$\boxed{\text{Addendum}}$: sulla teoria di Cartan

alla lezione VII

TOPOLOGIA E GEOMETRIA DIFFERENZIALE

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$$\begin{cases} d\underline{e}_1 = \omega_{12}\, \underline{e}_2 + \omega_{13}\underline{e}_3 \\ d\underline{e}_2 = \omega_{21}\, \underline{e}_1 + \omega_{23}\underline{e}_3 = -\omega_{12}e_1 + \omega_{23}e_3 \end{cases}$$

vogliamo verificare in dettaglio che

$\boxed{\omega_{12} \text{ dipende solo da } \langle\, ,\, \rangle \text{ : è legata ai simboli di Christoffel :}}$

Si ricordi (v. geometria)

$$\underline{r}_{uu} = \Gamma^1_{11}\, \underline{r}_u + \Gamma^2_{11}\, \underline{r}_v + e\,\underline{N} \qquad \text{etc.}$$
$$\| $$
$$(\underline{r}_u)_u$$

$$\langle \underline{r}_u , (\underline{r}_u)_u \rangle = \Gamma^1_{11} E + \Gamma^2_{11} F$$

Siano $u, v$ coord. loc. ortogonali: $F = 0$ $\quad (\langle \underline{r}_u, \underline{r}_v\rangle = 0)$

$$\underline{e}_1 = \frac{\underline{r}_u}{\sqrt{E}} \qquad \frac{\partial \underline{e}_1}{\partial u} = \frac{\partial}{\partial u}\left(\frac{\underline{r}_u}{\sqrt{E}}\right) \quad \perp e_1$$

(poiché è un versore)

$$\underline{e}_2 = \frac{\underline{r}_v}{\sqrt{G}}$$

$$(d\,\underline{e}_1)\left(\frac{\partial}{\partial u}\right)$$
$$\|$$
$$\frac{\partial e_1}{\partial u} = \underbrace{\omega_{12}\left(\frac{\partial}{\partial u}\right)}_{=\alpha} e_2 + \omega_{13}\left(\frac{\partial}{\partial u}\right)e_3$$

calcoliamo $\alpha := \omega_{12}\left(\frac{\partial}{\partial u}\right)$

$$\frac{\partial}{\partial u}\left(\frac{r_u}{\sqrt{E}}\right) = \alpha\, \underline{e}_2 + \omega_{13}\left(\frac{\partial}{\partial u}\right)e_3$$

$$\frac{\partial}{\partial u}\Bigg(\underbrace{\frac{\underline{r}_u}{\sqrt{E}}}_{e_1}\Bigg) = \frac{r_{uu}}{\sqrt{E}} - r_u \frac{\partial}{\partial u}\underbrace{\frac{1}{\sqrt{E}}}_{E^{-\frac{1}{2}}} = \left(\frac{r_{uu}}{\sqrt{E}} - \underline{r}_u\left(\frac{1}{2}\cdot E^{-\frac{3}{2}} E_u\right)\right)$$

$$\underline{e}_2$$

$$\left\langle \frac{\underline{r}_{uu}}{\sqrt{E}} - \frac{1}{2}\underline{r}_u E^{-\frac{3}{2}} E_u, \frac{\underline{r}_v}{\sqrt{G}} \right\rangle \qquad \langle r_u, r_v \rangle = F = 0$$

$$= \left\langle \frac{\underline{r}_{uu}}{\sqrt{EG}}, \underline{r}_v \right\rangle$$

$$= \frac{\Gamma^2_{11} \cdot G}{\sqrt{EG}} = \Gamma^2_{11} \sqrt{\frac{G}{E}}$$

$$\omega_{12}\left(\frac{\partial}{\partial u}\right) = \Gamma^2_{11} \sqrt{\frac{G}{E}}$$

Calcoliamo, per completezza, anche $\beta = \omega_{12}\left(\frac{\partial}{\partial v}\right)$

$$\frac{\partial}{\partial v}\underbrace{\left(\frac{\underline{r}_u}{\sqrt{E}}\right)}_{\underline{e}_1} = \beta\, \underline{e}_2 + \omega_{23}\left(\frac{\partial}{\partial v}\right)\underline{e}_3$$

$$/\!/$$

$$\frac{\underline{r}_{uv}}{\sqrt{E}} - \underline{r}_u \cdot \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{E}}\right) = \frac{\underline{r}_{uv}}{\sqrt{E}} - \frac{1}{2}E^{-\frac{3}{2}} E_v \underline{r}_u$$

il prodotto scalare con $\underset{\displaystyle \frac{\underline{r}_v}{\sqrt{G}}}{\overset{\underline{e}_2}{\shortparallel}}$ fornisce:

$$\left\langle \frac{\underline{r}_{uv}}{\sqrt{EG}}, \underline{r}_v \right\rangle = \frac{\Gamma^2_{12}}{\sqrt{EG}} G = \Gamma^2_{12}\sqrt{\frac{G}{E}}$$

$\|$ è poi noto che i $\Gamma$ dipendono solo dalla prima forma fondamentale e dalle sue derivate parziali prime