Lectures on

DIFFERENTIAL GEOMETRY AND TOPOLOGY

Mauro Spera

Lecture X

Topological manifolds

A topological space M is called topological manifold of dimension n (or, topological n-manifold) if

1. M is Hausdorff
2. M has a countable basis (also: second countable)
3. M is locally euclidean:

$\forall m \in M$, $\exists\ U \ni m$ neighbourhood of m i.e. an open set containing m

$\exists\ V \subset \mathbb{R}^n$ (V homeomorphic to an open ball in $\mathbb{R}^n$, the latter being equipped with the standard topology)

n independent of m

such that $\varphi : U \to V$ is a homeomorphism.

In words: every point in M admits a neighbourhood homeomorphic to an open ball in $\mathbb{R}^n$ (with n fixed)

$\varphi : U \to V$ is called local chart carta locale (also, local patch, coordinate system)

Notes: ◆ Hausdorff: any two points admit disjoint neighbourhoods

◆ A basis in a topological space $(X, \mathcal{T})$ is a subset $\mathcal{B} \subset \mathcal{T}$ ($\subset \mathcal{P}(X)$) such that $\forall A \in \mathcal{T}$, $A = \bigcup_{\lambda \in \Lambda} B_\lambda$

Λ an index set.

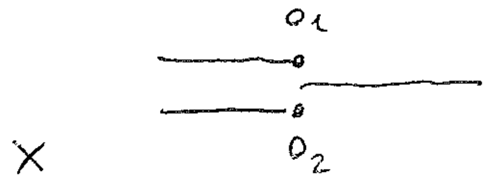
In $\mathbb{R}^n$, open balls with rational radii and rational centres (i.e. with rational coordinates) give rise to a countable basis thereof.

Observe that, if $B_1 \in \mathcal{B}$, $B_2 \in \mathcal{B}$, $B_1 \cap B_2 \in \mathcal{T}$ and there exists $B \in \mathcal{B}$ such that $B \subset B_1 \cup B_2$, since $B_1 \cap B_2 = \bigcup_{\lambda \in \Lambda} B_\lambda$ for suitable $B_\lambda \in \mathcal{B}$.

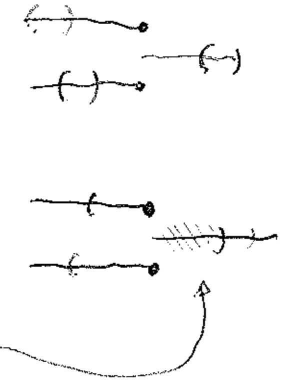
One can prove that, given on a set X a family $\mathcal{B} \subset \mathcal{P}(X)$ containing $\emptyset$ and X, and such that $\bigcup_{B \in \mathcal{B}} B = X$, and $\forall B_1, B_2 \in \mathcal{B}$ $\exists\ \underset{\in \mathcal{B}}{B} \subset B_1 \cap B_2$, then

$\exists!$ topology $\mathcal{T}$ admitting $\mathcal{B}$ as a basis: the open sets in $\mathcal{T}$ are unions of sets in $\mathcal{B}$...

Notice that $3 \not\Rightarrow 1$

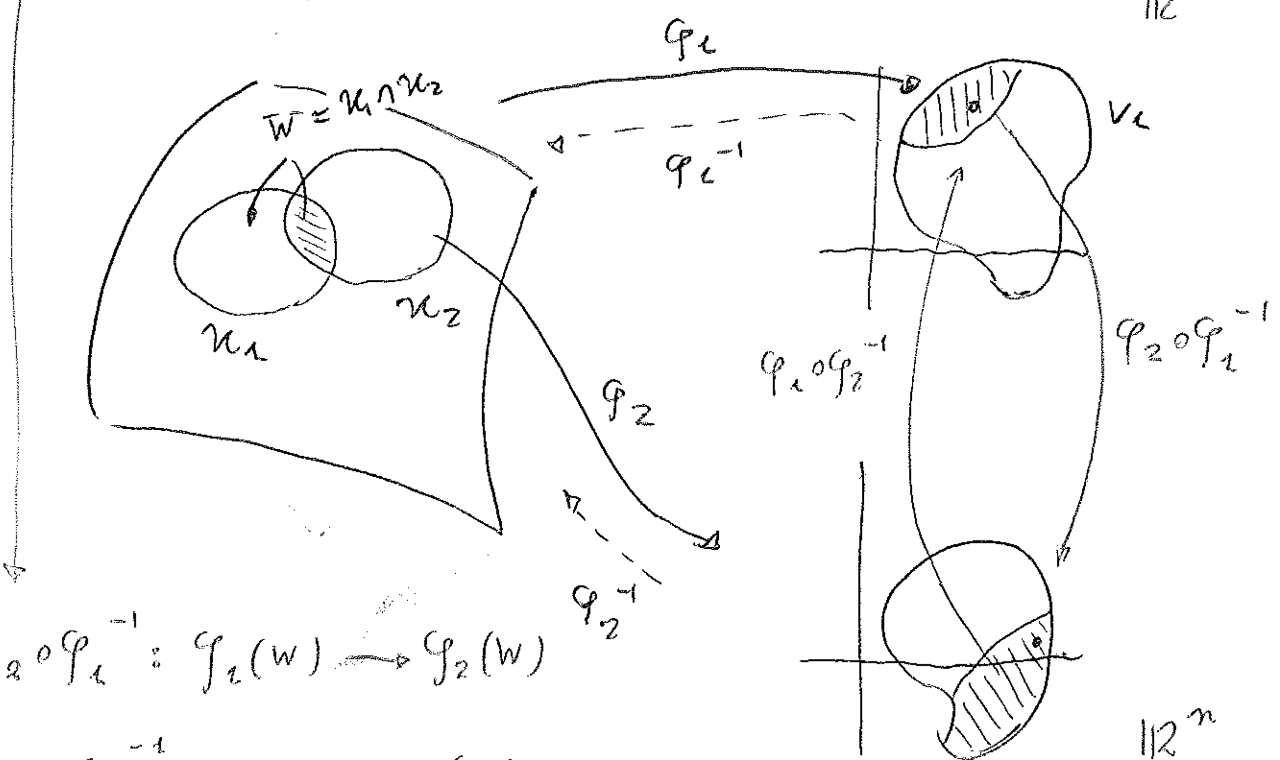


basis :



one obtains a topology that obviously makes it locally euclidean. X is not Hausdorff since O_1 and O_2 cannot be separated by disjoint neighbourhoods.

In order to get a differentiable manifold, we require the overlap maps (also: transition maps, coordinate change maps ...) to be smooth (weaker requirements are possible) :



$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(W) \longrightarrow \varphi_2(W)$$

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Therefore, a differentiable manifold (of dimension n) M is a topological space which is Hausdorff, has countable basis, equipped with an atlas

$$\mathcal{A} := \{ (\mathcal{U}_\alpha, \varphi_\alpha) \}_{\alpha \in \Omega}$$

differentiable structure — index set

i.e. a collection of local charts fulfilling the following properties

(i) $\displaystyle\bigcup_{\alpha \in \Omega} \mathcal{U}_\alpha = M$

(namely, $\{\mathcal{U}_\alpha\}_{\alpha \in \Omega}$ is an open covering (or cover) of M,

(ii) $\varphi_\alpha : \mathcal{U}_\alpha \to V_\alpha$ is a homeomorphism

local chart — V_α: ball in $\mathbb{R}^n$

(iii) and, if $\mathcal{U}_\alpha \cap \mathcal{U}_\beta =: W_{\alpha\beta} \neq \emptyset$

the overlap maps (transition maps)

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(W_{\alpha\beta}) \longrightarrow \varphi_\beta(W_{\alpha\beta})$$

open in $\mathbb{R}^n$ — open in $\mathbb{R}^n$

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(W_{\alpha\beta}) \longrightarrow \varphi_\beta(W_{\alpha\beta})$$

are smooth

they are maps between open sets in $\mathbb{R}^n$, so the concept of smoothness is meaningful for them ...

One could be more sophisticated.

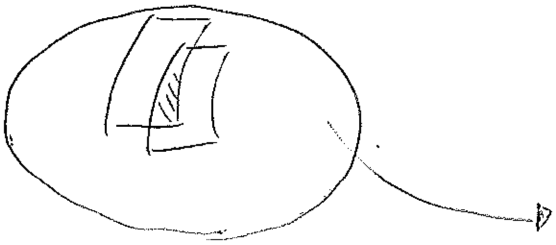
Two atlases are said to be compatible (or equivalent) if their union is still an atlas.

A maximal atlas is the union of all atlases compatible with a fixed atlas (existence follows from Zorn's lemma). In theoretical investigations it turns out to be convenient to work with a maximal atlas: it gives us a sort of universal receptacle of charts where from we can take those suiting our needs.

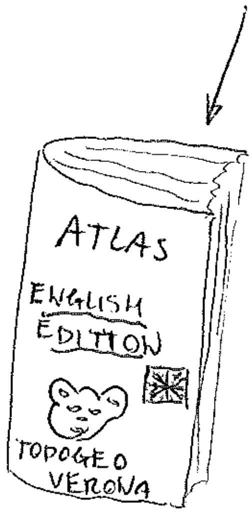
More formally, a differentiable manifold of dimension n / n-dimensional differentiable manifold is a pair $(M, [\mathcal{A}])$, with M a topological n-manifold and $[\mathcal{A}]$ the equivalence class determined by a maximal atlas: this is also called a differentiable structure

Note. One can speak of C^k-manifolds or C^ω-manifolds (transition charts being real-analytic)). Upon replacing $\mathbb{R}^n$ with $\mathbb{C}^n$, and requiring (bi-)holomorphy / holomorphicity (complex analyticity) one arrives at the notion of complex manifold of dimension n. If $n = 1$, one obtains a Riemann surface (historically, the latter concept is due to H. Weyl (1913))

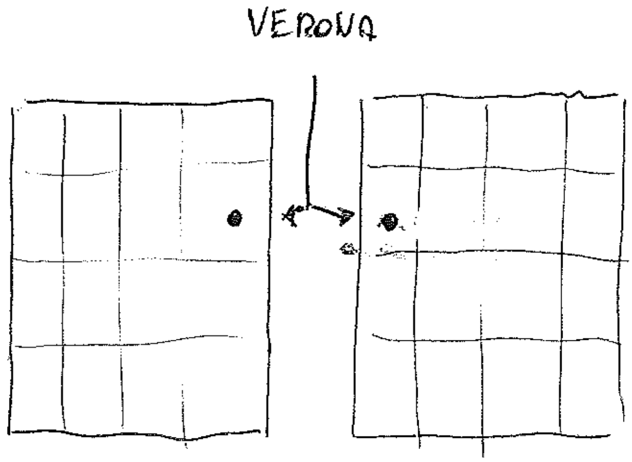
4 Basic motivation : cartography



not a maximal one!



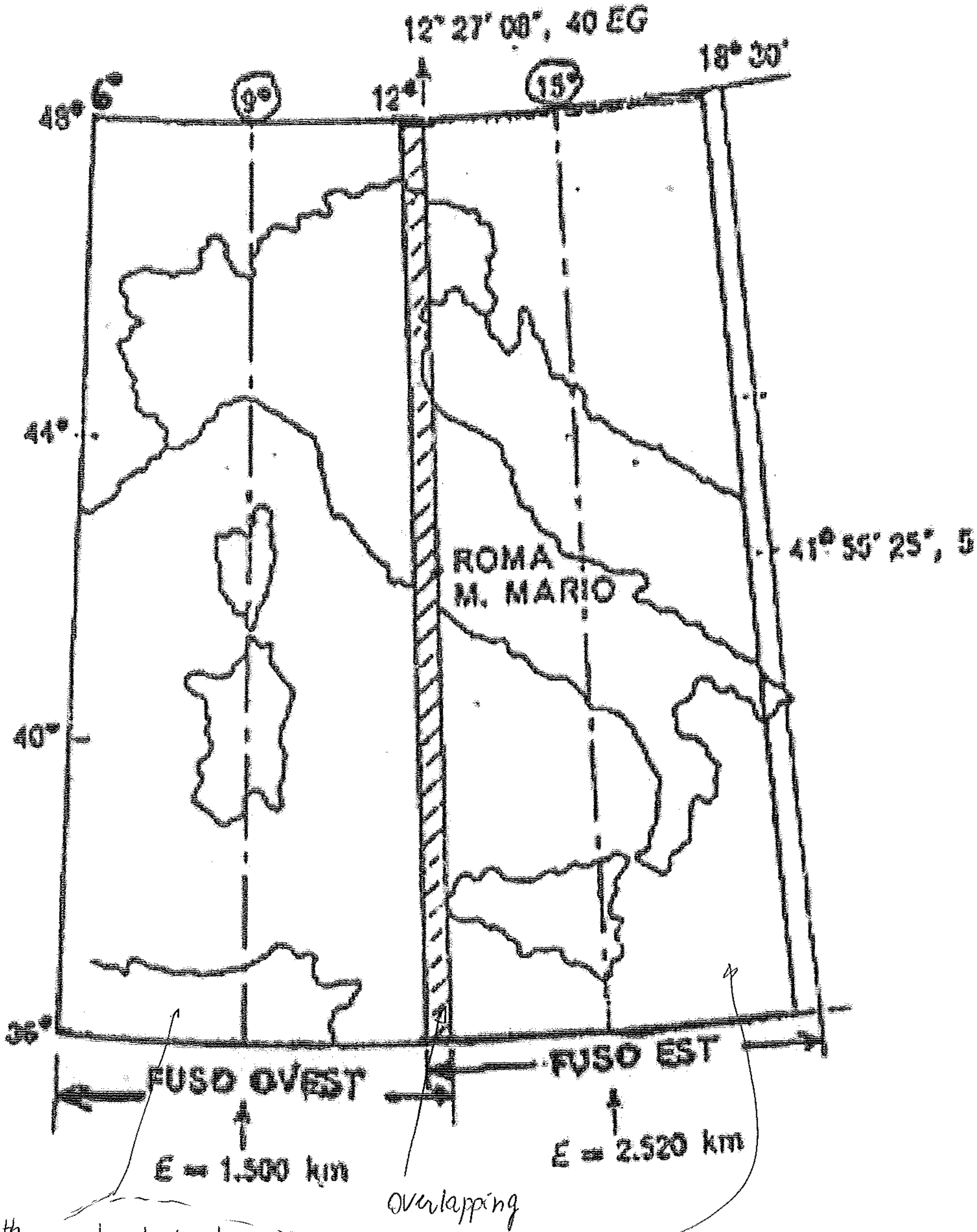
terrestrial ellipsoid
(with enhanced eccentricity)



A transition map is involved, invisible to the ... final user

★ Gauss-Boaga projection

italian version of the UTM projection



★ Another (equivalent) definition of smooth manifold without starting from a topological space.

Let M be a set, such that $\exists\ f_\alpha : \mathcal{U}_\alpha \to M$, $\alpha \in \mathcal{Q}$, $\mathcal{U}_\alpha$ open, $\mathcal{U}_\alpha \subset \mathbb{R}^n$

no topology on it, a priori

f_α injective

[observe that charts go in the opposite direction, but this is not important]

such that

1. $\displaystyle\bigcup_{\alpha \in \mathcal{Q}} f_\alpha(\mathcal{U}_\alpha) = M$

2. $\forall\ \alpha, \beta \in \mathcal{Q}$ such that $f_\alpha(U_\alpha) \cap f_\beta(U_\beta) \equiv W_{\alpha\beta} \neq \phi$,

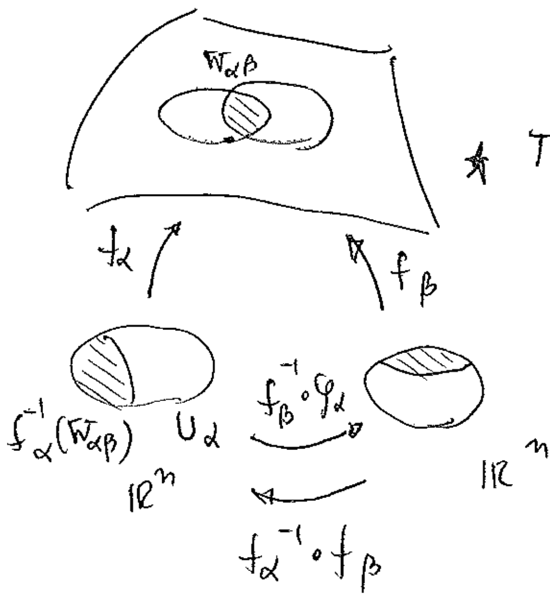
$f_\alpha^{-1}(W_{\alpha\beta})$ and $f_\beta^{-1}(W_{\alpha\beta})$ are open in $\mathbb{R}^n$ and such that

$$f_\alpha^{-1} \circ f_\beta \quad \text{and} \quad f_\beta^{-1} \circ f_\alpha \quad \text{are smooth}$$

well defined in view of injectivity

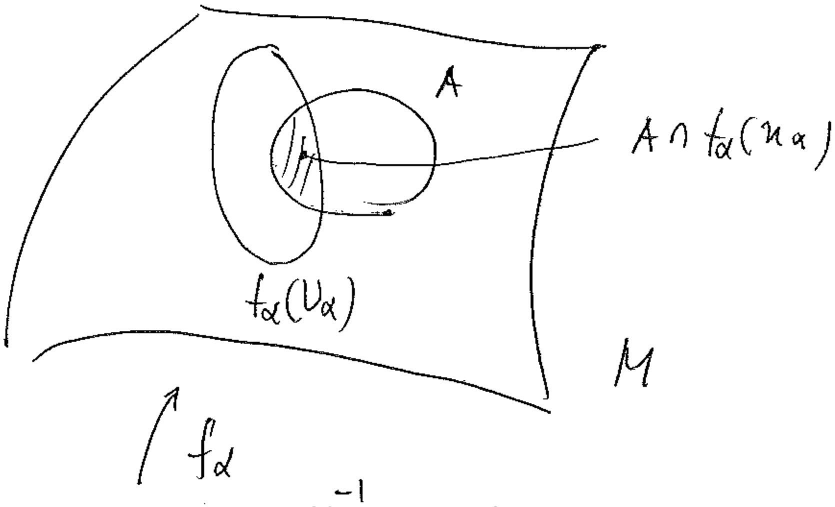
3. The above family is maximal with respect to the properties 1 and 2

$$\mathcal{A} = \{ (\mathcal{U}_\alpha, f_\alpha) \}_{\alpha \in \mathcal{Q}} \quad \text{atlas (diff. structure)}$$



★ This gives us a natural topology $\mathcal{T}$ on M:

$A \subset M$ is open iff $f_\alpha^{-1}(A \cap f_\alpha(U_\alpha))$ is open in $\mathbb{R}^n$



$f_\alpha^{-1}(A \cap f_\alpha(U_\alpha))$ is required to be open in $\mathbb{R}^n$

★ One checks that $\mathcal{T}$ fulfils the axioms of a topology.
($\mathcal{T}$ contains ϕ, M and is closed under arbitrary unions and finite intersections)

The extra requirements : Hausdorff + countable basis are then postulated.

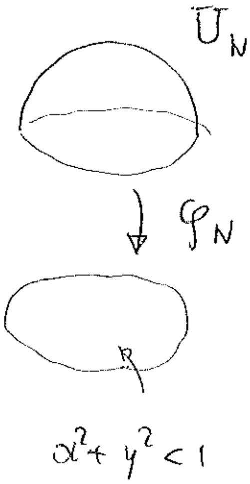
Hausdorff → uniqueness of limits

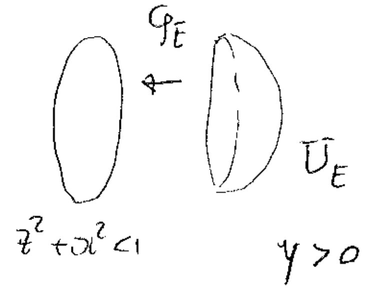
countable basis → existence of partitions of unity, see below

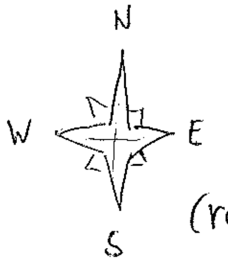
This approach is useful in applications, in cases there is no a priori topology to be imposed on set.

★ Examples

1. The sphere $S^2 = \{ (x,y,z) \in \mathbb{R}^3 \,/\, x^2+y^2+z^2 = 1 \}$ equipped with the relative topology (inherited from the standard one in $\mathbb{R}^3$)



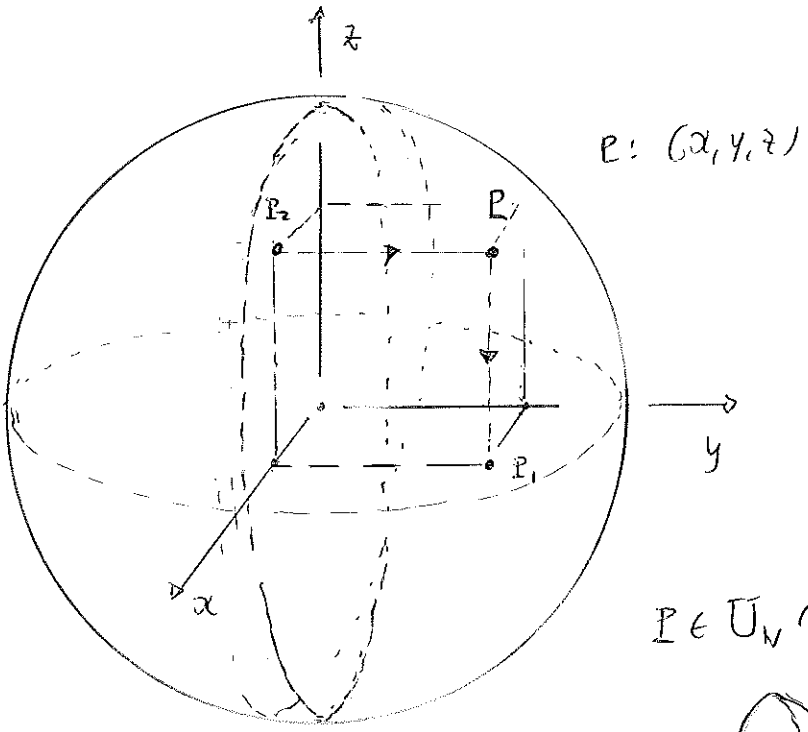




(rosa dei venti)

compass-card

bussola



$P \in U_N \cap U_E$

$\bullet \xrightarrow{\varphi_E^{-1}} \bullet \xrightarrow{\varphi_N} \bullet$

compasso: compasses

a punte fisse: dividers

$(z,x) \xmapsto{\varphi_E^{-1}} (x,y,z)$

$(x,y,z) \xmapsto{\varphi_N} (x,y)$

$y = \sqrt{1-x^2-z^2}$

$(z,x) \xmapsto{\varphi_N \circ \varphi_E^{-1}} (x, \sqrt{1-x^2-z^2})$

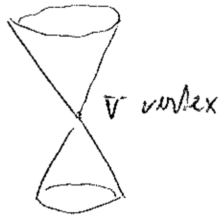
$(z,x) \in \varphi_E(U_E) = \{ z^2+x^2<1 \}$

$(x, \sqrt{1-x^2-z^2}) \in \varphi_N(U_N) = \{ x^2+y^2<1 \}$

★ $\varphi_N \circ \varphi_E^{-1}$ is smooth, with smooth inverse

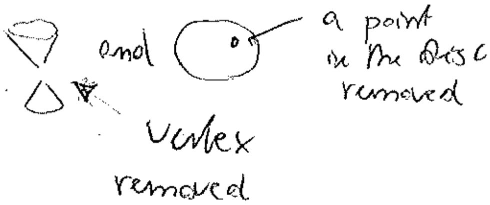
2. A "non-example" : $x^2+y^2-z^2=0$

(cone in $\mathbb{R}^3$, endowed with relative topology)



This is not a smooth manifold, and not even a topological manifold : v does not possess a neighbourhood homeomorphic to an open disc !

Why? Were it, then



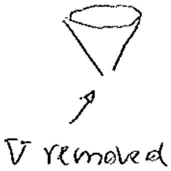
would be homeomorphic, but this is false (the latter space is connected, the former is not)

2'. 

is a topological manifold (C^0)

$x^2+y^2-z^2=0$

$z \geq 0$

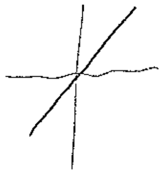
2''. 

is a smooth manifold

(and a smooth submanifold of $\mathbb{R}^3$ as well, as we have already seen)

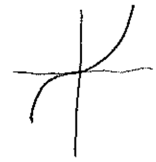
3. A remark on the concept of differentiable structure

$M_1 = (\mathbb{R}, t)$ $\qquad \varphi_1(t) = t$



↑ atlas consisting of a single chart ↓

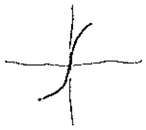
$M_2 = (\mathbb{R}, t^3)$ $\qquad \varphi_2(t) = t^3$



✱ The two atlases are not compatible (upon requiring $k > 0$) degree of differentiability

$\varphi_1^{-1}\varphi_2 : \quad t \xmapsto{\varphi_1^{-1}} t \xmapsto{\varphi_2} t^3$ is smooth

$\varphi_2^{-1}\varphi_1 : \quad t \longmapsto t^{\frac{1}{3}} \xmapsto{\varphi_1} t^{\frac{1}{3}}$ is not smooth (nor C^k $k \geq 1$)



Therefore one has $\mathbb{R}$ equipped with different differentiable structures (they are however equivalent in a suitable sense.) ✱ The situation is really complicated in general:

Jungle of topological manifolds: ✱ For dim $M \leq 3$ $\exists!$ differentiable structure (Munkres, Moise)

✱ In dimension > 3, $\exists$ M which do not admit any differentiable structure.

✱ Exotic spheres (Milnor, Kervaire): on S^7 there exist 28 inequivalent differentiable structures

✱ Fake $\mathbb{R}^4$: on $\mathbb{R}^4$, there exists an uncountable set of inequivalent differentiable structures (Freedman)

We refrain from further delving into these fascinating topics.

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Lecture X

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$\exists\, \tilde{V} \subset \mathbb{R}^n$ ($\tilde{V}$ homeomorphic to an open ball in $\mathbb{R}^n$, the latter being equipped with the standard topology)

n independent of m

such that $\varphi : U \to \tilde{V}$ is a homeomorphism.

In words: every point in M admits a neighbourhood homeomorphic to an open ball in $\mathbb{R}^n$ (with n fixed)

$\varphi : U \to V$ is called local chart (carta locale) (also, local patch, coordinate system)

Notes: * Hausdorff: any two points admit disjoint neighbourhoods

* A basis in a topological space $(X, \mathcal{T})$ is a subset $\mathcal{B} \subset \mathcal{T}$ ($\subset \mathcal{P}(X)$) such that $\forall\, A \in \mathcal{T}$, $A = \bigcup_{\lambda \in \Lambda} B_\lambda$

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Observe that, if $B_1 \in \mathcal{B}$, $B_2 \in \mathcal{B}$, $B_1 \cap B_2 \in \mathcal{T}$ and there exists $B \in \mathcal{B}$ such that $B \subset B_1 \cup B_2$, since $B_1 \cap B_2 = \bigcup_{\lambda \in \Lambda} B_\lambda$ for suitable $B_\lambda \in \mathcal{B}$.

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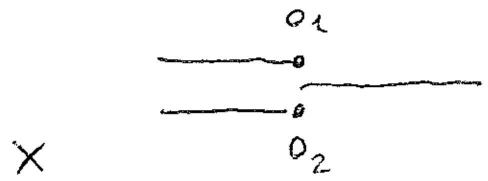
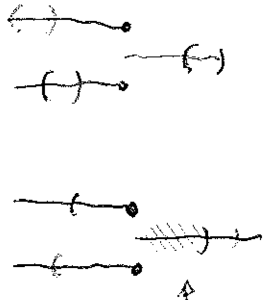


basis :



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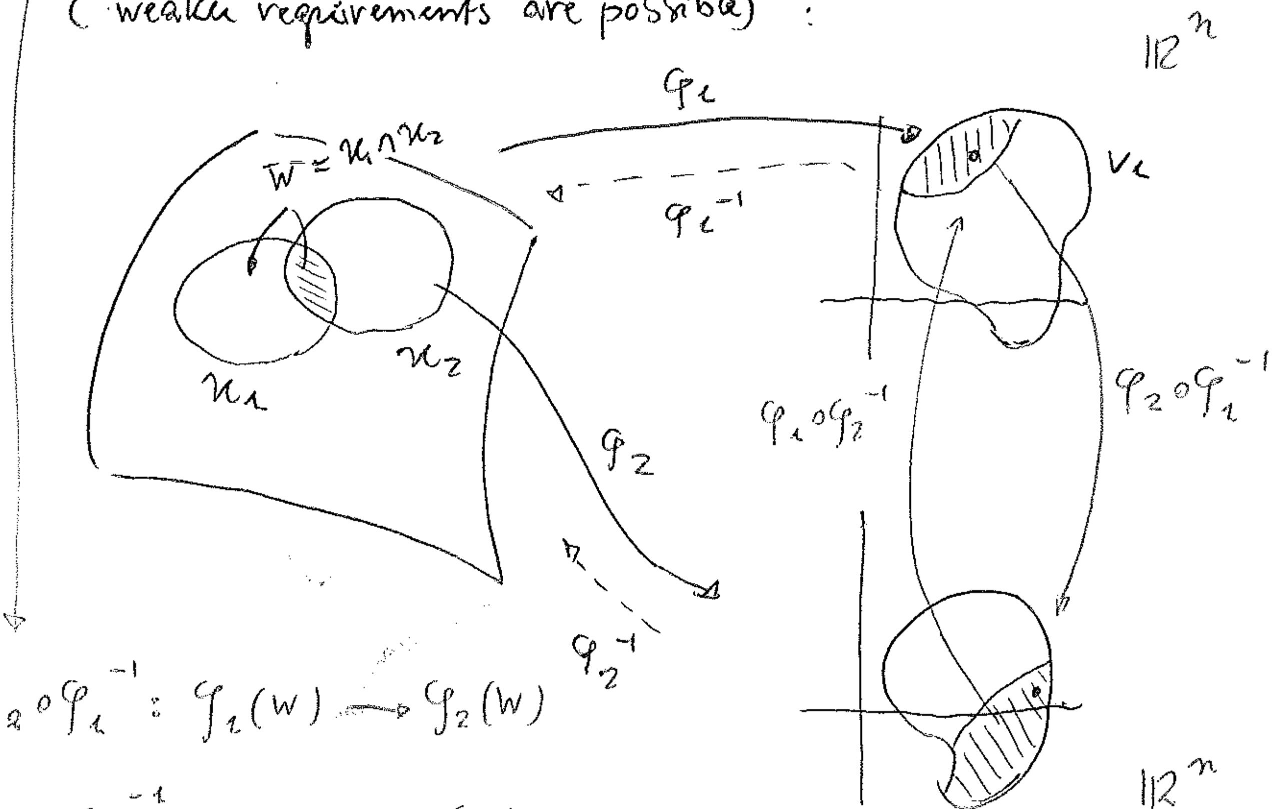

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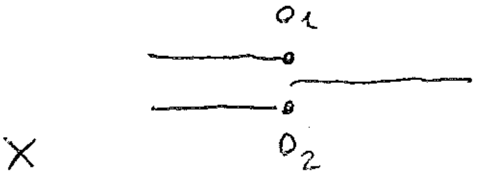

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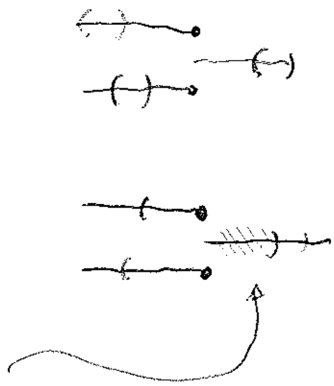

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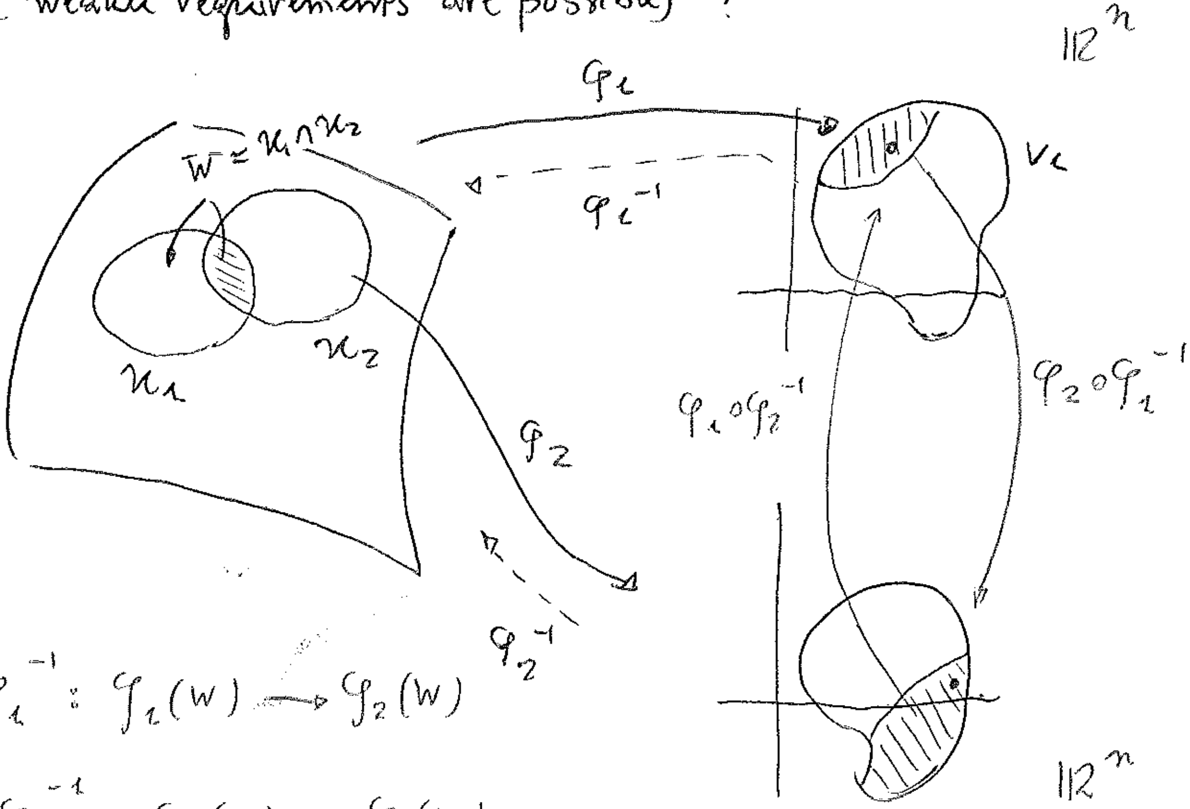

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the overlap maps (transition maps)

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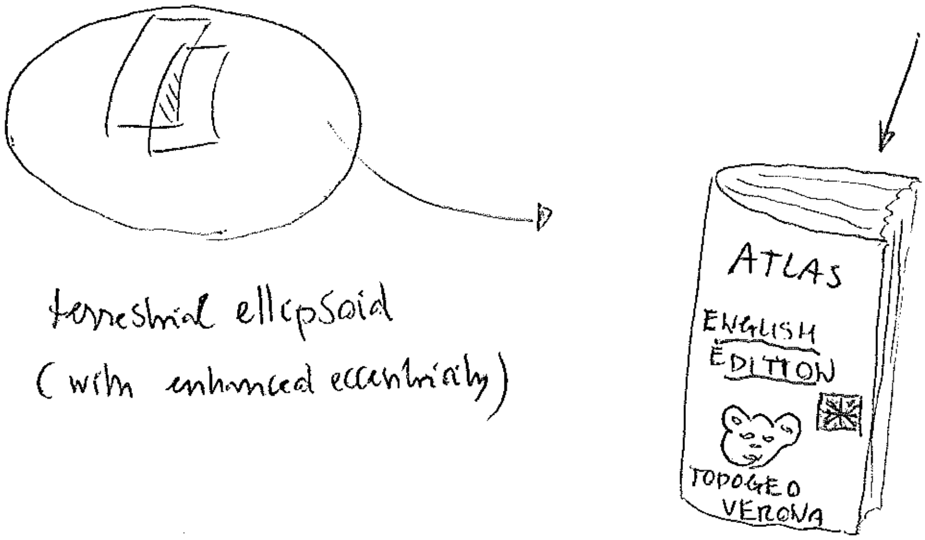
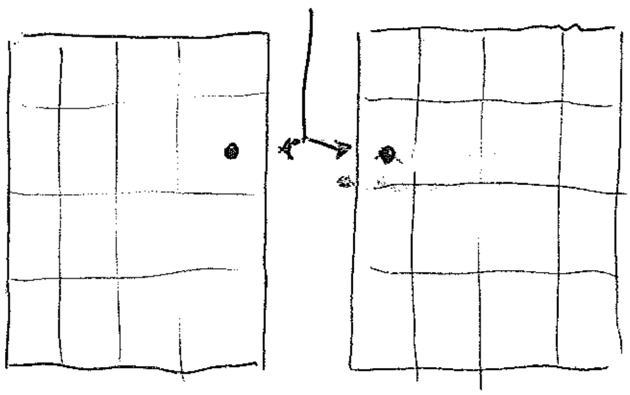
4 Basic motivation : Cartography

not a maximal one!



terrestrial ellipsoid
(with enhanced eccentricity)

VERONA



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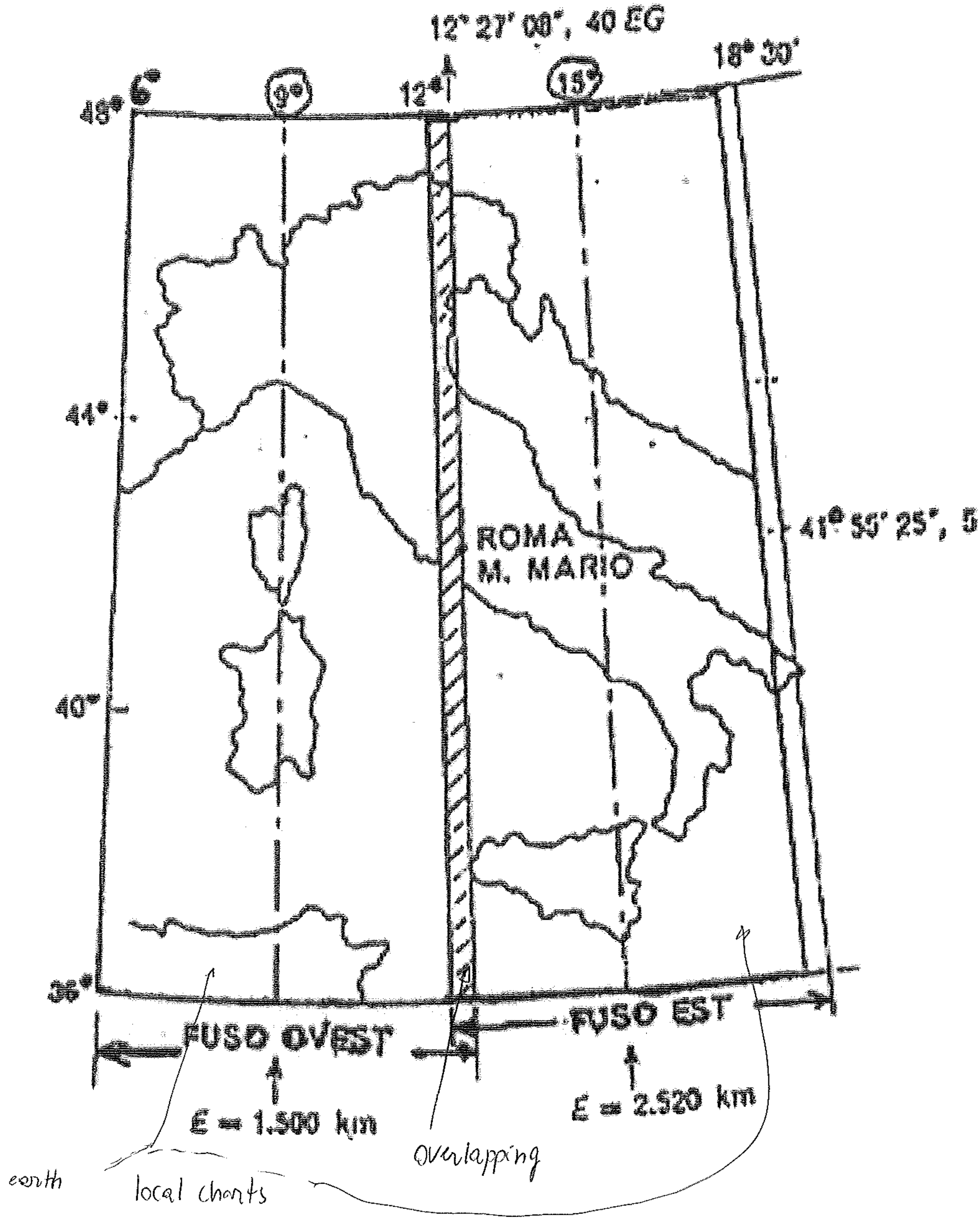

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3. M is locally euclidean:

$\forall m \in M$, $\exists U \ni m$ neighbourhood of m

i.e. an open set containing m

$\exists V \subset \mathbb{R}^n$ (V homeomorphic to an open ball in $\mathbb{R}^n$, the latter being equipped with the standard topology)

n independent of m

such that $\varphi : U \to V$ is a homeomorphism.

In words: every point in M admits a neighbourhood homeomorphic to an open ball in $\mathbb{R}^n$ (with n fixed)

$\varphi : U \to V$ is called local chart

carta locale

(also, local patch, coordinate system)

Notes: ◆ Hausdorff: any two points admit disjoint neighbourhoods

◆ A basis in a topological space $(X, \mathcal{T})$ is a subset $\mathcal{B} \subset \mathcal{T}$ $(\subset \mathcal{P}(X))$ such that $\forall A \in \mathcal{T}$, $A = \bigcup_{\lambda \in \Lambda} B_\lambda$

Λ an index set.

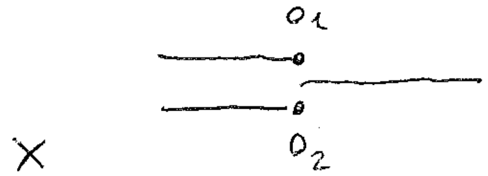
In $\mathbb{R}^n$, open balls with rational radii and rational centres (i.e. with rational coordinates) give rise to a countable basis thereof.

Observe that, if $B_1 \in \mathcal{B}$, $B_2 \in \mathcal{B}$, $B_1 \cap B_2 \in \mathcal{T}$ and there exists $B \in \mathcal{B}$ such that $B \subset B_1 \cup B_2$, since $B_1 \cap B_2 = \bigcup_{\lambda \in \Lambda} B_\lambda$ for suitable $B_\lambda \in \mathcal{B}$.

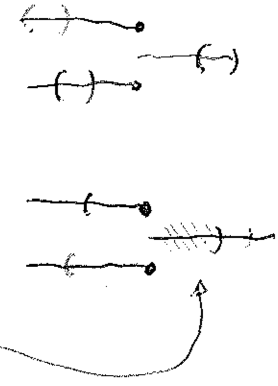
One can prove that, given on a set X a family $\mathcal{B} \subset \mathcal{P}(X)$ containing $\emptyset$ and X, and such that $\bigcup_{B \in \mathcal{B}} B = X$, and $\forall B_1, B_2 \in \mathcal{B}$ $\exists \underset{\mathcal{B}}{\underset{\cap}{B}} \subset B_1 \cap B_2$, then

$\exists!$ topology $\mathcal{T}$ admitting $\mathcal{B}$ as a basis: the open sets in $\mathcal{T}$ are unions of sets in $\mathcal{B}$...

Notice that $3 \not\Rightarrow 1$

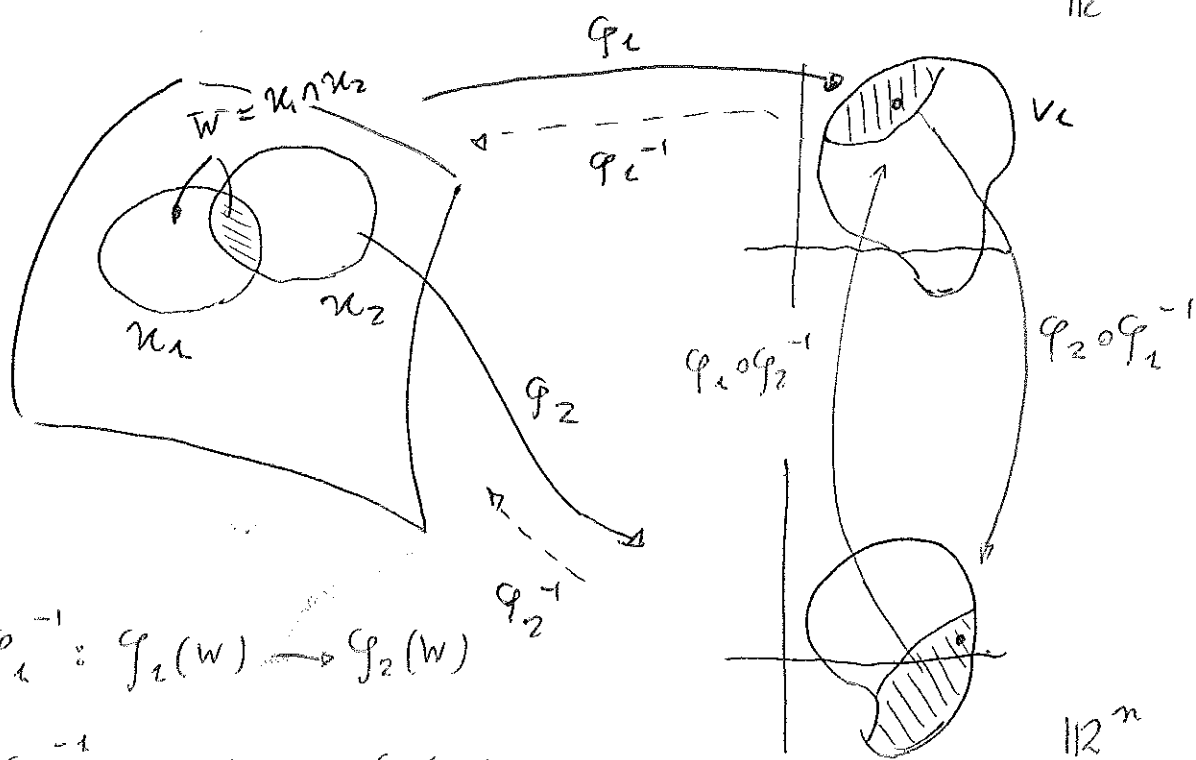


basis:



one obtains a topology that obviously makes it locally euclidean. X is not Hausdorff since O_1 and O_2 cannot be separated by disjoint neighbourhoods.

In order to get a differentiable manifold, we require the overlap maps (also: transition maps, coordinate change maps ...) to be smooth (weaker requirements are possible):



$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(W) \longrightarrow \varphi_2(W)$$

$$\varphi_1 \circ \varphi_2^{-1} : \varphi_2(W) \longrightarrow \varphi_1(W)$$

Therefore, a differentiable manifold (of dimension n) M is a topological space which is Hausdorff, has countable basis, equipped with an atlas

$$\mathcal{A} := \{ (\mathcal{U}_\alpha, \varphi_\alpha) \}_{\alpha \in \Omega}$$

differentiable structure — index set

i.e. a collection of local charts fulfilling the following properties

(i) $\bigcup_{\alpha \in \Omega} \mathcal{U}_\alpha = M$

(namely, $\{ \mathcal{U}_\alpha \}_{\alpha \in \Omega}$ is an open covering (or cover) of M,

(ii) $\varphi_\alpha : \mathcal{U}_\alpha \to V_\alpha$ is a homeomorphism

local chart — $V_\alpha \subset$ ball in $\mathbb{R}^n$

(iii) and, if $\mathcal{U}_\alpha \cap \mathcal{U}_\beta =: W_{\alpha\beta} \neq \emptyset$

the overlap maps
transition maps

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \underset{\text{open in } \mathbb{R}^n}{\varphi_\alpha(W_{\alpha\beta})} \longrightarrow \underset{\text{open in } \mathbb{R}^n}{\varphi_\beta(W_{\alpha\beta})}$$

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(W_{\alpha\beta}) \longrightarrow \varphi_\beta(W_{\alpha\beta})$$

are smooth

they are maps between open sets in $\mathbb{R}^n$, so the concept of smoothness is meaningful for them ...

One could be more sophisticated.

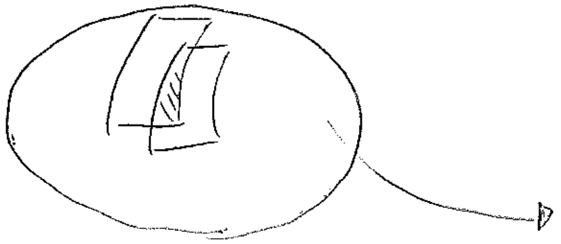
Two atlases are said to be compatible (or equivalent) if their union is still an atlas.

A maximal atlas is the union of all atlases compatible with a fixed atlas (existence follows from Zorn's lemma). In theoretical investigations it turns out to be convenient to work with a maximal atlas: it gives us a sort of universal receptacle of charts wherefrom we can take those suiting our needs.

More formally, a differentiable manifold of dimension n [n-dimensional differentiable manifold] is a pair $(M, [\mathcal{A}])$, with M a topological n-manifold and $[\mathcal{A}]$ the equivalence class determined by a maximal atlas: this is also called a differentiable structure

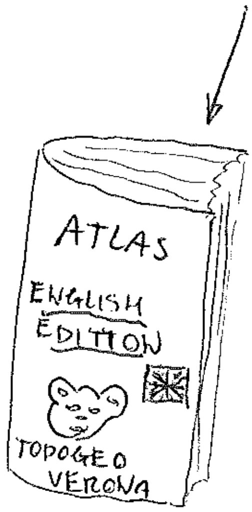
Note. One can speak of C^k-manifolds or C^ω-manifolds (transition charts being real-analytic)). Upon replacing $\mathbb{R}^n$ with $\mathbb{C}^n$, and requiring (bi-)holomorphy (holomorphicity) (complex analiticity) one abuts at the notion of complex manifold of dimension n. If $n=1$, one obtains a Riemann surface (historically, the latter concept is due to H. Weyl (1913))

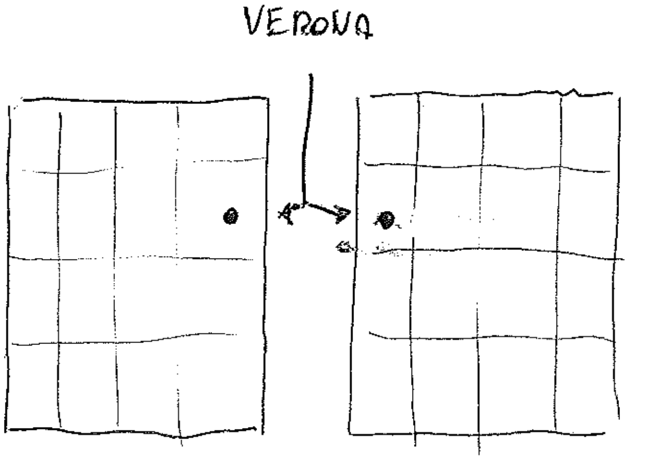
* Basic motivation : cartography



terrestrial ellipsoid
(with enhanced eccentricity)

not a maximal one!

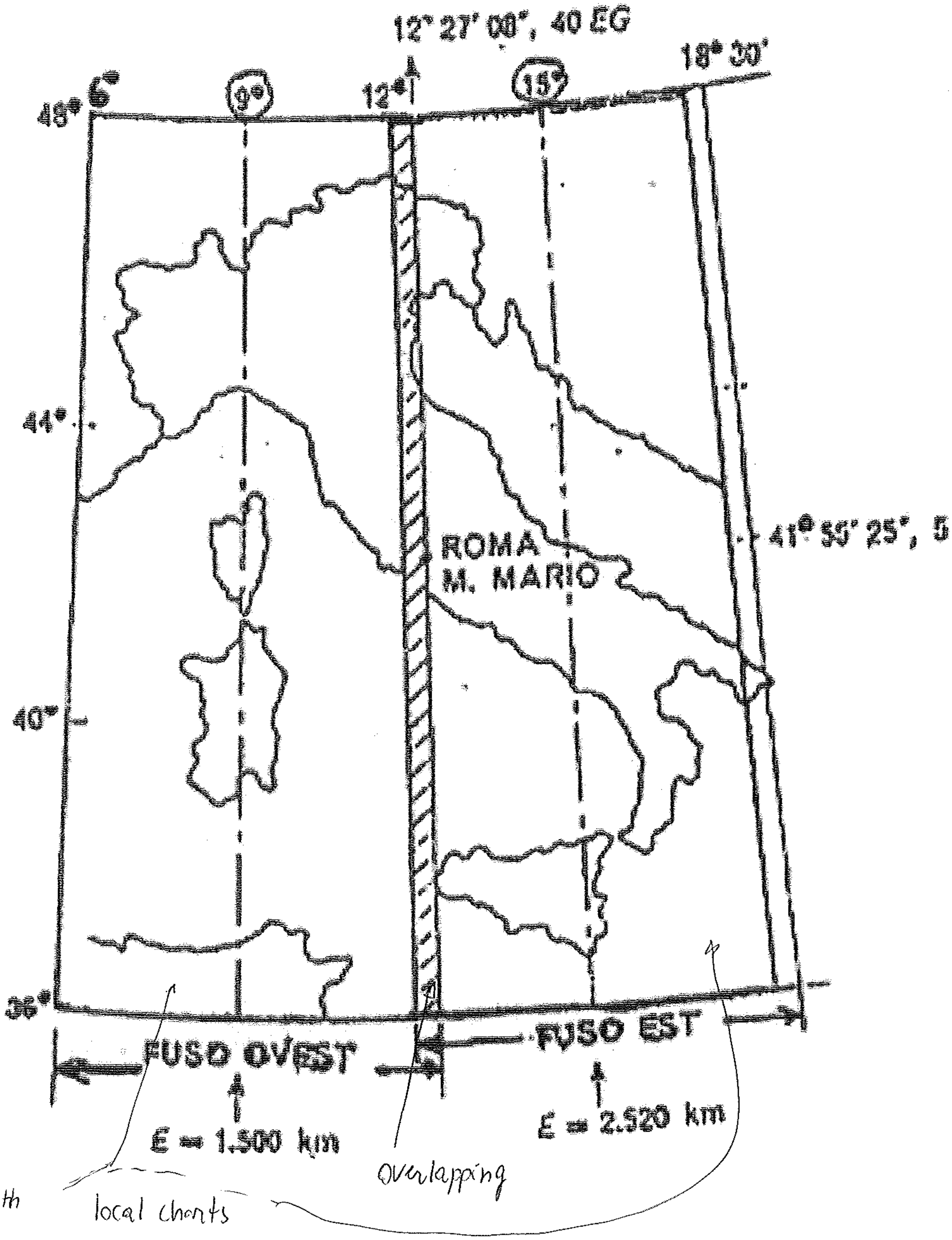




A transition map is involved, invisible to the
... final user

★ Gauss-Boaga projection

italian version of the UTM projection



★ Another (equivalent) definition of smooth manifold without starting from a topological space.

Let M be a set (no topology on it, a priori), such that $\exists\ f_\alpha : \mathcal{U}_\alpha \to M$, $\alpha \in \mathcal{O}$, $\mathcal{U}_\alpha \subset \mathbb{R}^n$ open,

f_α injective

[observe that charts go in the opposite direction, but this is not important]

such that

1. $\bigcup_{\alpha \in \mathcal{O}} f_\alpha(\mathcal{U}_\alpha) = M$

2. $\forall\ \alpha, \beta \in \mathcal{O}$ such that $f_\alpha(U_\alpha) \cap f_\beta(U_\beta) \equiv W_{\alpha\beta} \neq \phi$,

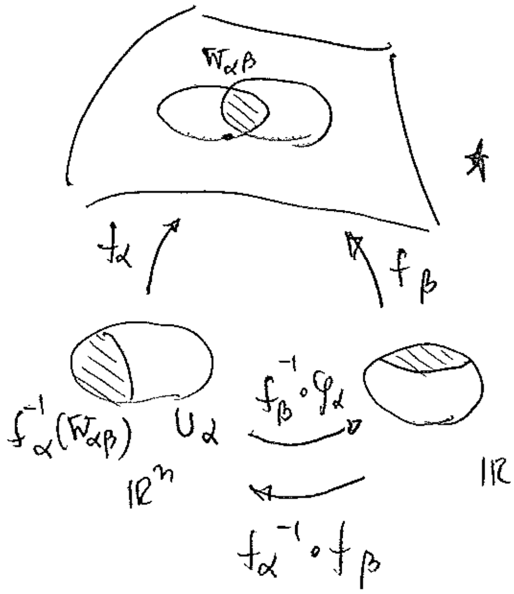
$f_\alpha^{-1}(W_{\alpha\beta})$ and $f_\beta^{-1}(W_{\alpha\beta})$ are open in $\mathbb{R}^n$ and such that

$$f_\alpha^{-1} \circ f_\beta \quad \text{and} \quad f_\beta^{-1} \circ f_\alpha \quad \text{are smooth}$$

well defined in view of injectivity

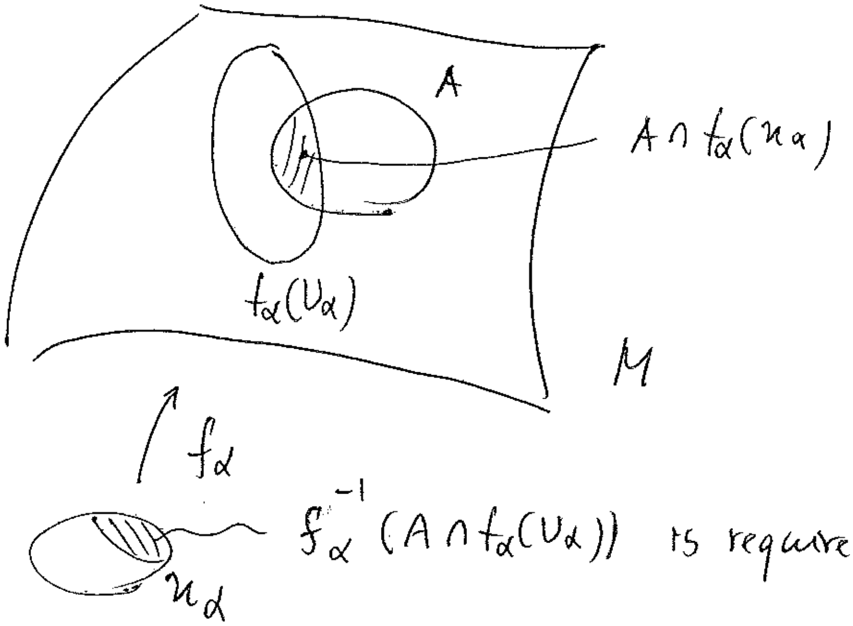
3. The above family is maximal with respect to the properties 1 and 2

$$\mathcal{A} = \{ (\mathcal{U}_\alpha, f_\alpha) \}_{\alpha \in \mathcal{O}} \quad \text{atlas (diff. structure)}$$



★ This gives us a natural topology $\mathcal{T}$ on M:

$A \subset M$ is open iff $f_\alpha^{-1}(A \cap f_\alpha(U_\alpha))$ is open in $\mathbb{R}^n$



$f_\alpha^{-1}(A \cap f_\alpha(\mathcal{U}_\alpha))$ is required to be open in $\mathbb{R}^n$

★ One checks that $\mathcal{T}$ fulfils the axioms of a topology.
($\mathcal{T}$ contains $\emptyset$, M and is closed under arbitrary unions and finite intersections)

The extra requirements: Hausdorff + countable basis are then postulated.

uniqueness of limits

existence of partitions of unity, see below

This approach is useful in applications, in cases there is no a priori topology to be imposed on set.