2D Wavelets

Hints on advanced Concepts

Advanced concepts

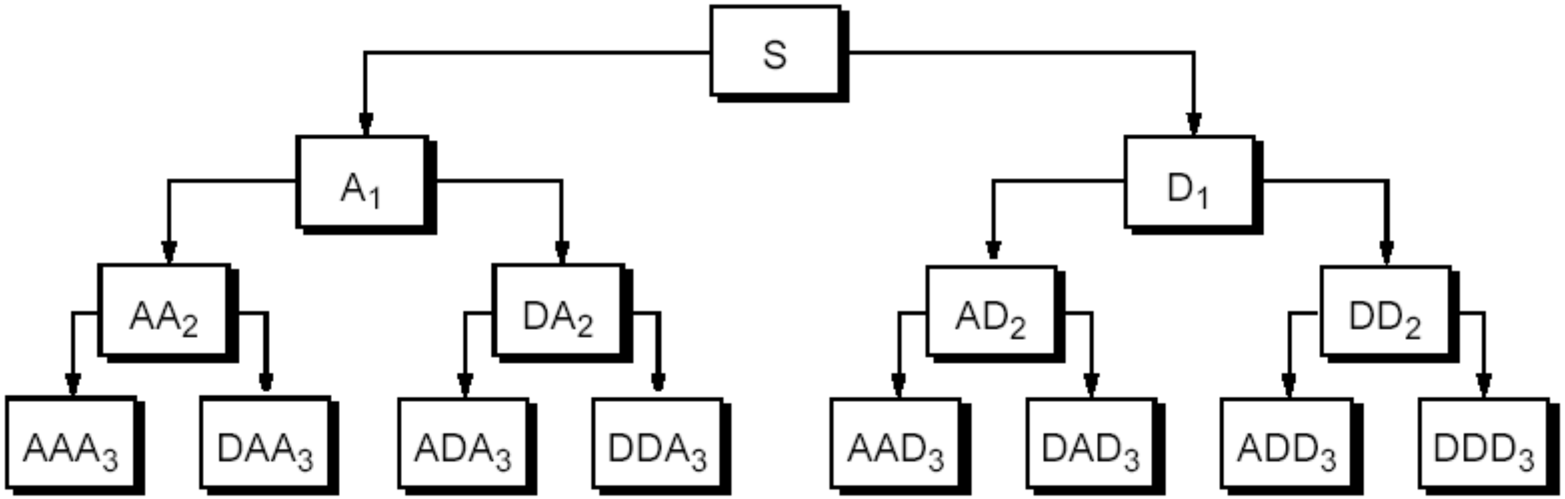
- Wavelet packets
- Laplacian pyramid
- Overcomplete bases
  - Discrete wavelet frames (DWF)
    - Algorithme à trous
  - Discrete dyadic wavelet frames (DDWF)

- Overview on edge sensitive wavelets
  - Contourlets

Wavelet packets

Wavelet packets

Both the approximation and the detail subbands are further decomposed



Packet tree

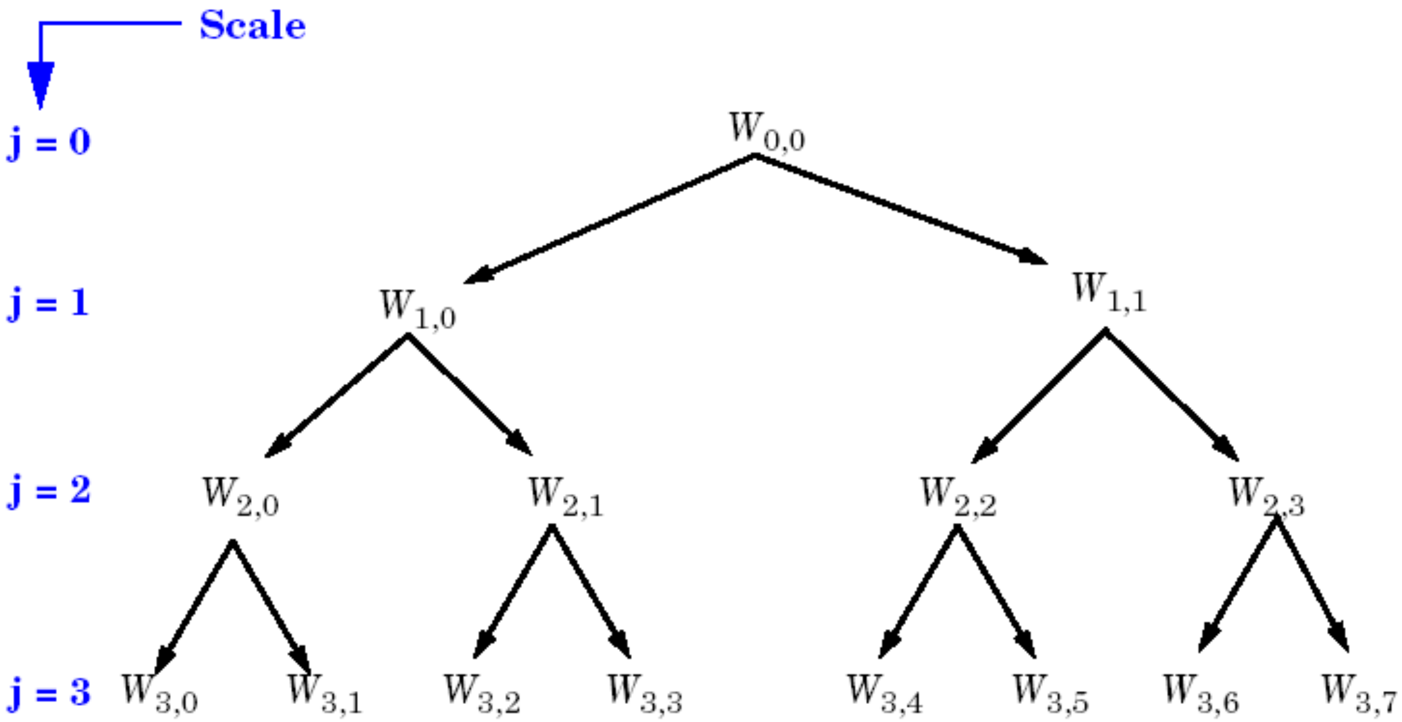
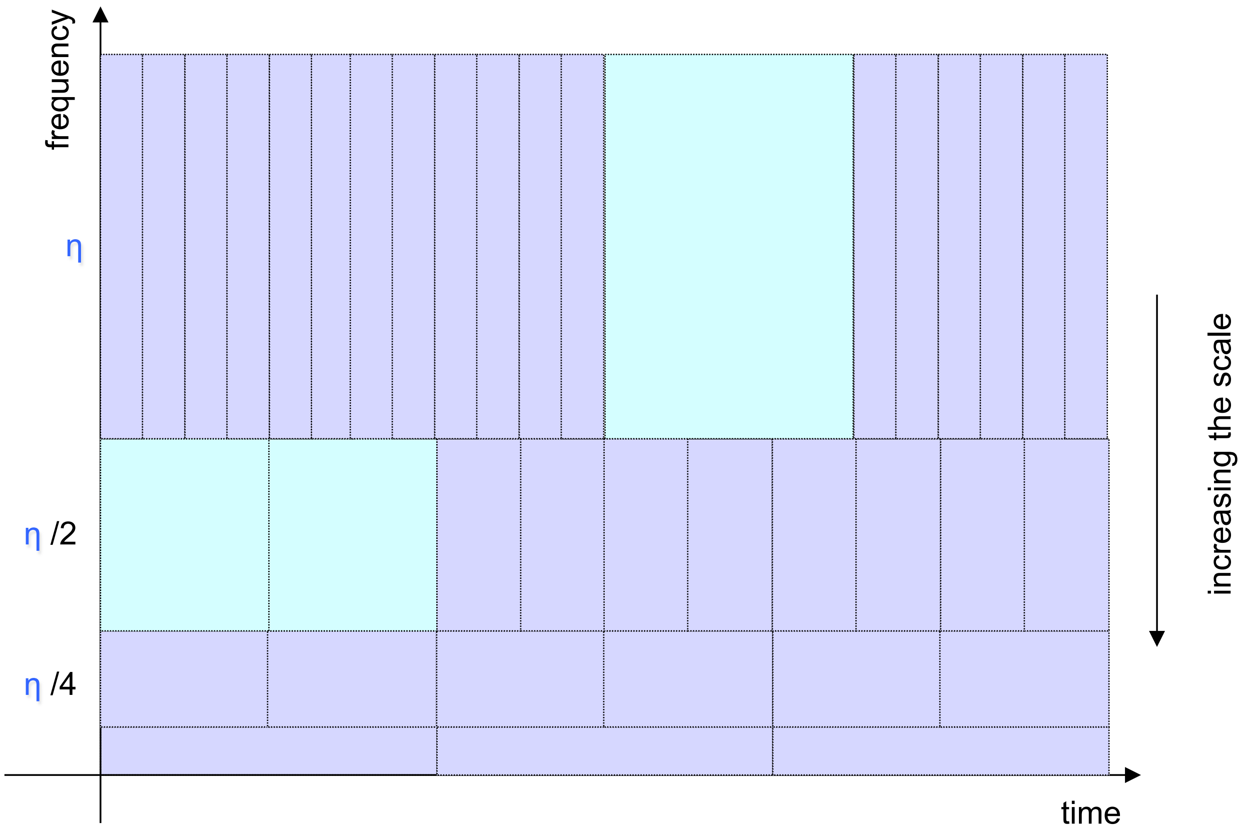
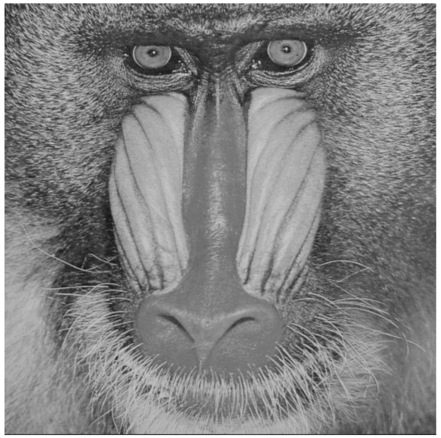


Figure 6-40: Wavelet Packets Organized in a Tree; Scale j Defines Depth and Frequency n Defines Position in the Tree

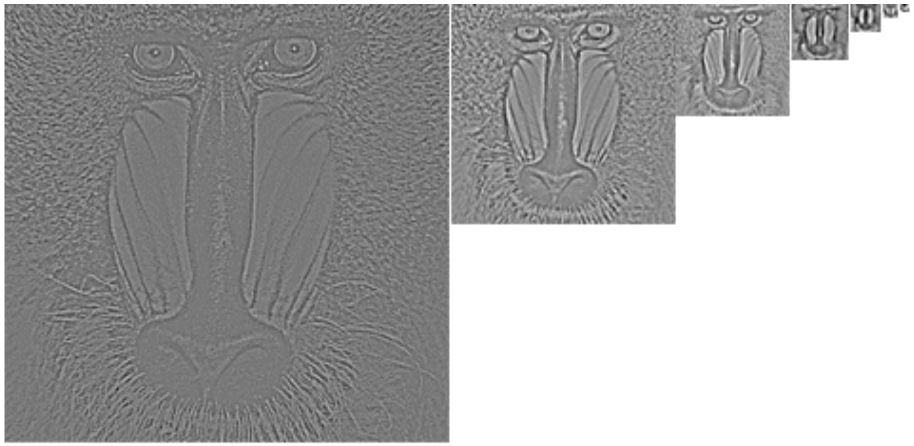
Wavelet Packets

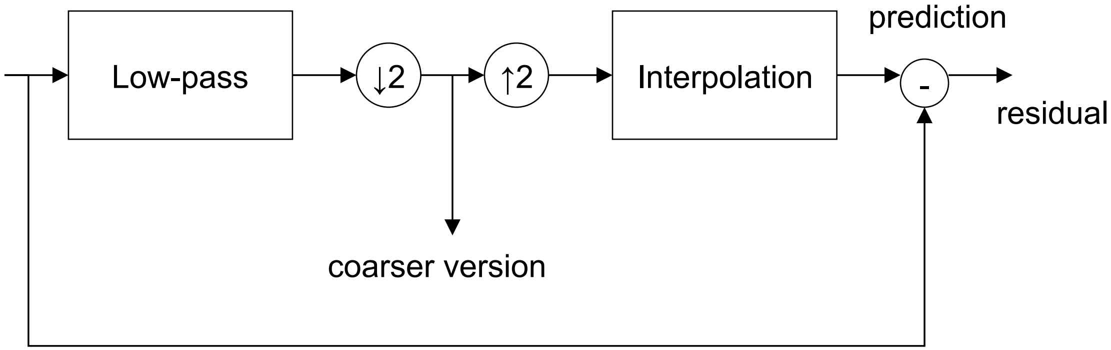


Laplacian Pyramid



residuals



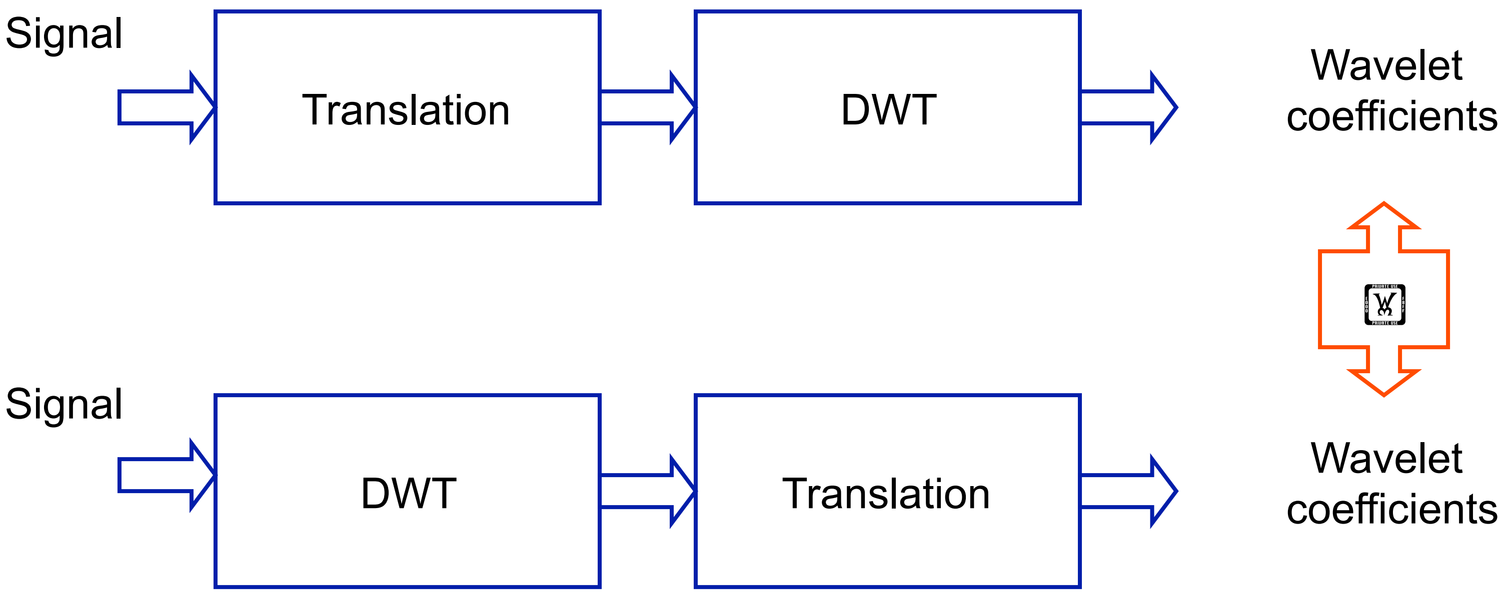


Overcomplete bases

Translation Covariance

Translation covariance $T\{DWT\{f(x,y)\}\} = DWT\{T\{f(x,y)\}\}$

If translation covariance does *not* hold: $T\{DWT\{f(x,y)\}\} \neq DWT\{T\{f(x,y)\}\}$



NOT good for signal analysis

Rationale

- In pattern recognition it is important to build representations that are translation invariant. This means that when the pattern is translated the descriptors should be translated but not modified in the value.
- CWTs and windowed FT provide translation covariance, while sampling the translation parameter might destroy translation covariance unless some conditions are met.
- Intuition: either the sampling step is very small compared to the translation or the translation is a multiple of the sampling step.

Translation invariant representations

Translation-Invariant Representations There are several strategies for maintaining the translation invariance of a wavelet transform. If the sampling interval $a^j u_0$ is small enough then the samples of $f \star \bar{\psi}_{a^j}(t)$ are approximately translated when f is shifted. The dyadic wavelet transform presented in Section 5.5 is a translation-invariant representation that does not sample the translation factor u. This creates a highly redundant signal representation.

To reduce the representation size while maintaining translation invariance, one can use an adaptive sampling scheme, where the sampling grid is automatically translated when the signal is translated. For each scale a^j, $Wf(u, a^j) = f \star \bar{\psi}_{a^j}(u)$ can be sampled at locations u where $|Wf(a^j, u)|$ is locally maximum. The resulting representation is translation invariant since the local maxima positions are translated when f and hence $f \star \bar{\psi}_{a^j}$ are translated. This adaptive sampling is studied in Section 6.2.2.

Translation covariance

- The signal descriptors should be covariant with translations
  - Continuous WT and windowed FT are translation covariant.

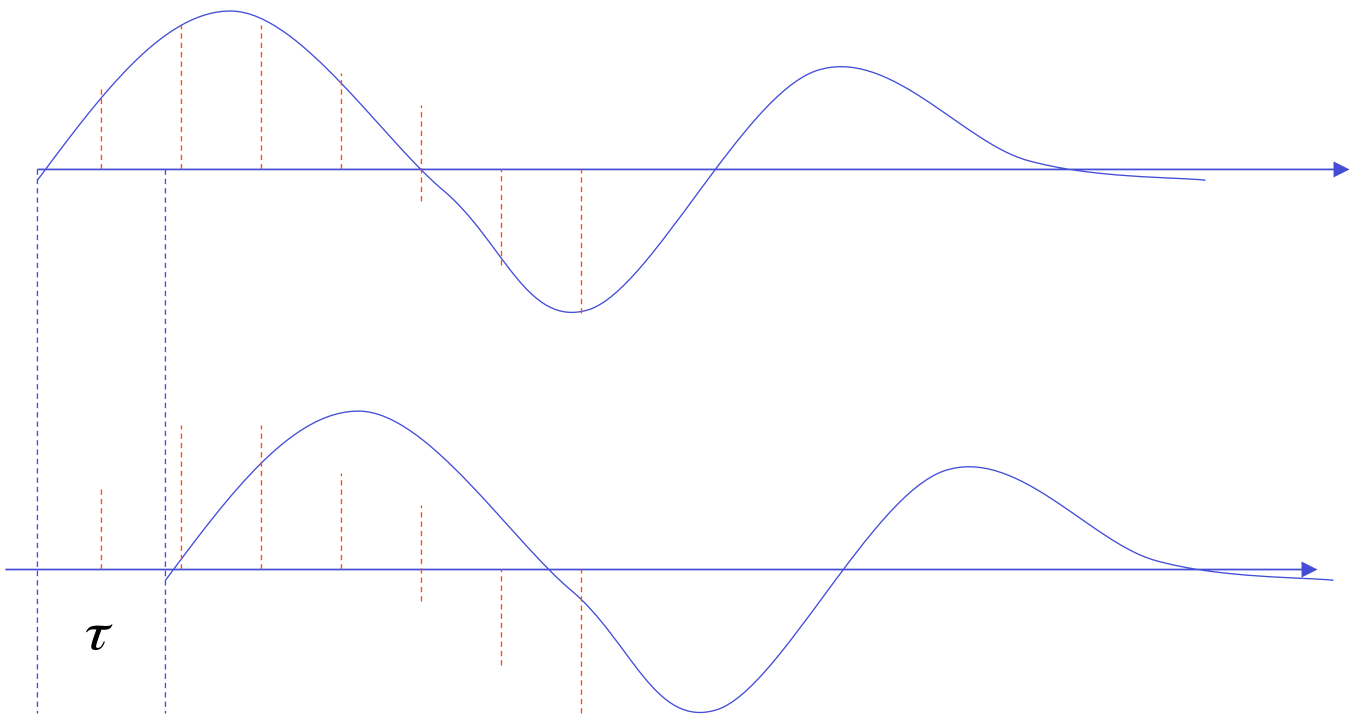
$$f_\tau(t) = f(t-\tau)$$

$$Wf(u,s) = \int f(t)\frac{1}{\sqrt{s}}\psi\left(\frac{t-u}{s}\right)dt = f * \overline{\psi}_s(u)$$

$$Wf_\tau(u,s) = \int f(t-\tau)\frac{1}{\sqrt{s}}\psi\left(\frac{t-u}{s}\right)dt = Wf(u-\tau,s)$$

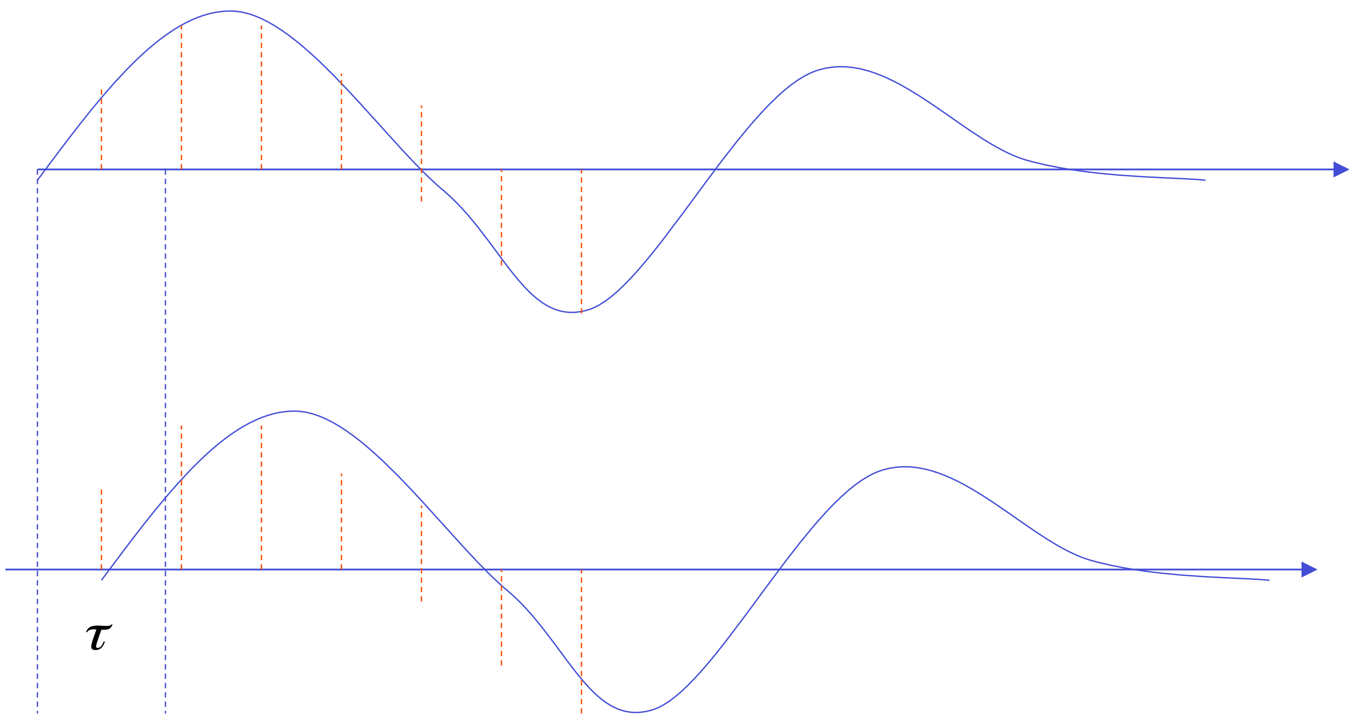
  - Wavelet frames (DWF) are constructed by sampling continuous transforms over *uniform time grids*.
  - The sampling grid removes the translation covariance because the translation factor τ is a priori not equal to the translation interval

Sampling and translation covariance



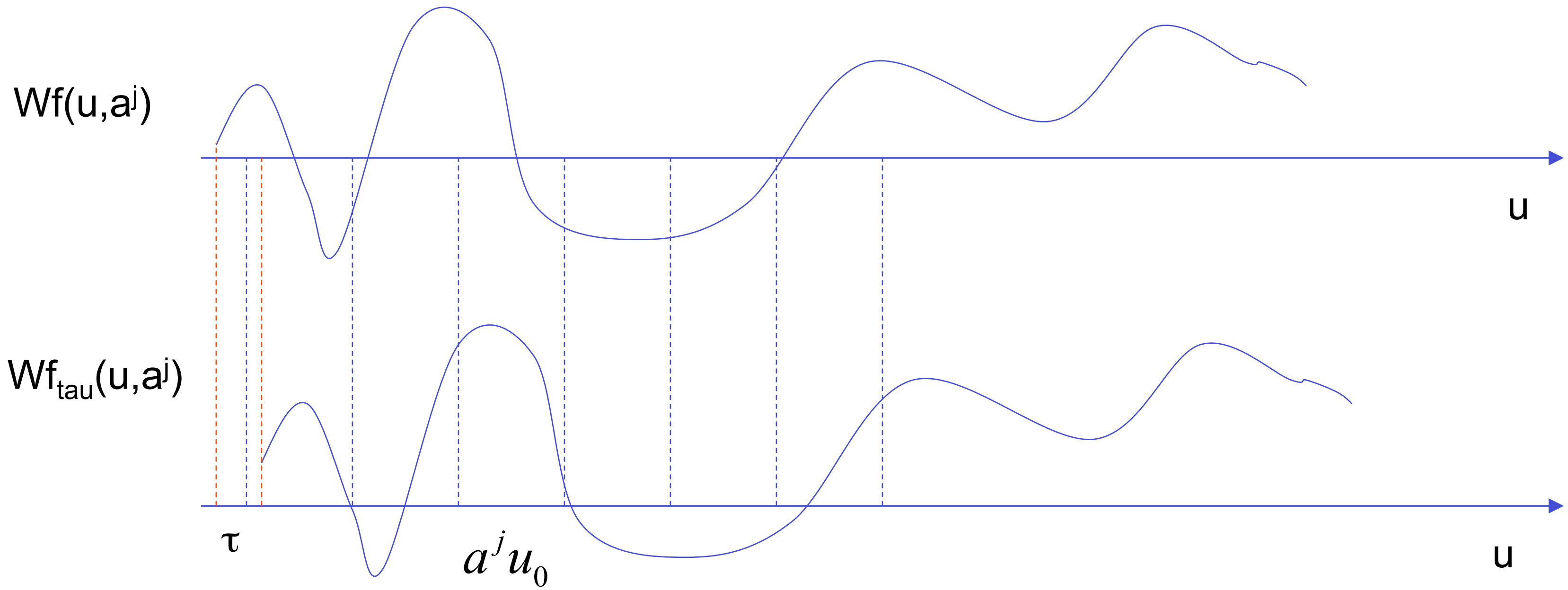
If the translation does not correspond to a multiple of the sampling step, a different set of samples will be obtained by keeping the same sampling grid

Sampling and translation covariance



Translation invariant representation

- If the sampling interval $a^j u_0$ is <u>small enough</u> than the samples of $f * \overline{\psi}_{a^j}(t)$ are approximately translated when f is shifted.
- Translation covariance holds if $\tau = k u_0 a^j$ namely it is a multiple of the sampling interval



Translation invariant representation

- Uniformly sampling the translation parameter destroys covariance unless the translation is very small
- Translation invariant representations can be obtained by sampling the scale parameter s but not the translation parameter u

Dyadic Wavelet Transform

- Sampling scheme
  - Dyadic scales
  - Integer translations

$$Wf(u,2^j) = \int f(t)\frac{1}{\sqrt{2^j}}\psi\left(\frac{t-u}{2^j}\right)dt = f * \overline{\psi}_{2^j}(u)$$

$$\overline{\psi}_{2^j}(t) = \psi_{2^j}(-t) = \frac{1}{\sqrt{2^j}}\psi\left(\frac{-t}{2^j}\right)$$

  - If the frequency axis is completely covered by dilated dyadic wavelets, then it defines a **complete** and **stable** representation
    - The normalized dyadic wavelet transform operator has the same properties of a frame operator, thus both an analysis and a reconstruction wavelets can be identified

- Special case: ***algorithme à trous***

Algorithme à trous

- Similiar to a fast biorthogonal WT without subsampling
- Fast dyadic transform
  - The samples of the discrete signal $a_0[n]$ are considered as averages of some function weighted by some scaling kernels φ(t-n)

$$a_0[n] = \langle f(t), \varphi(t-n) \rangle$$

For any j ≥ 0 we denote

$$a_{2^j}[n] = \langle f(t), \varphi_{2^j}(t-n) \rangle$$

The dyadic wavelet coefficients are computed for j > 0 over the integer grid

$$d_j[n] = \langle f(t), \psi_{2^j}(t-n) \rangle = Wf(n, 2^j)$$

  - For any filter $x[n]$, we denote by $x_j[n]$ the filters obtained by inserting 2^j-1 zeros between each sample of $x[n]$ → create holes (*trous*, in French)

$$\bar{x}_j[n] = x_j[-n]$$

Algorithme a trous

- Proposition

For any $j \geq 0$

$$a_{j+1}[n] = a_j * \bar{h}_j[n]$$

$$d_{j+1}[n] = a_j * \bar{g}_j[n]$$

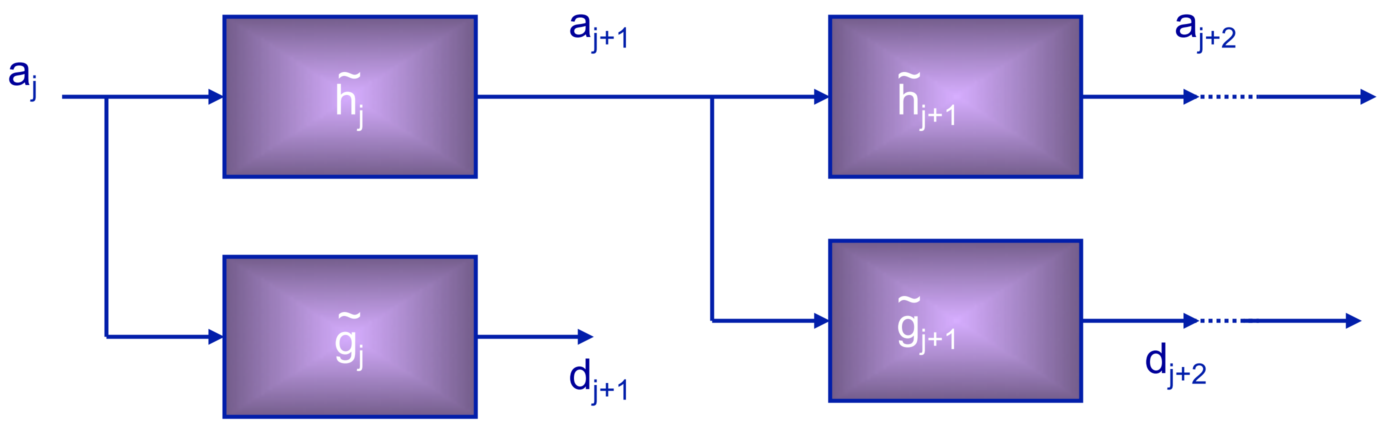
$$a_j[n] = \frac{1}{2}\left(a_{j+1} * \tilde{h}_j[n] + d_{j+1} * \tilde{g}_j[n]\right)$$

The dyadic wavelet representation of a_0 is defined as the set of wavelet coefficients up to the scale 2^J plus the remaining low-pass frequency information a_J

$$\left[a_J, \{d_j\}_{1 \leq j \leq J}\right]$$

– Fast filterbank implementation

Analysis



$$\tilde{h}[n] = h_0 \quad h_1 \quad h_2$$

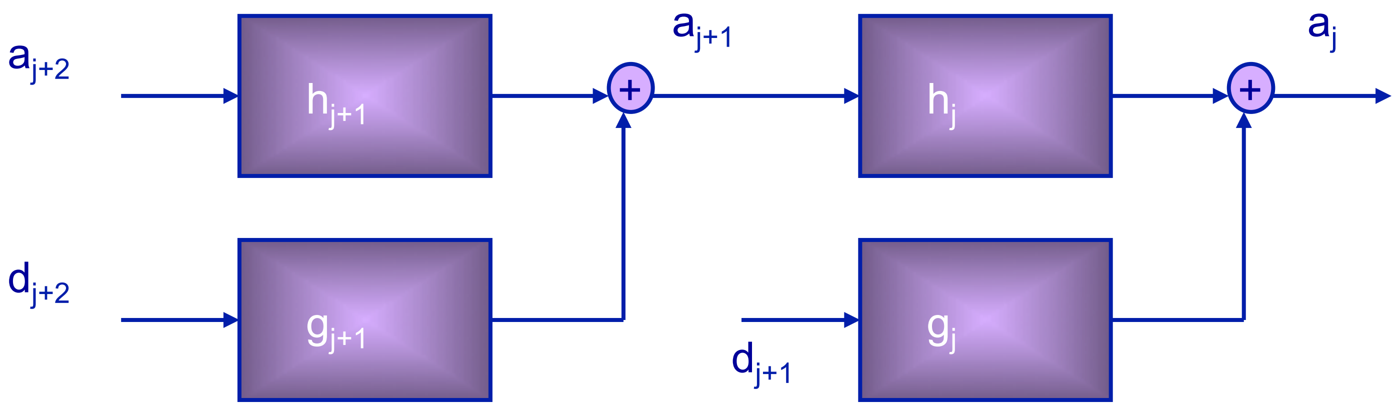
$$\tilde{h}_1[n] = h_0 \quad 0 \quad h_1 \quad 0 \quad h_2$$

$$\tilde{h}_2[n] = h_0 \quad 0 \quad 0 \quad 0 \quad h_1 \; 0 \quad 0 \quad 0 \quad h_2$$

2^j-1"trous"

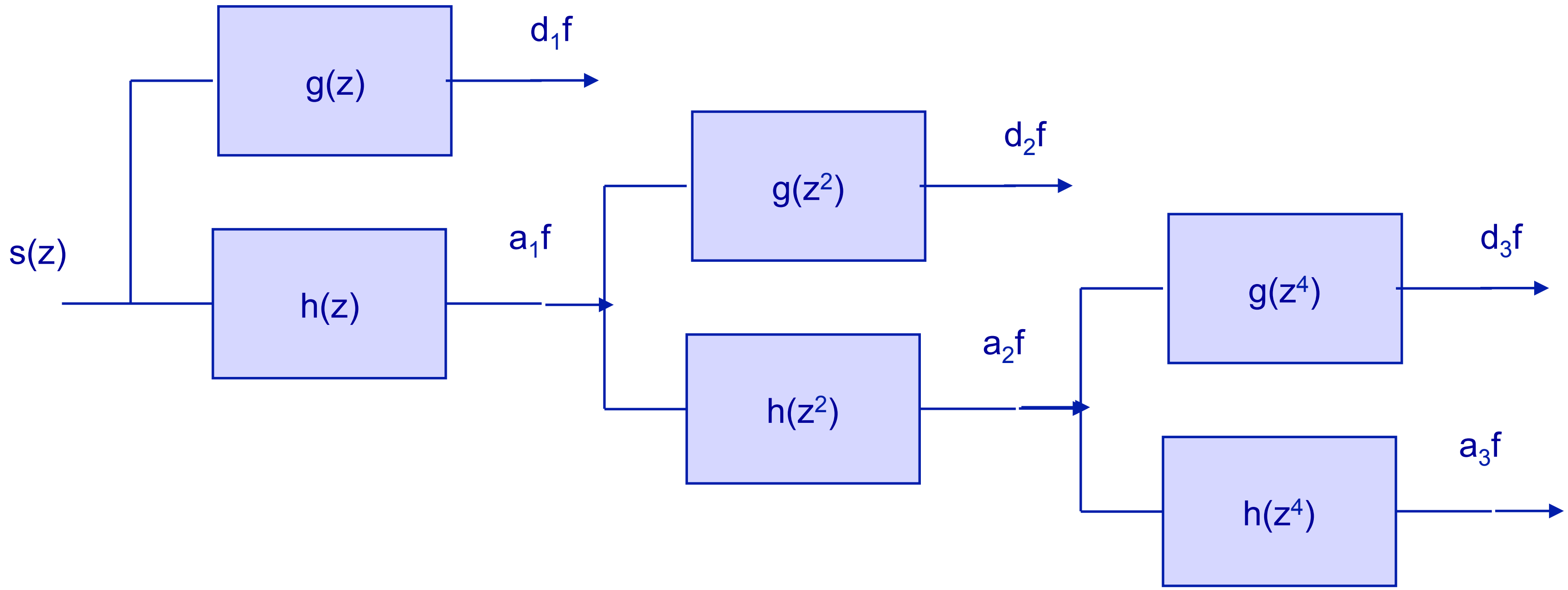
No subsampling!!

Synthesis



Overcomplete wavelet representation: $[a_J, \{d_j\}_{1 \leq j \leq J}]$

Algorithme a trous



DWT vs DWF

- DWT
  - Non-redundant
  - Signal il subsampled
  - Not translation invariant
  - Total number of coefficients:
    $N_x N_y$

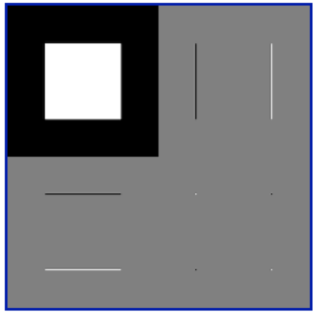
- Compression

- DWF
  - Redundant (in general)
  - Signal is not subsapled
  - Filters are upsampled
  - Translation invariant
  - Total number of coefficients:
    $(3J+1)N_x N_y$

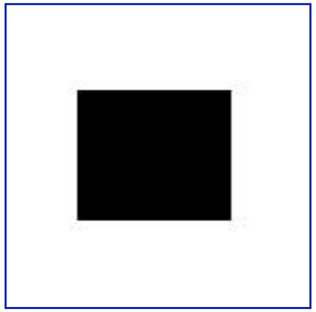
- Feature extraction

Discrete WT *vs* Dyadic WT

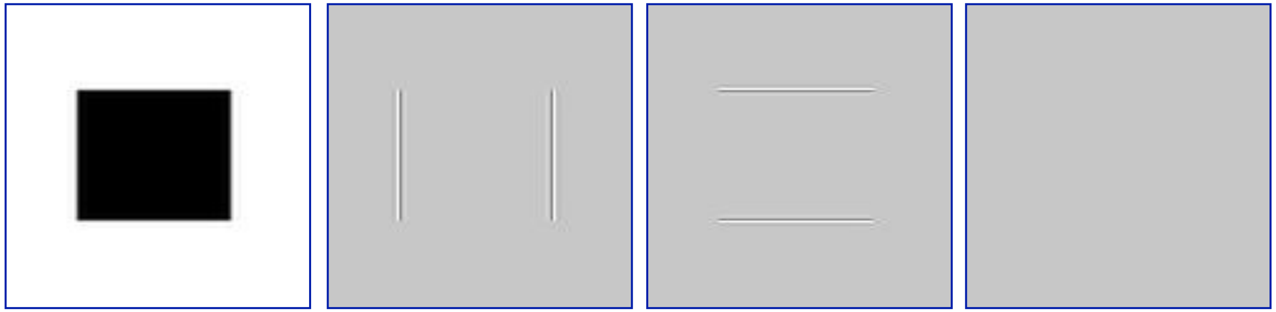
DWT



Original

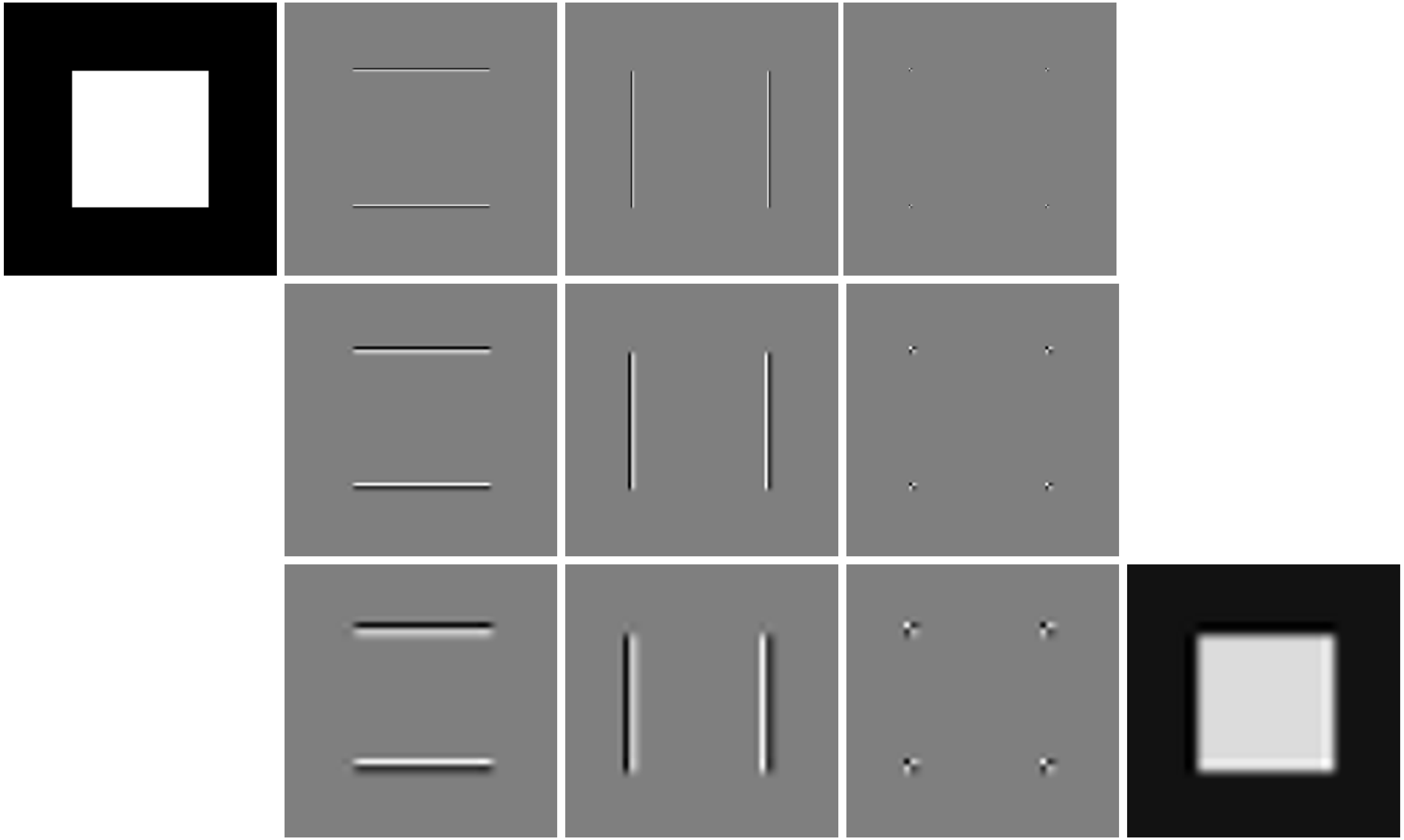


Dyadic WT

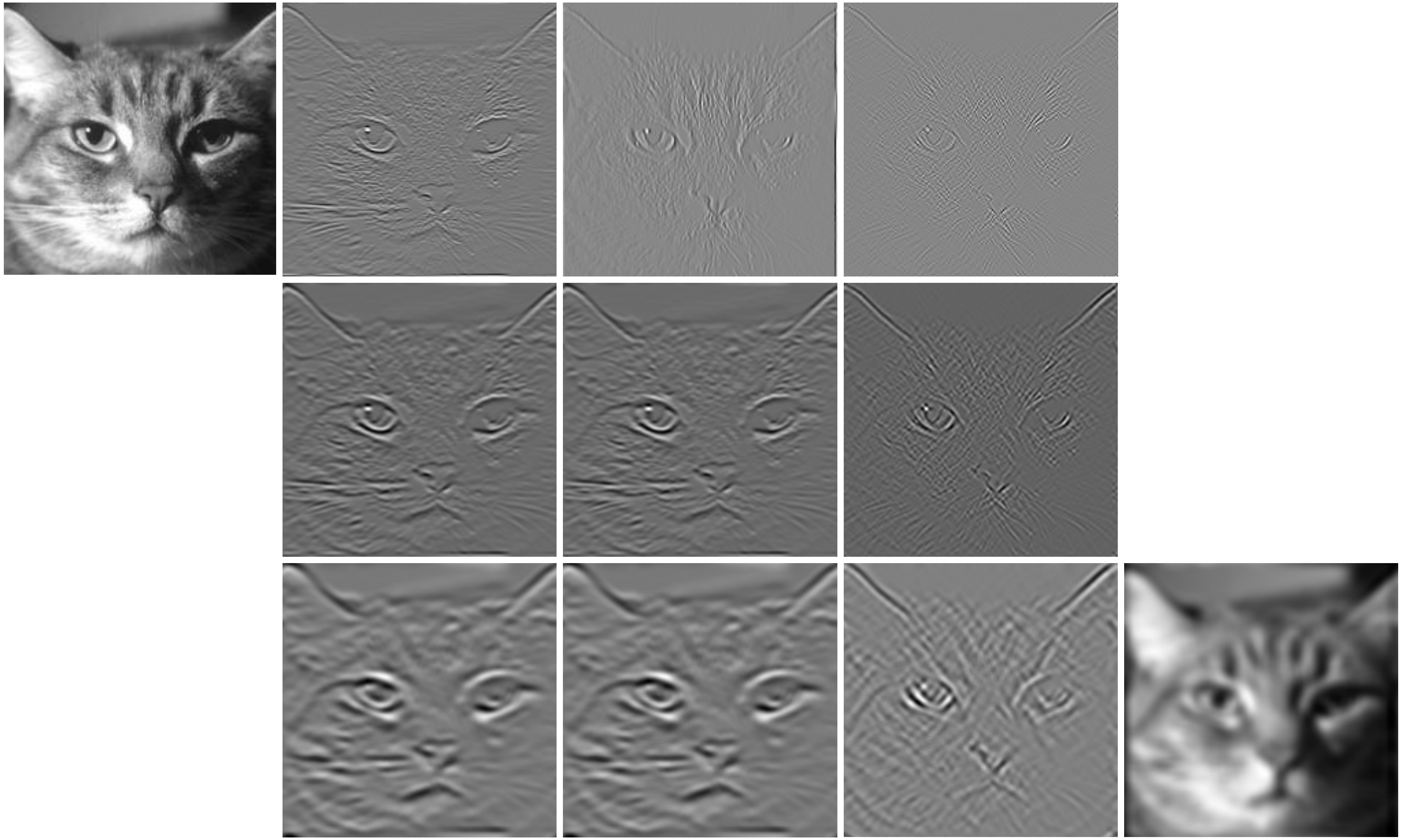


LL HL LH HH

Example 1



Example 2



Rotation covariance

- *Oriented* wavelets $\{\psi^k(x,y)\}_{1 \le k \le K}$
  - In 2D, a dyadic WT is computed with several wavelets which have different spatial orientations
    - We denote

$$\psi_{2^j}{}^k(x,y) = \frac{1}{2^j}\psi^k\left(\frac{x}{2^j},\frac{y}{2^j}\right)$$

    - The WT in the direction k is defined as

$$W^k f(u,v,2^j) = \left\langle f(x,y), \psi_{2^j}{}^k(x-u,y-v)\right\rangle = f * \overline{\psi}_{2^j}{}^k(u,v)$$

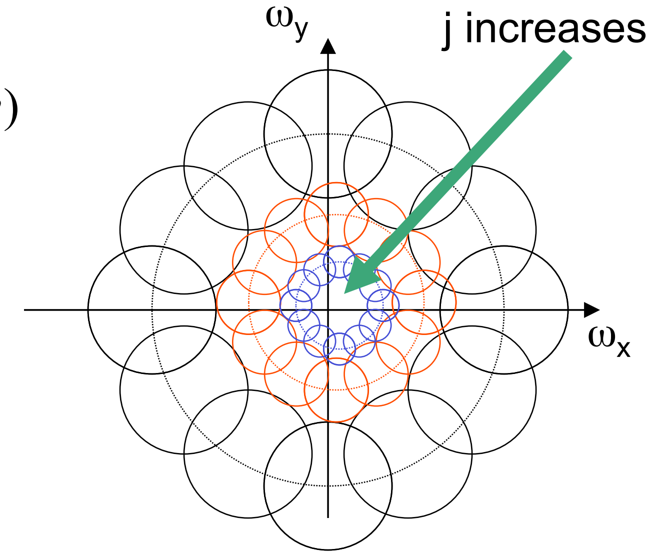
  - One can prove that this is a *complete* and *stable* representation if there exist A>0 and B>0 such that

$$\forall(\omega_x,\omega_y) \in \mathbf{R}^2 - \{0,0\}, A \le \sum_{k=1}^{K}\sum_{j=-\infty}^{+\infty}\left|\hat{\psi}\left(2^j\omega_x, 2^j\omega_y\right)\right|^2 \le B$$

Oriented wavelets

– Then, there exists a reconstruction wavelet family such that

$$f(x,y) = \sum_{j=-\infty}^{+\infty} \frac{1}{2^j} \sum_{k=1}^{K} W^k f(w,y,2^j) * \tilde{\psi}_{2^j}^k(x,y)$$



- Gabor wavelets

$$\psi^k(x,y) = g(x,y) e^{-i\eta(x\cos\alpha_k + y\sin\alpha_k)}$$

$$g(x,y) = \frac{1}{2\pi} e^{-\left(\frac{x^2+y^2}{2}\right)} \rightarrow \text{Gaussian} \rightarrow \text{Gabor wavelets}$$

$$\hat{\psi}_{2^j}^k(\omega_x,\omega_y) = \sqrt{2^j}\, \hat{g}(2^j \omega_x - \eta\cos\alpha_k, 2^j \omega_y - \eta\sin\alpha_k)$$

– In the Fourier plane the energy of the Gabor wavelet is mostly concentrated in a neighborhood proportional to 2^{-j}.

$$\left(2^{-j}\eta\cos\alpha_k, 2^{-j}\eta\sin\alpha_k\right)$$

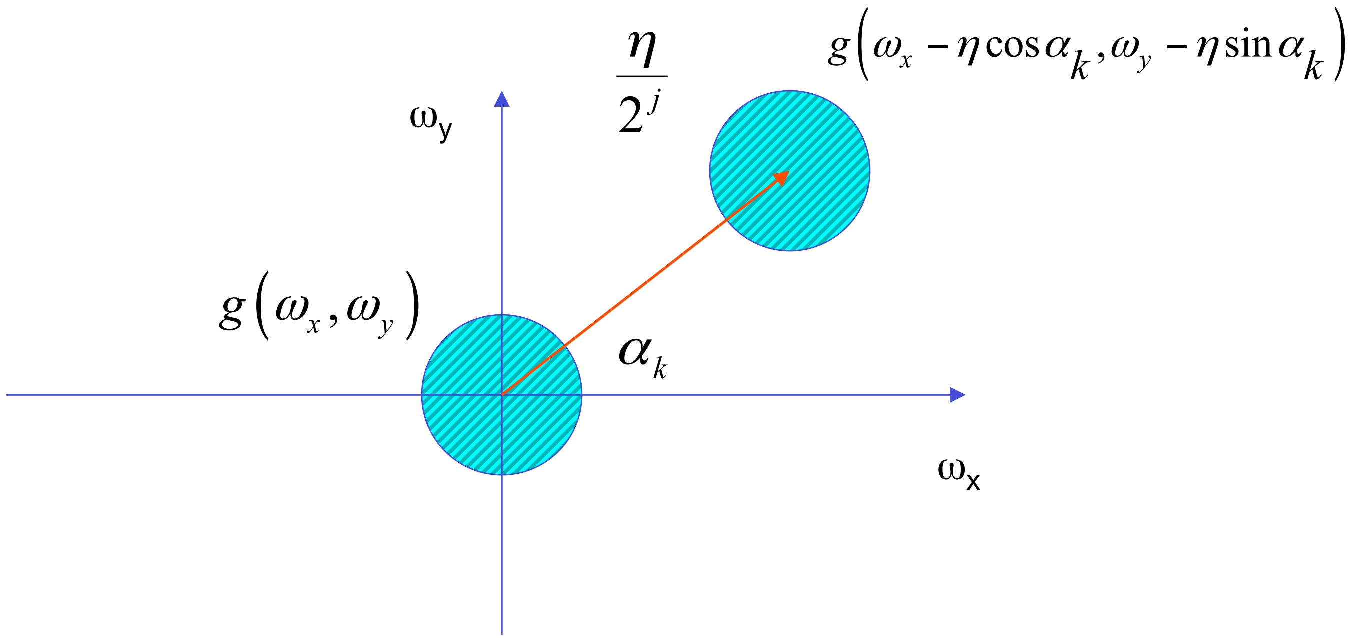
Gabor wavelets

$$\psi^k(x,y) = g(x,y)e^{-i\eta(x\cos\alpha_k + y\sin\alpha_k)},$$

$g(x,y)$ Gaussian $\rightarrow$ Gabor wavelets

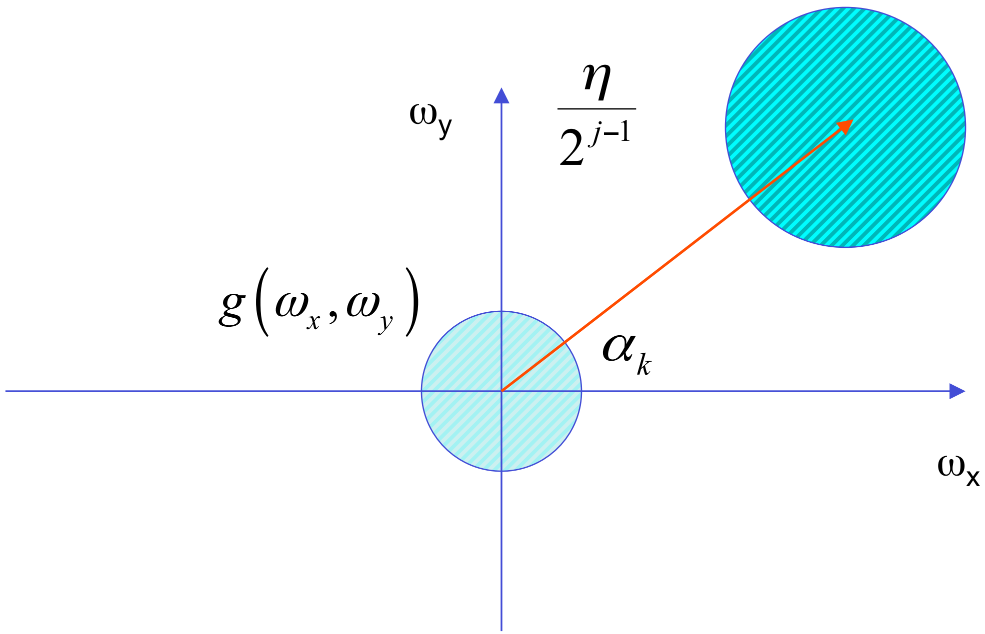
then

$$\psi^k(\omega_x,\omega_y) = g\left(\omega_x - \eta\cos\alpha_k, \omega_y - \eta\sin\alpha_k\right)$$

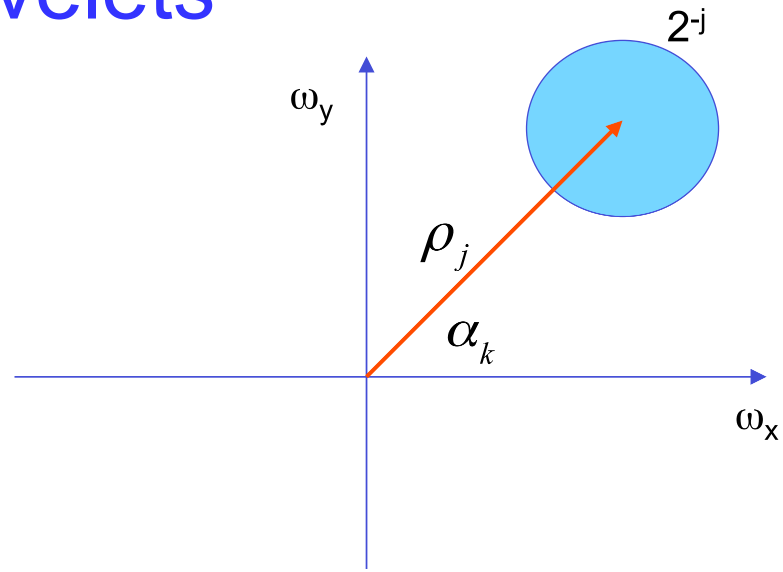


Gabor wavelets

$$\hat{\psi}_{2^j}{}^k(\omega_x,\omega_y) = \sqrt{2^j}\,\hat{g}(2^j\omega_x - \eta\cos\alpha_k, 2^j\omega_x - \eta\sin\alpha_k)$$



Gabor wavelets



$$\rho_j = \frac{\eta}{2^j}\sqrt{(\cos\alpha_k)^2 + (\sin\alpha_k)^2} = \frac{\eta}{2^j}$$

$$\theta_k = tg^{-1}(\alpha_k)$$

- Other directional wavelet families
  - Dyadic Frames of Directional Wavelets [Vandergheynst 2000]
  - Curvelets [Donoho&Candes 1995]
  - Steerable pyramids [Simoncelli-95]
  - Contourlets [Do&Vetterli 2002]

Dyadic Frames of Directional Wavelets

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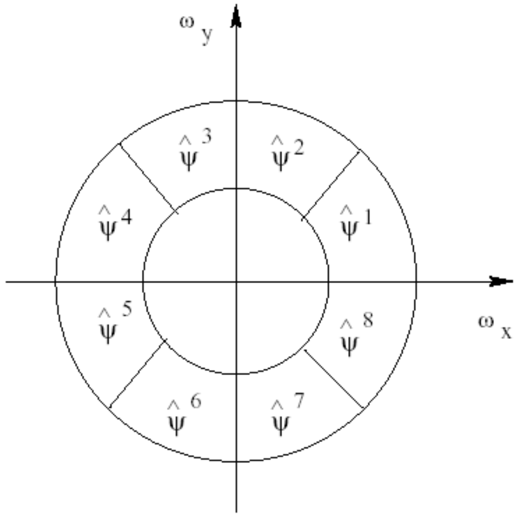
http://people.epfl.ch/pierre.vandergheynst

Dyadic Directional WF

K oriented wavelets $\{\psi^k_{\vec{b},2^j,}, 1 \leq k \leq K\}$



Scaled and translated wavelets:

$$\psi^k_{\vec{b},2^j}(\vec{x}) = \frac{1}{2^{2j}}\psi^k\left(2^{-j}(\vec{x}-\vec{b})\right)$$

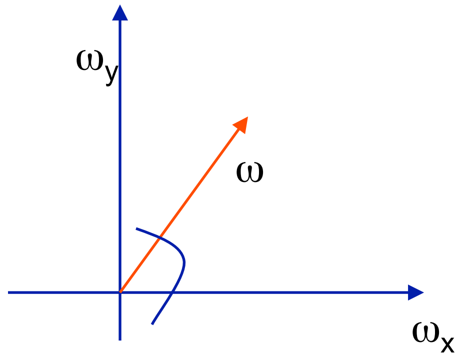
$$Wf^k(\vec{b},2^j) = \langle \psi^k_{2^j,\vec{b}}, f\rangle$$

Dyadic Directional Wavelet Frames

- Directional selectivity at any desired angle at any scale
  - Not only horizontal, vertical and diagonal as for DWT and DWF
  - *Rotation covariance for multiples of 2π/K*
- Recipe
  - Build a family of *isotropic* wavelets such that the Fourier transform of the mother wavelet expressed in polar coordinates is separable

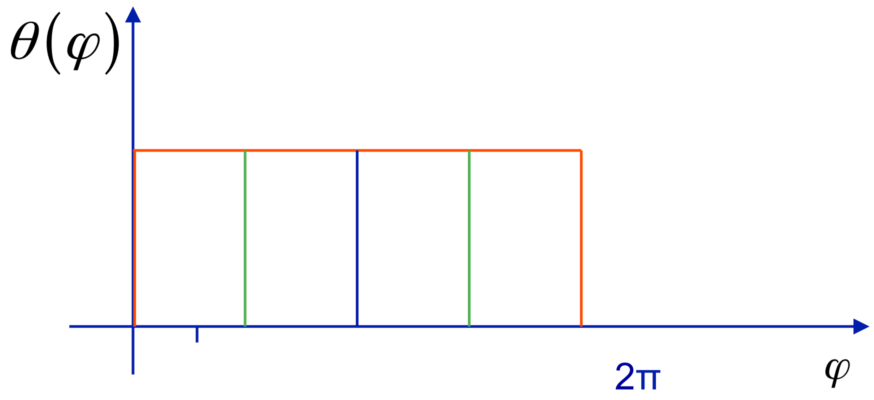
$$\hat{\Psi}(\omega,\varphi)=\Gamma(\omega)\Theta(\varphi)$$

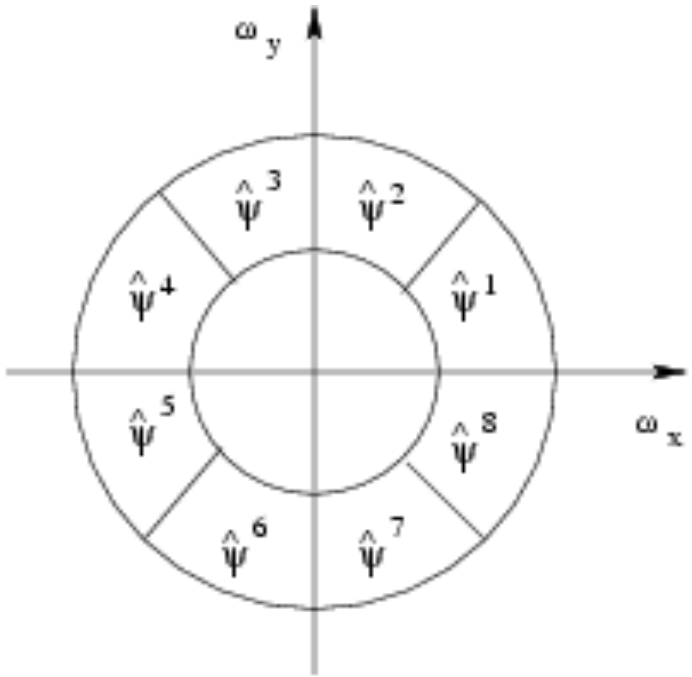
$$\omega=\sqrt{\left(\omega_x^{\,2}+\omega_y^{\,2}\right)}$$



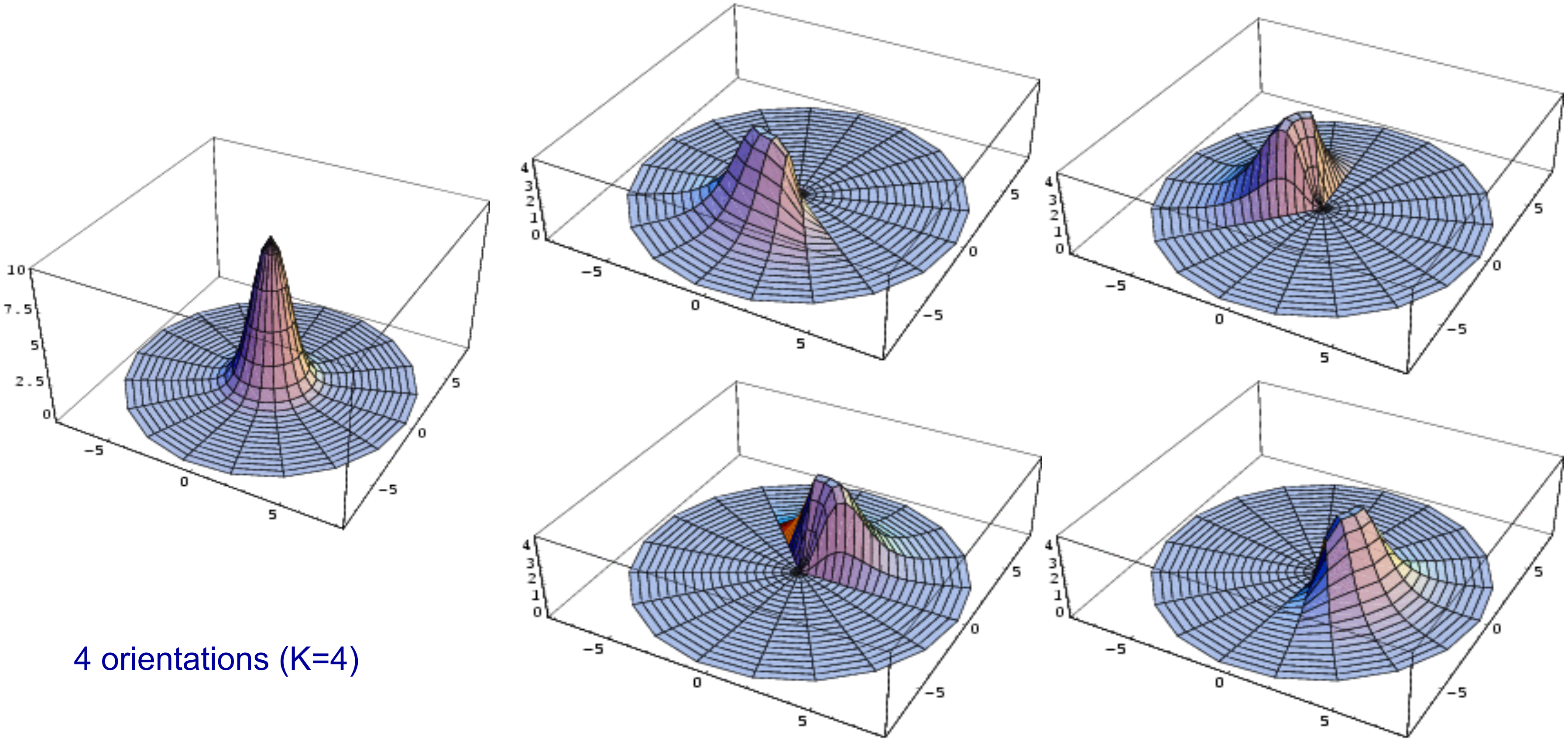
  - *Split each isotropic wavelet in a set of oriented wavelets* by an angular window
    - Express the angular part Θ(φ) as a sum of window functions centered at θ_k

Partitions of the F-domain



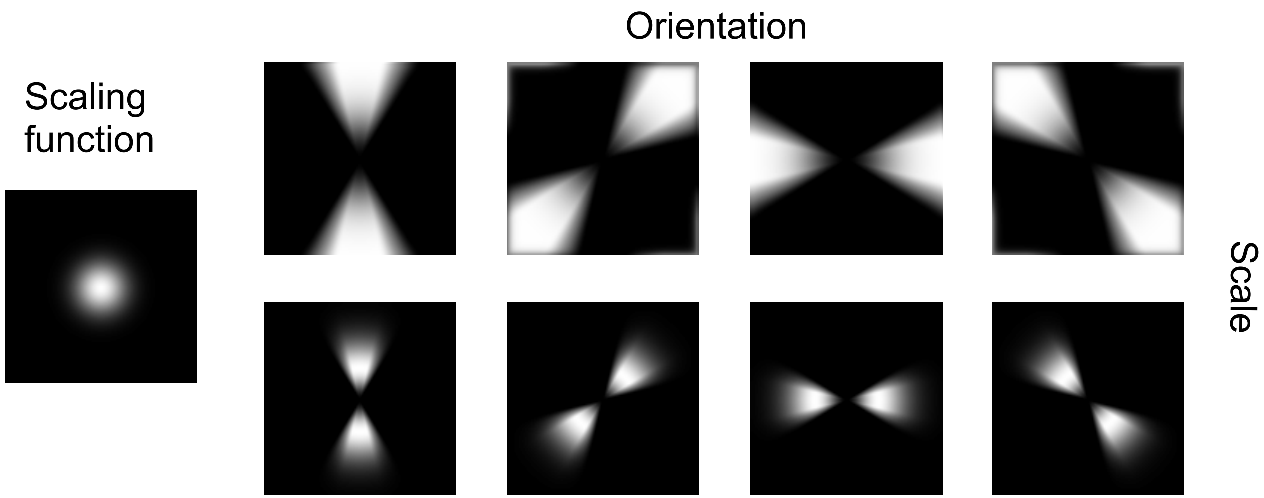


Dyadic Directional Wavelet Frames



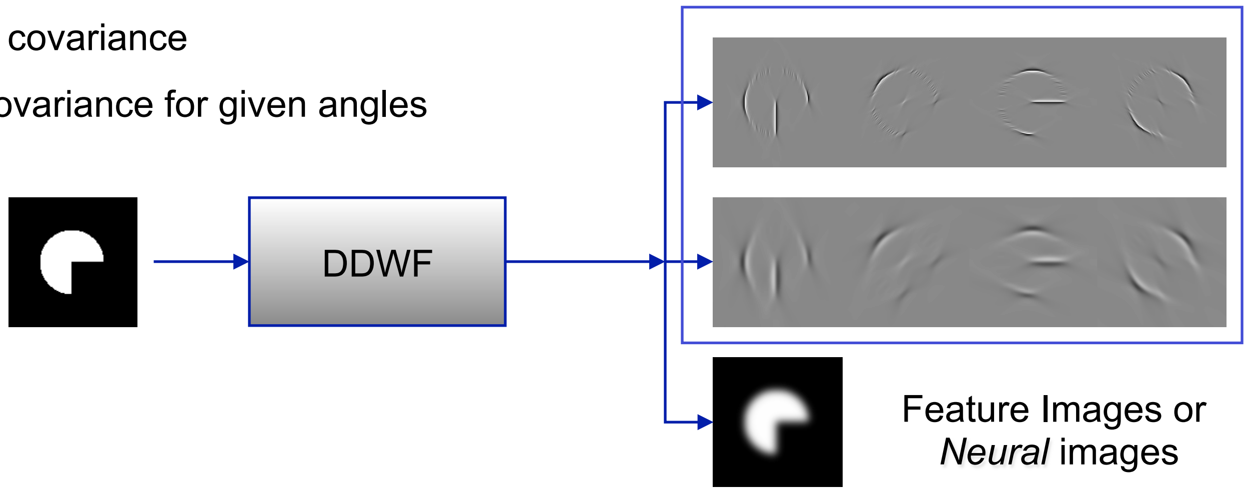
4 orientations (K=4)

Dyadic Directional Wavelet Frames



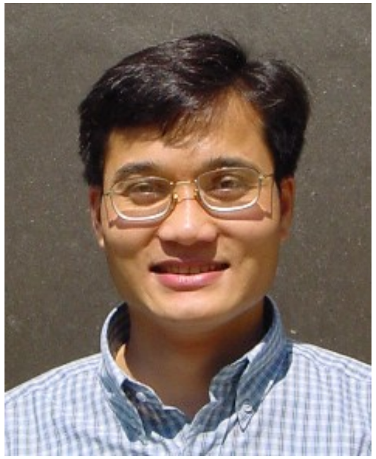
Properties

- Overcomplete
- Translation covariance
- Rotation covariance for given angles



Contourlets

Brief overview



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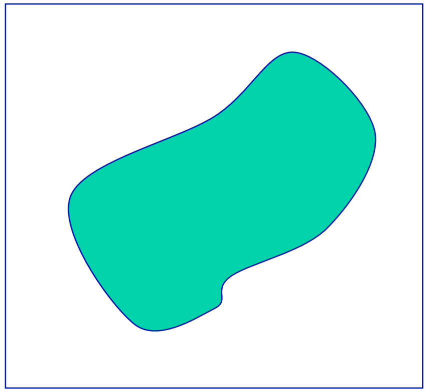
Martin Vetterli, LCAV-EPFL

Contourlets

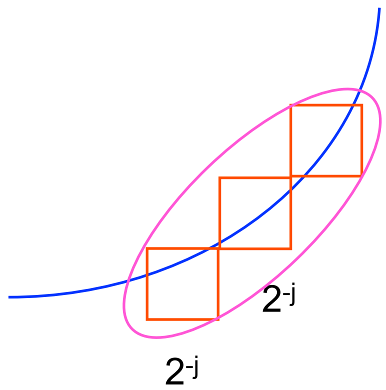
- Goal
  - Design an efficient linear expansion for 2D signals, which are smooth away from discontinuities across smooth curves
  - *Efficiency means Sparseness*



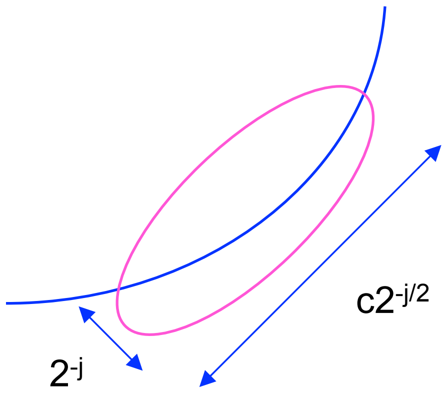
  - [Do&Vetterli] Piecewise smooth images with smooth contours
  - Inspired to *curvelets* [Donoho&Candes]

Curvelets

- Basic idea
  - Curvelets can be interpreted as a grouping of nearby wavelet basis functions into linear structures so that they can *capture the smooth discontinuity* curve more efficiently



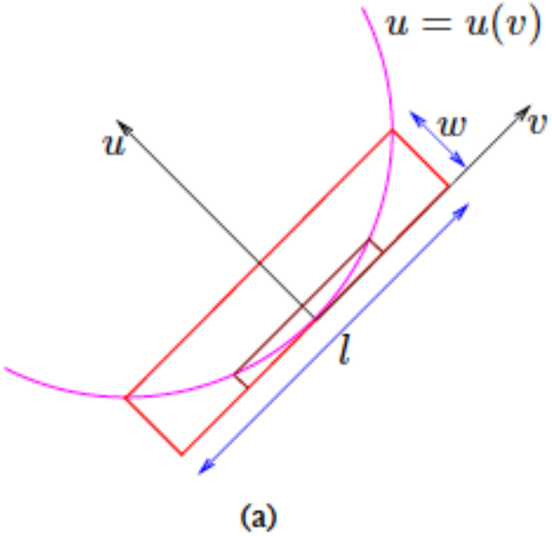
wavelets



curvelets

  - More efficient in capturing the geometry -> more concise (sparse) representation

Parabolic scaling



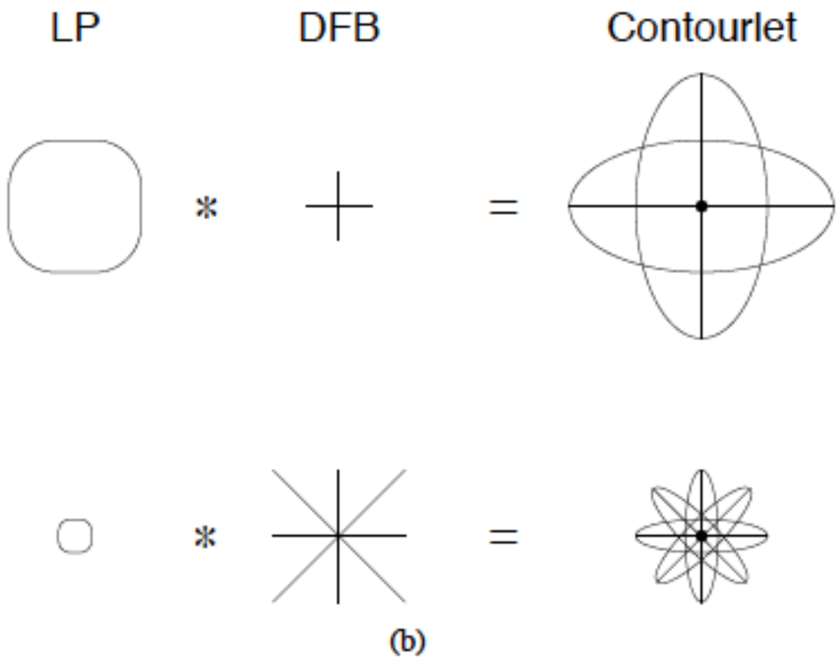
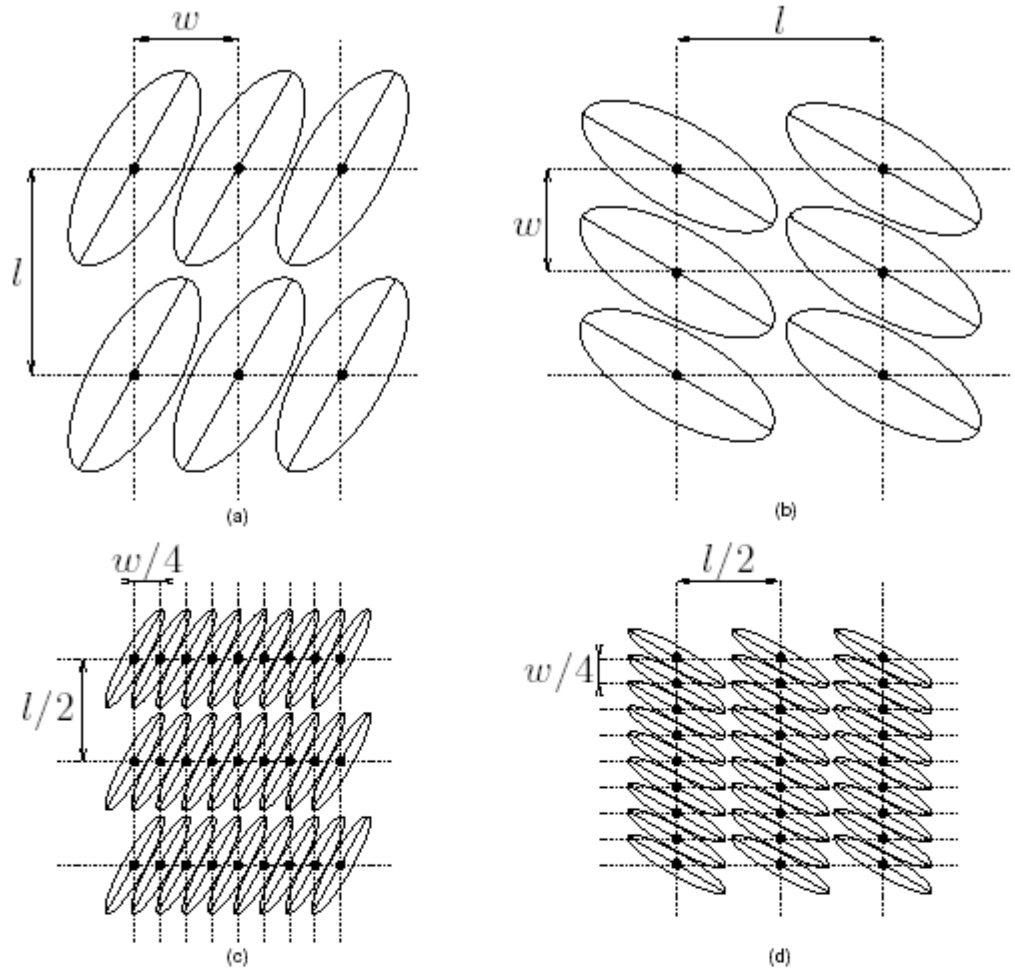


Fig. 10. Parabolic scaling relation for curves. (a) The rectangular supports of the basis functions that fit a curve exhibit the quadratic relation: $width \propto length^2$. (b) Illustrating the evolution of the support sizes of contourlet functions that satisfy the parabolic scaling.

M. Do and M. Vetterli, The Contourlet Transform: An Efficient Directional Multiresolution Image Representation, IEEE-TIP

Curvelets



Embedded grids of approximations in spatial domain.

Upper line represents the coarser scale and the lower line the finer scale.

Two directions (almost horizontal and almost vertical) are considered.

Each subspace is spanned by a shift of a curvelet prototype function.

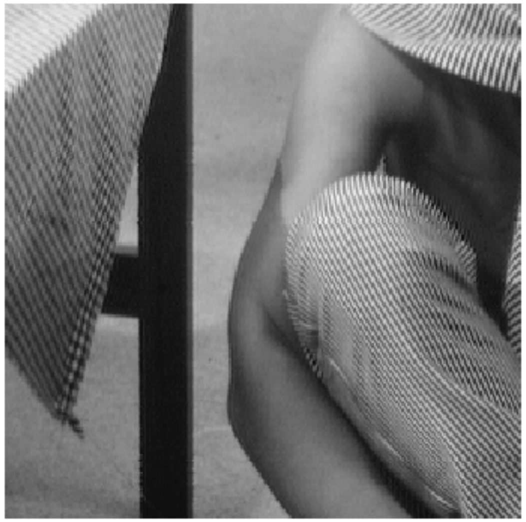
The sampling interval matches with the support of the prototype function, for example width w and length l, so that the shifts would tile the R^2 plan.

The functions are designed to obey the key anisotropy scaling relation:

$$width \propto length^2$$

Close resemblance with complex cells' (orientation selective RF)!

For a given rate, a better representation of edges is reached



(a) Original image

(b) DWT2: PSNR = 24.34 dB

(c) PDFB: PSNR = 25.70 dB

Summary of useful relations

- If f is real

$$\hat{f}(\omega) = \hat{f}(e^{j\omega})$$
$$\hat{f}(\omega+\pi) = \hat{f}(e^{j(\omega+\pi)})$$
$$\hat{f}(-\omega) = \hat{f}(e^{-j\omega}) = \hat{f}^*(\omega)$$

$$\hat{t}(\omega) = \hat{f}^*(\omega+\pi) = \hat{f}(e^{-j(\omega+\pi)}) \rightarrow t[n] = (-1)^{-n} f[-n]$$
$$\hat{g}(\omega) = e^{-j\omega}\hat{f}^*(\omega+\pi) = e^{-j\omega}\hat{f}(e^{-j(\omega+\pi)}) \rightarrow g[n] = (-1)^{1-n} f[1-n]$$

Conclusions

- Multiresolution representations are the fixed point of vision sciences and signal processing
- Different types of wavelet families are suitable to model different image features
  - Smooth functions -> isotropic wavelets
  - Contours and geometry -> Curvelets
- Adaptive basis
  - More flexible tool for image representation
  - Could be related to the RF of highly specialized neurons

References

- A Wavelet tour of Signal Processing, S. Mallat, Academic Press
- Papers
  - *A theory for multiresolution signal decomposition, the wavelet representation*, S. Mallat, IEEE Trans. on PAMI, 1989
  - *Dyadic Directional Wavelet Transforms: Design and Algorithms*, P. Vandergheynst and J.F. Gobbers, IEEE Trans. on IP, 2002
  - *Contourlets*, M. Do and M. Vetterli (Chapter)