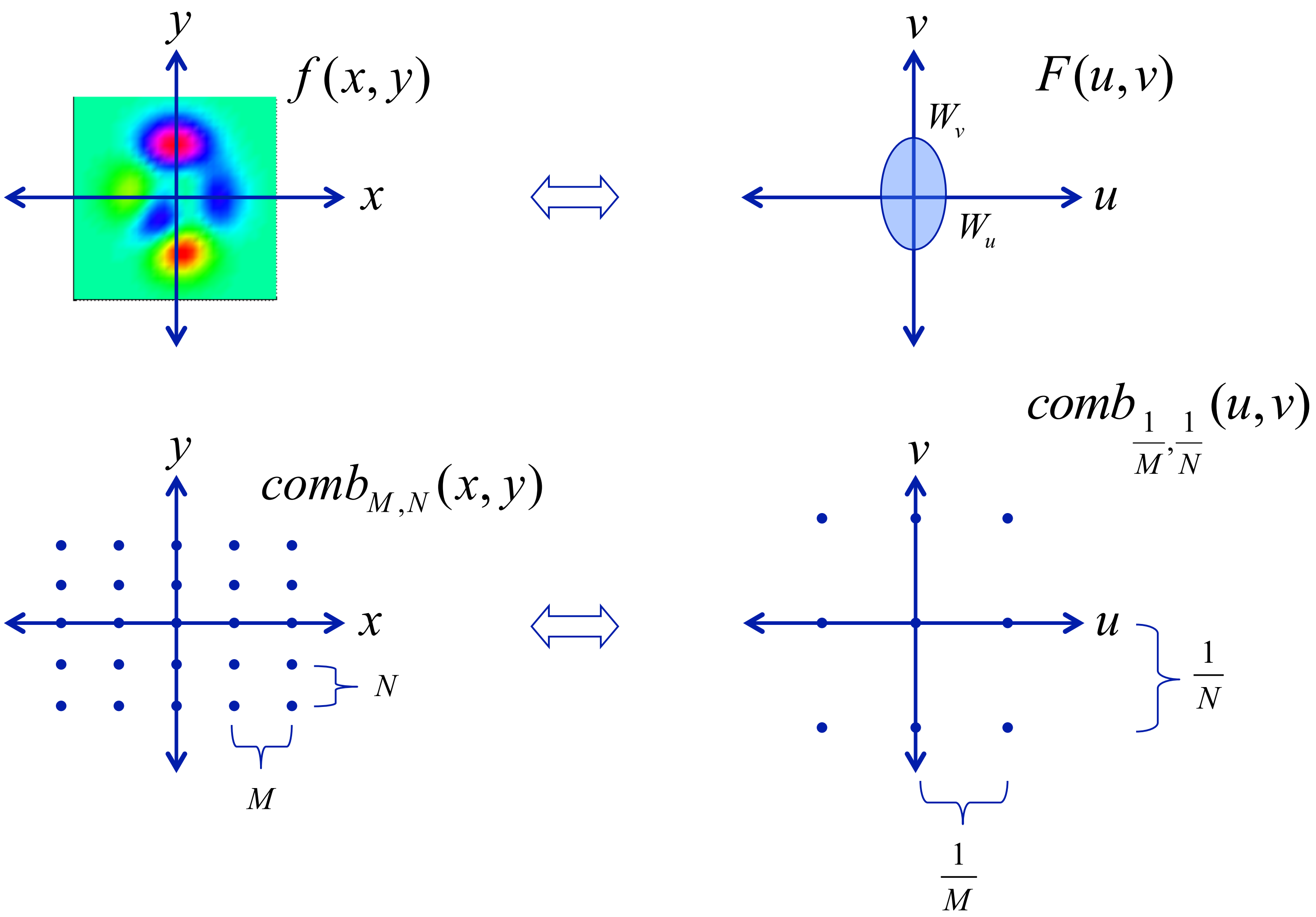


- With anti-aliasing filter:

$$\left[f(x)*h(x)\right]comb_M(x) \Leftrightarrow$$

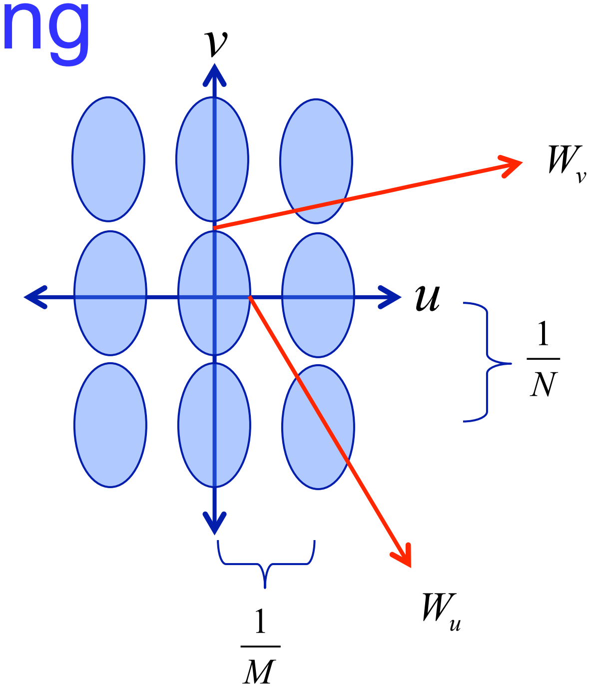


# Sampling in 2D (images)



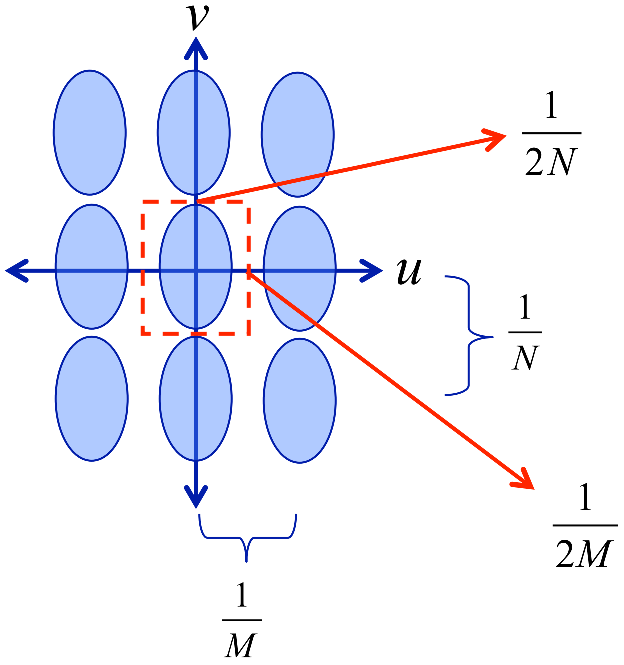
# Sampling

$f(x,y)comb_{M,N}(x,y)$ $\Longleftrightarrow$



*No aliasing if* $\frac{1}{M} > 2W_u$ *and* $\frac{1}{N} > 2W_v$

# Interpolation (low pass filtering)



*Ideal reconstruction filter:*

$$H(u,v) = \begin{cases} MN, \text{ for } u \le \dfrac{1}{2M} \text{ and v} \le \dfrac{1}{2N} \\ 0, \text{ otherwise} \end{cases}$$

# Anti-Aliasing

```
a=imread('barbara.tif');
```

# Anti-Aliasing

```
a=imread( ‘barbara.tif’ );
b=imresize(a,0.25);
c=imresize(b,4);
```

# Anti-Aliasing

```
a=imread( ‘barbara.tif’ );
b=imresize(a,0.25);
c=imresize(b,4);

H=zeros(512,512);
H(256-64:256+64, 256-64:256+64)=1;

Da=fft2(a);
Da=fftshift(Da);
Dd=Da.*H;
Dd=fftshift(Dd);
d=real(ifft2(Dd));
```

# Discrete Fourier Transform (DFT)

Applies to **finite length** discrete time (sampled) signals and time series

*The easiest way to get to it*

Time is a discrete variable (t=n)

Frequency is a discrete variable (f=k)

# DFT

- The DFT can be considered as a generalization of the CTFT to discrete series
- It is the FT of a discrete (sampled) function of one variable

$$F[k] = \frac{1}{N}\sum_{n=0}^{N-1} f[n]e^{-j2\pi kn/N}$$

$$f[n] = \sum_{k=0}^{N-1} F[k]e^{j2\pi kn/N}$$

  - The 1/N factor is put either in the analysis formula or in the synthesis one, or the 1/sqrt(N) is put in front of both.
- Calculating the DFT takes about $N^2$ calculations

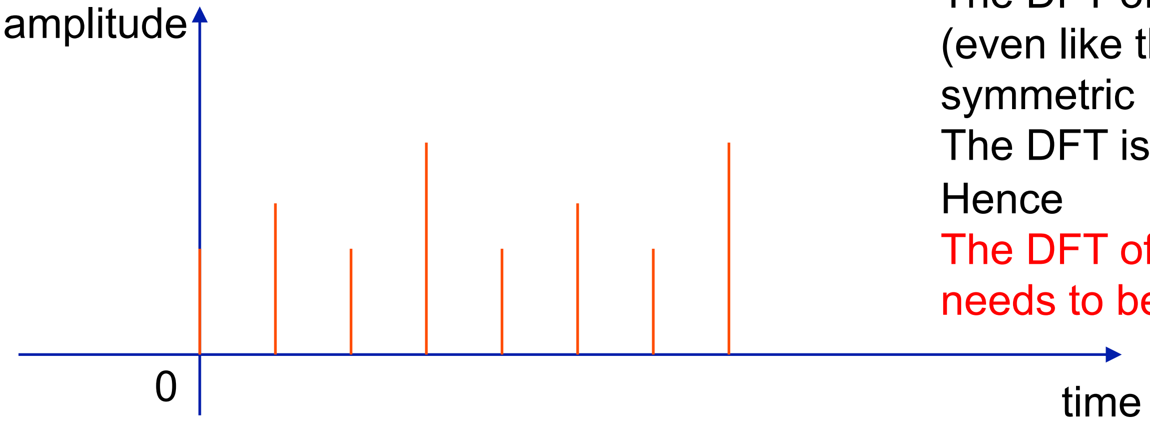
# In practice..

- In order to calculate the DFT we start with k=0, calculate F(0) as in the formula below, then we change to u=1 etc

$$F[0] = \frac{1}{N}\sum_{n=0}^{N-1} f[n]e^{-j2\pi 0n/N} = \frac{1}{N}\sum_{n=0}^{N-1} f[n] = \overline{f}$$

- F[0] is the average value of the function f[n] in k=0
  - This is also the case for the CTFT
- The transformed function F[k] has the same number of terms as f[n] and always exists
- The transform is always reversible by construction so that we can always recover f given F

# Highlights on DFT properties



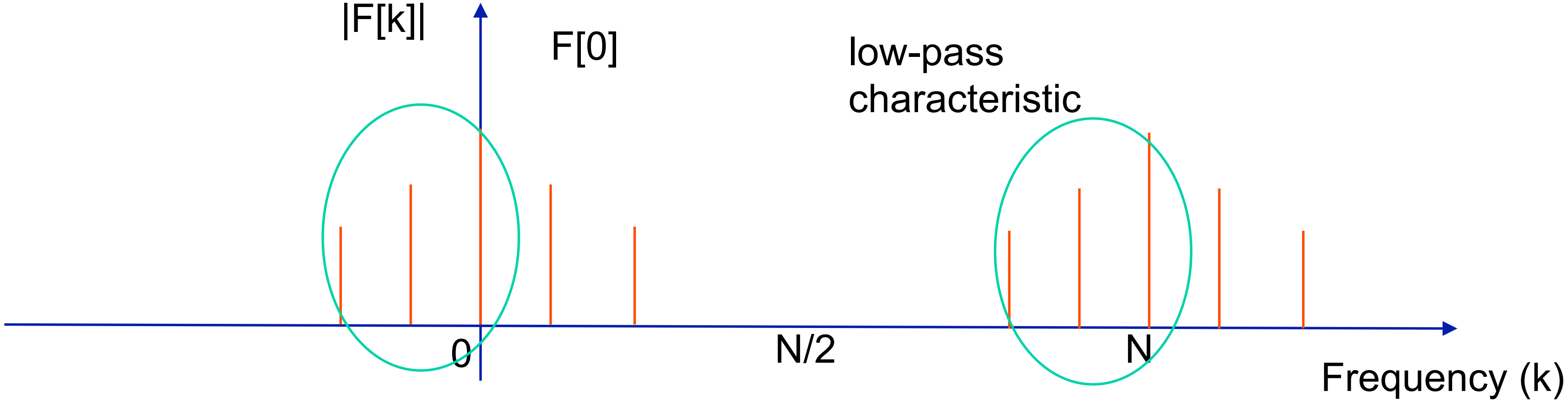
The DFT of a real signal is symmetric

The DFT of a real symmetric signal (even like the cosine) is real and symmetric

The DFT is N-periodic

Hence

The DFT of a real symmetric signal only needs to be specified in [0, N/2]



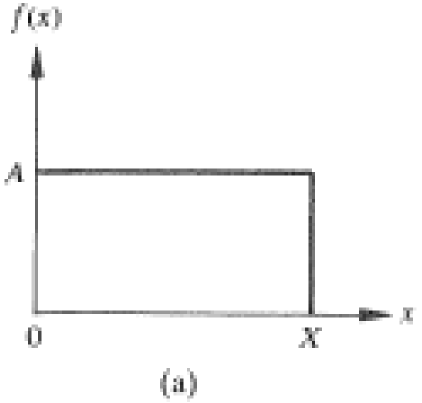
# Visualization of the basic repetition

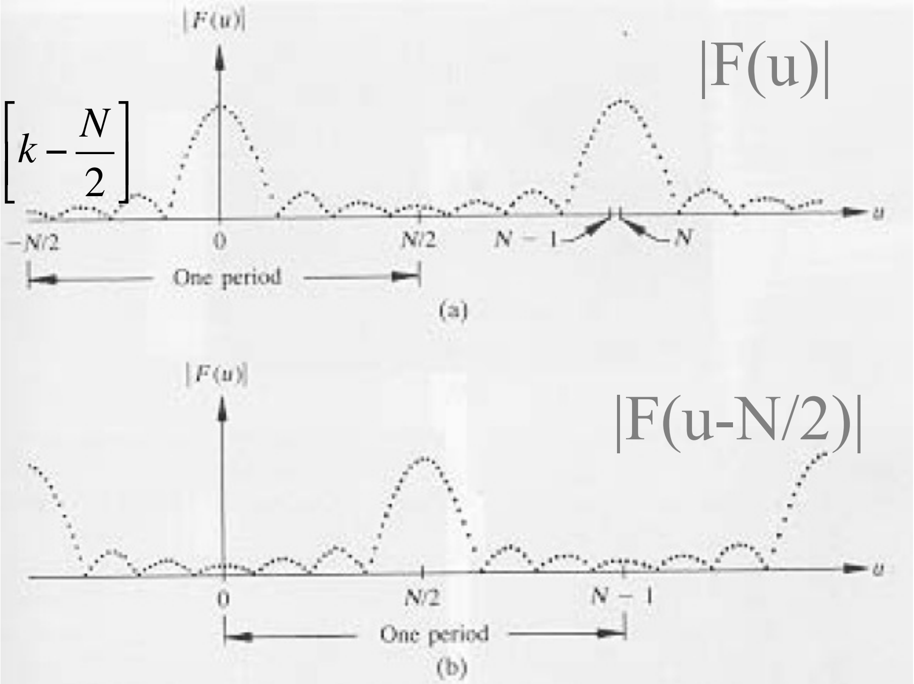
- To show a full period, we need to translate the origin of the transform at **u=N/2** (or at **(N/2,N/2)** in 2D)

$$f[n]e^{2\pi u_0 \frac{n}{N}} \rightarrow f[k-u_0]$$

$$u_0 = \frac{N}{2}$$

$$f[n]e^{2\pi \frac{N}{2N} n} = f[n]e^{\pi n} = (-1)^n f[n] \rightarrow f\left[k-\frac{N}{2}\right]$$





# DFT

- About N$^2$ multiplications are needed to calculate the DFT
- The transform F[k] has the same number of components of f[n], that is N
- The DFT always exists for signals that do not go to infinity at any point
- Using the Eulero’s formula

$$e^{j\theta} = \cos\theta + j\sin\theta.$$

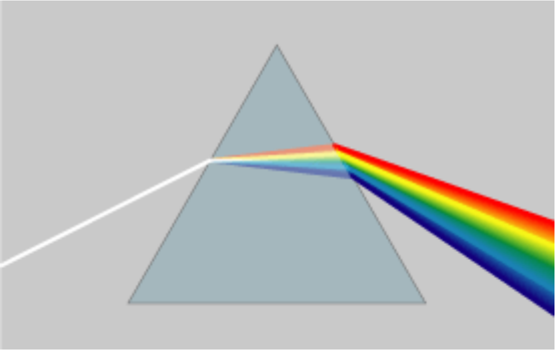
$$F[k] = \frac{1}{N}\sum_{n=0}^{N-1} f[n]e^{-j2\pi kn/N} = \frac{1}{N}\sum_{n=0}^{N-1} f[n]\left(\cos\left(j2\pi kn/N\right) - j\sin\left(j2\pi kn/N\right)\right)$$

frequency component *k*

discrete trigonometric functions

# Going back to the intuition

- The FT decomposed the signal over its harmonic components and thus represents it as a sum of linearly independent complex exponential functions
- Thus, it can be interpreted as a “mathematical prism”



# DFT

- Each term of the DFT, namely each value of F[k], results of the contributions of all the samples in the signal (f[n] for n=1,..,N)
- The samples of f[n] are multiplied by trigonometric functions of different frequencies
- The domain over which F[k] lives is called frequency domain
- Each term of the summation which gives F[k] is called frequency component of harmonic component

# DFT is a complex number

- F[k] in general are complex numbers

$$F[k] = \mathrm{Re}\{F[k]\} + j\,\mathrm{Im}\{F[k]\}$$

$$F[k] = |F[k]| \exp\{j\mathrm{R}F[k]\}$$

$$\left\{ \begin{array}{l} |F[k]| = \sqrt{\mathrm{Re}\{F[k]\}^2 + \mathrm{Im}\{F[k]\}^2} \\ \\ \quad \mathrm{R}F[k] = \tan^{-1}\left\{ -\dfrac{\mathrm{Im}\{F[k]\}}{\mathrm{Re}\{F[k]\}} \right\} \end{array} \right\}$$

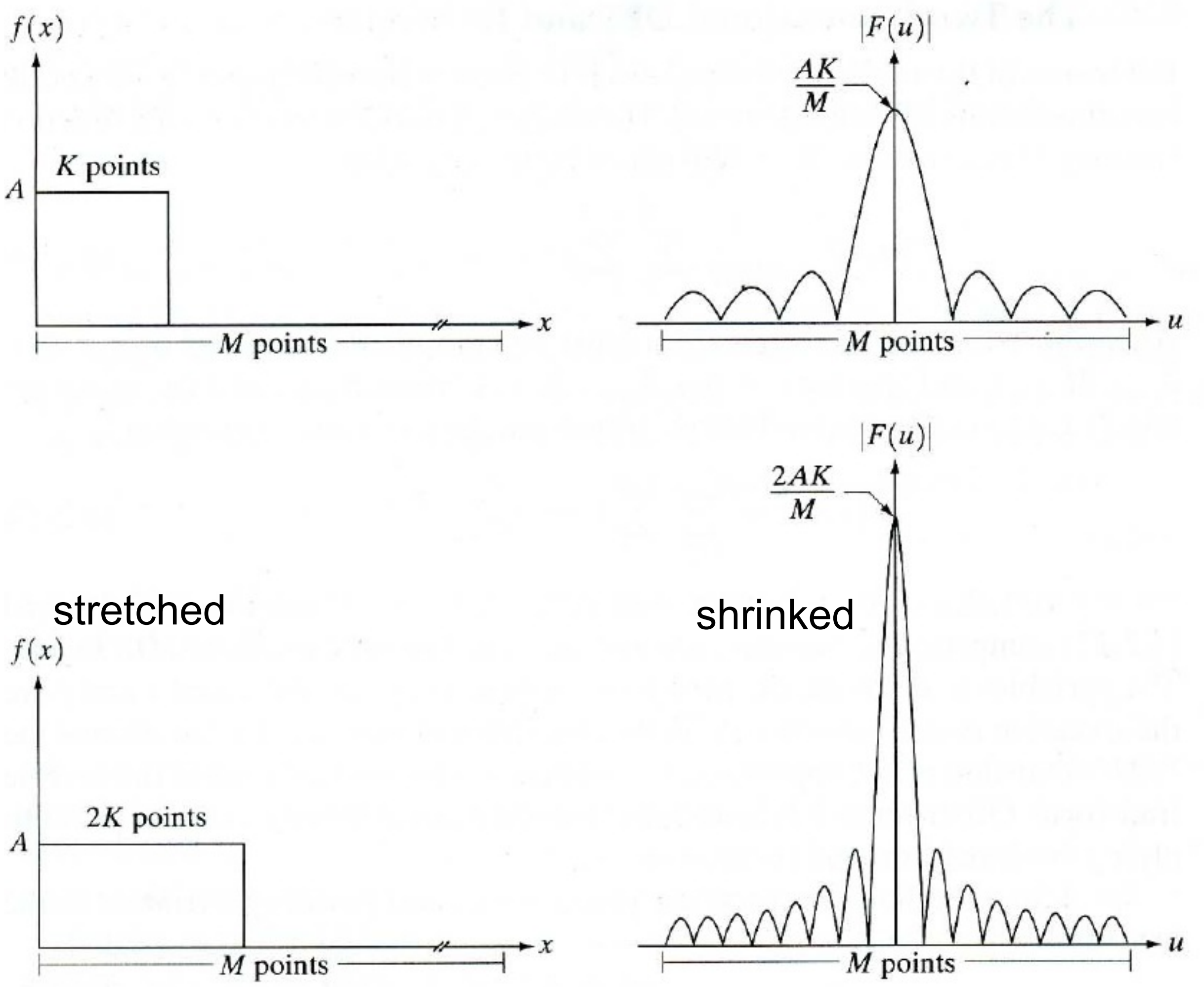
magnitude or spectrum

phase or angle

$$P[k] = |F[k]|^2$$

power spectrum

# Stretching vs shrinking



stretched

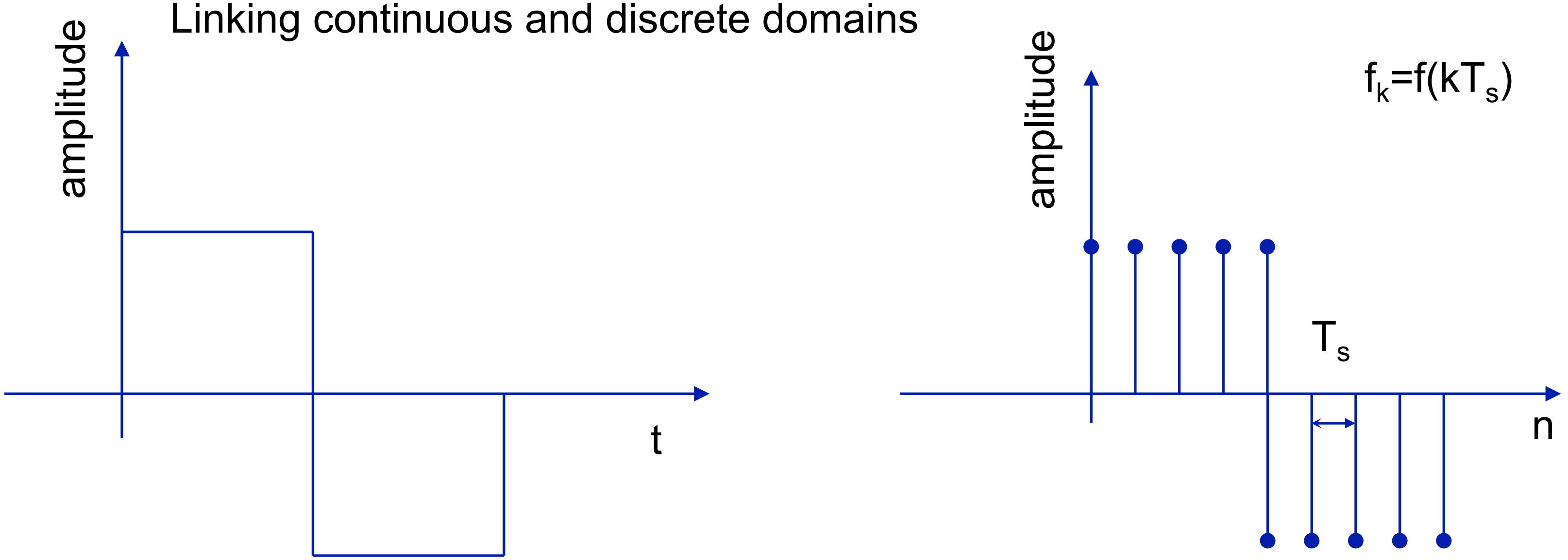
shrinked

a b
c d

**FIGURE 4.2** (a) A discrete function of $M$ points, and (b) its Fourier spectrum. (c) A discrete function with twice the number of nonzero points, and (d) its Fourier spectrum.

# Periodization vs discretization

Linking continuous and discrete domains



- DT (discrete time) signals can be seen as *sampled* versions of CT (continuous time) signals
- Both CT and DT signals can be of finite duration or periodic
- There is a duality between *periodicity* and *discretization*
  - Periodic signals have discrete frequency (sampled) transform
  - Discrete time signals have periodic transform
  - DT periodic signals have discrete (sampled) periodic transforms

# Increasing the resolution by Zero Padding

- Consider the analysis formula

$$F[k] = \frac{1}{N} \sum_{n=0}^{N-1} f[n] e^{-\frac{2\pi jkn}{N}}$$

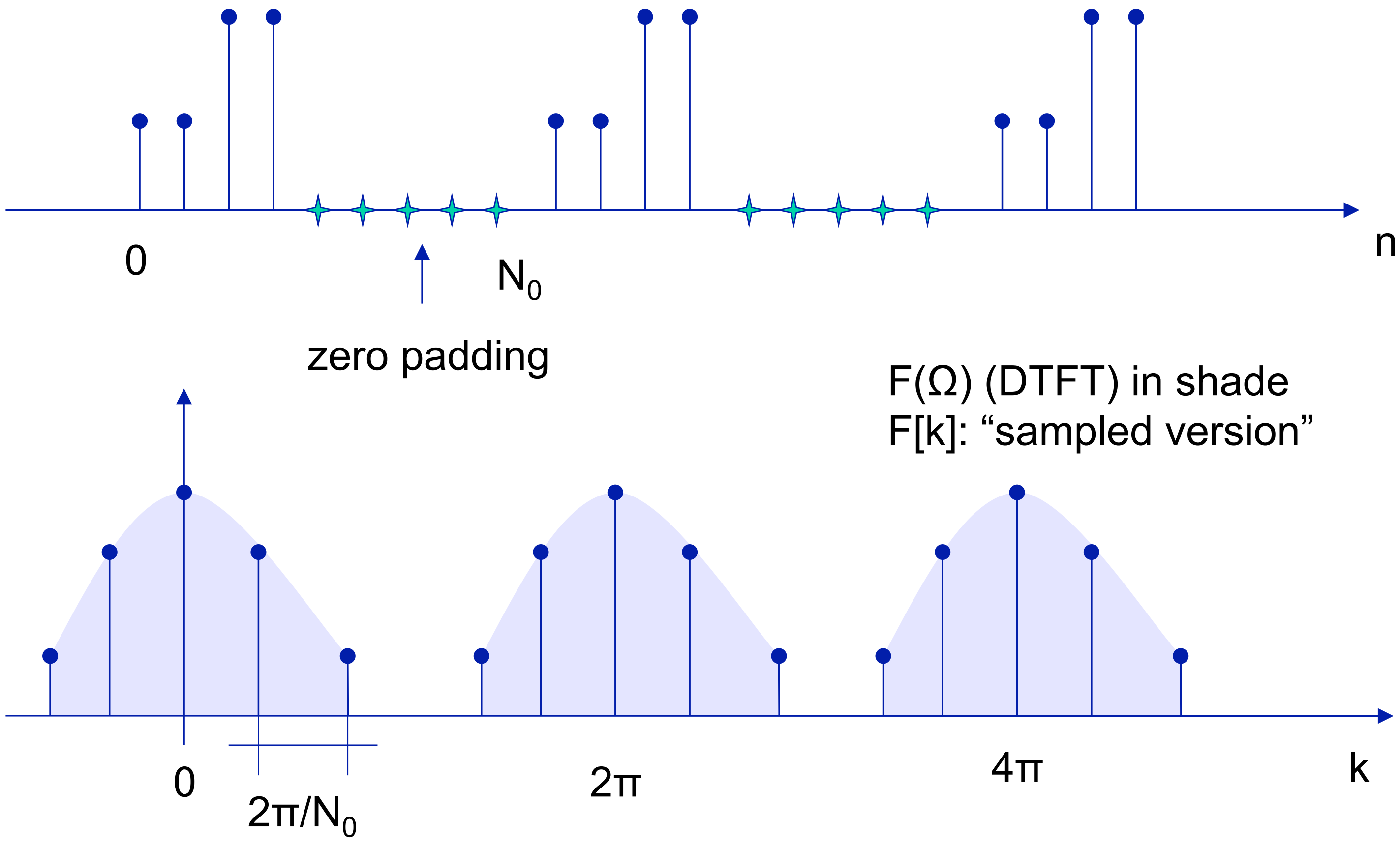
- If f[n] consists of n samples than F[k] consists of N samples as well, it is discrete (k is an integer) and it is periodic (because the signal f[n] is discrete time, namely n is an integer)
- The value of each F[k], for all k, is given by a weighted sum of the values of f[n], for n=1,..,N-1
- Key point: if we artificially increase the length of the signal adding M zeros on the right, we get a signal $f_1[m]$ for which m=1,…,N+M-1. Since

$$f_1[m] = \begin{cases} f[m] \quad for \quad 0 \le m < N \\ 0 \quad for \quad N \le m < N + M \end{cases}$$

# Increasing the resolution through ZP

- Then the value of each F[k] is obtained by a weighted sum of the “real” values of f[n] for 0≤k≤N-1, which are the only ones different from zero, but they happen at different “normalized frequencies” since the frequency axis has been rescaled. In consequence, F[k] is more “densely sampled” and thus features a higher resolution.

# Increasing the resolution by Zero Padding

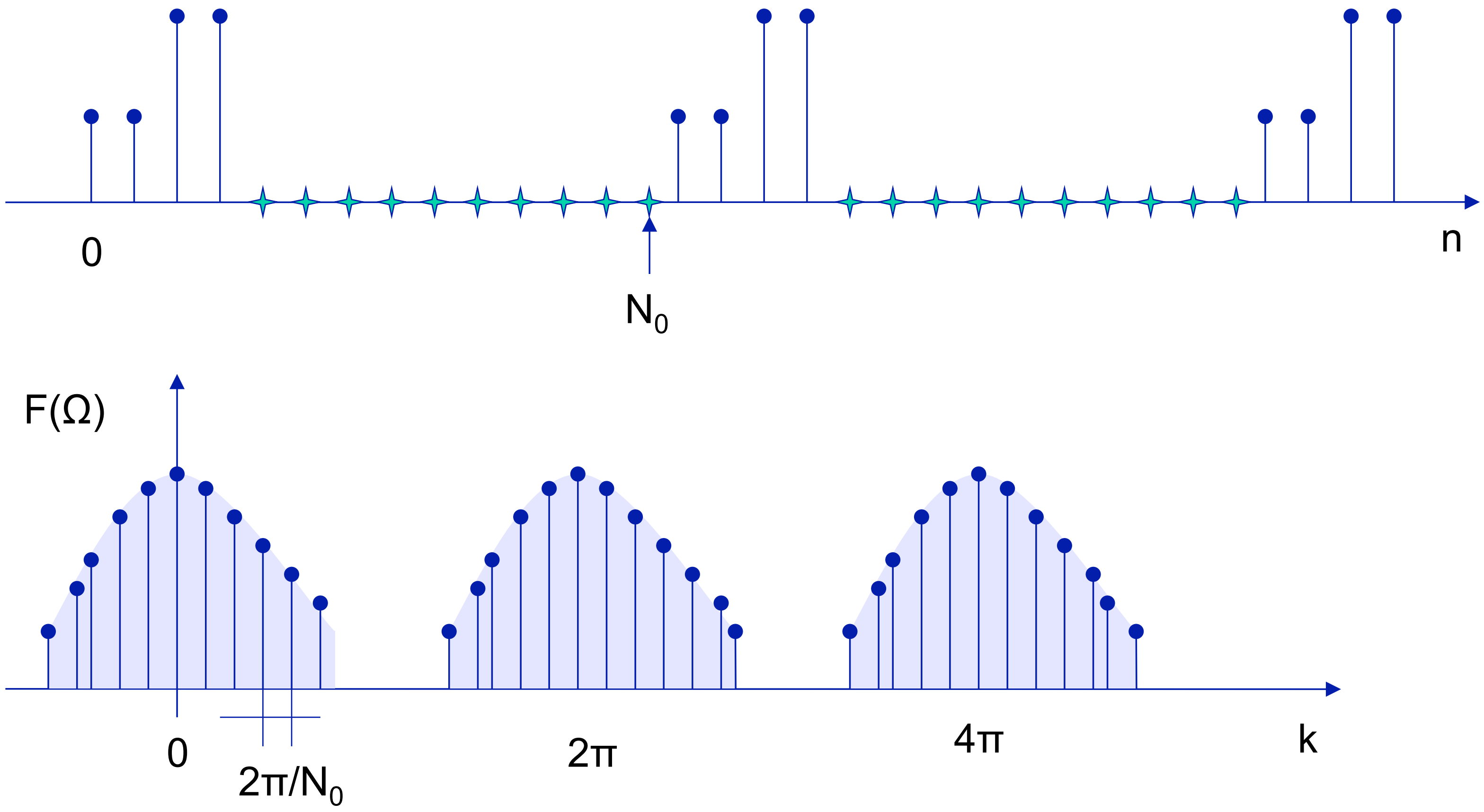


F(Ω) (DTFT) in shade
F[k]: “sampled version”

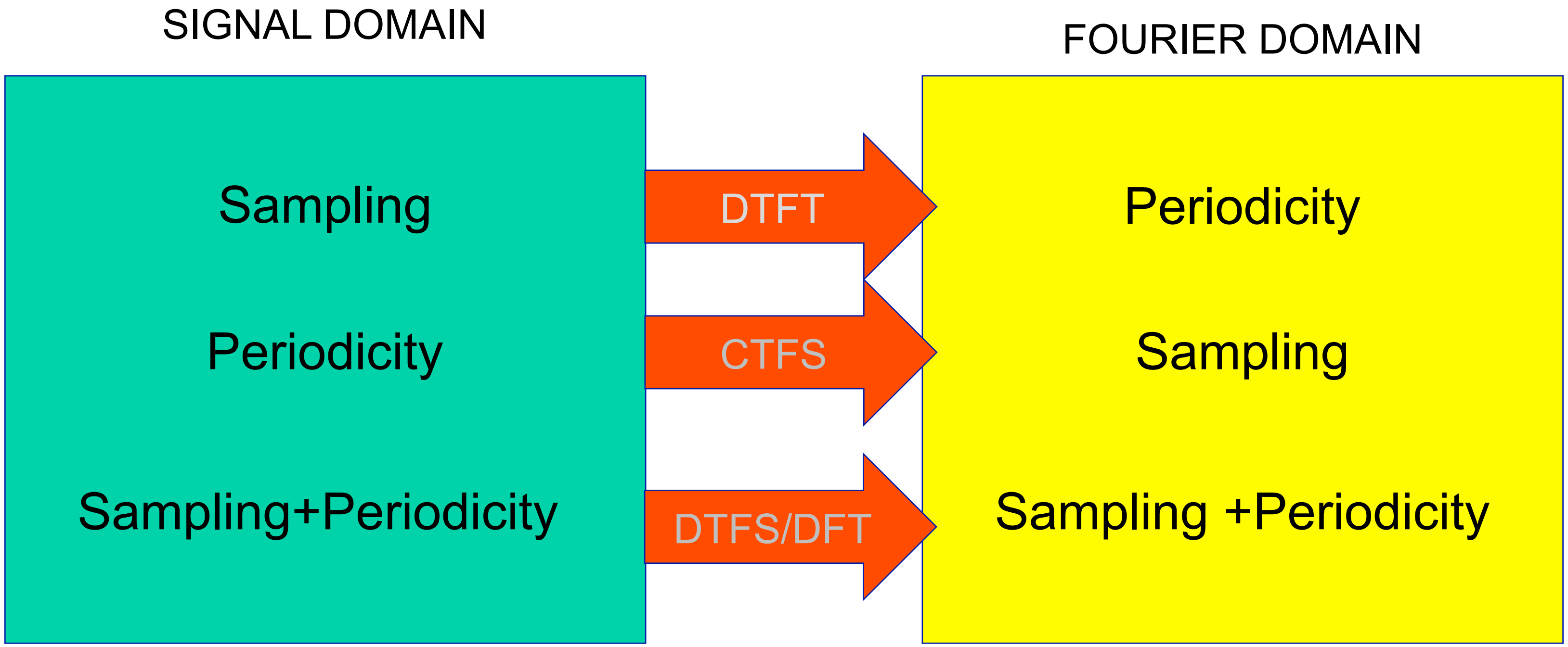
# Zero padding

Increasing the number of zeros augments the “resolution” of the transform since the samples of the DFT get “closer”

zero padding



# Summary of dualities



SIGNAL DOMAIN

FOURIER DOMAIN

Sampling

Periodicity

Sampling+Periodicity

DTFT

CTFS

DTFS/DFT

Periodicity

Sampling

Sampling +Periodicity

# Discrete Cosine Transform (DCT)

Applies to digital (sampled) finite length signals AND uses only **cosines**.

The DCT coefficients are all **real numbers**

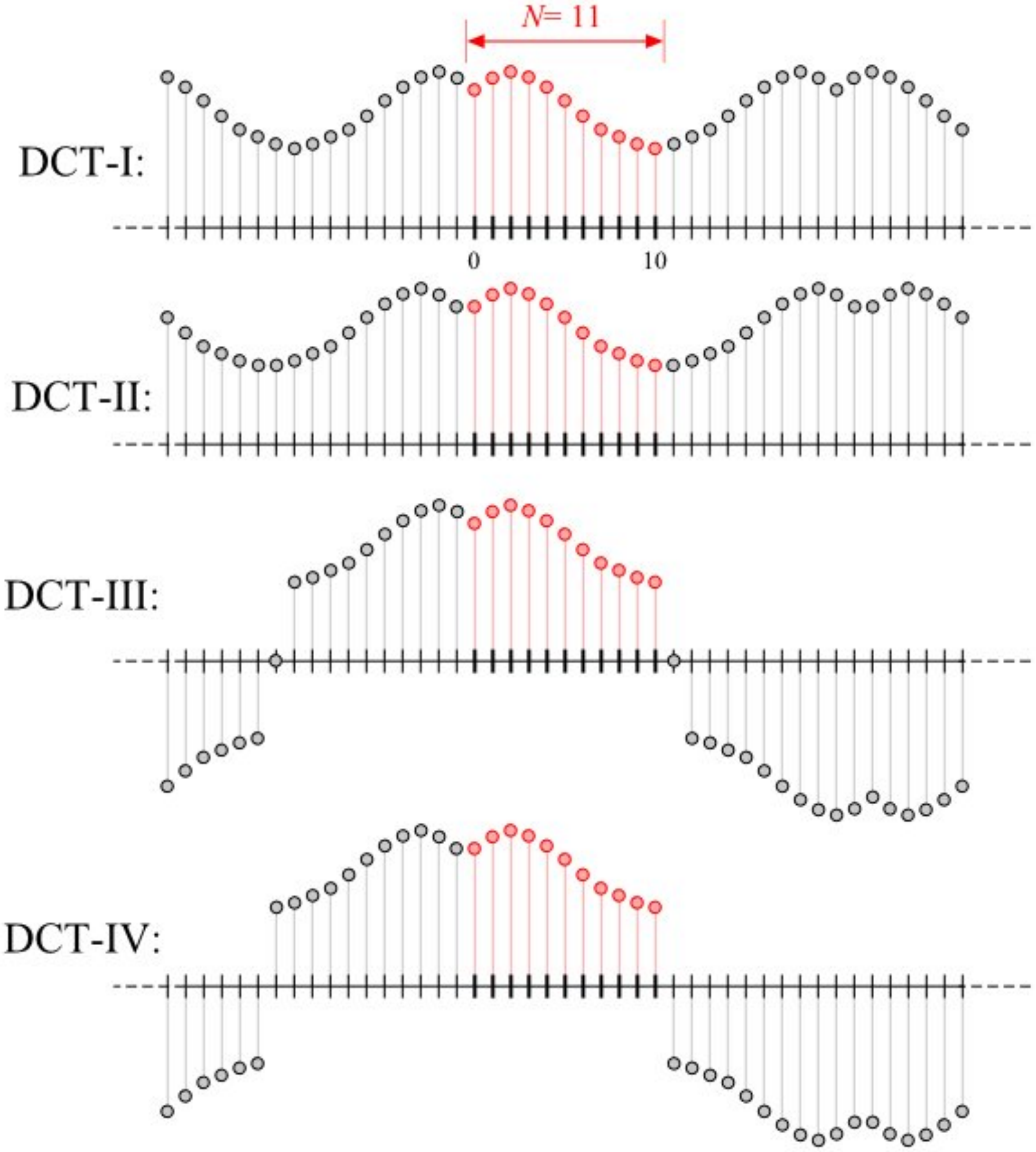
# Discrete *Cosine* Transform (DCT)

- Operate on finite discrete sequences (as DFT)
- A **discrete cosine transform** (**DCT**) expresses a sequence of finitely many data points in terms of a sum of cosine functions oscillating at different frequencies
- DCT is a Fourier-related transform similar to the DFT but using only real numbers
- DCT is equivalent to DFT of roughly twice the length, operating on real data with even symmetry (since the Fourier transform of a real and even function is real and even), where in some variants the input and/or output data are shifted by half a sample
- There are eight standard DCT variants, out of which four are common
- Strong connection with the Karunen-Loeven transform
  - VERY important for signal compression

# DCT

- DCT implies different boundary conditions than the DFT or other related transforms
- A DCT, like a cosine transform, implies an *even periodic* extension of the original function
- Tricky part
  - First, one has to specify whether the function is even or odd at *both* the left and right boundaries of the domain
  - Second, one has to specify around *what point* the function is even or odd
    - In particular, consider a sequence *abcd* of four equally spaced data points, and say that we specify an even *left* boundary. There are two sensible possibilities: either the data is even about the sample *a*, in which case the even extension is ***dcbabcd***, or the data is even about the point *halfway* between *a* and the previous point, in which case the even extension is ***dcbaabcd*** (*a* is repeated).

# Symmetries



# DCT

$$X_k = \sum_{n=0}^{N_0-1} x_n \cos\left[\frac{\pi}{N_0}\left(n+\frac{1}{2}\right)k\right] \qquad k = 0,\ldots.,N_0 - 1$$

$$x_n = \frac{2}{N_0}\left\{\frac{1}{2}X_0 + \sum_{k=0}^{N_0-1} X_k \cos\left[\frac{\pi k}{N_0}\left(k+\frac{1}{2}\right)\right]\right\}$$

- *Warning*: the normalization factor in front of these transform definitions is merely a convention and differs between treatments.
  - Some authors multiply the transforms by $(2/N_0)^{1/2}$ so that the inverse does not require any additional multiplicative factor.
    - Combined with appropriate factors of √2 (see above), this can be used to make the transform matrix orthogonal.