Lectures on

DIFFERENTIAL GEOMETRY AND TOPOLOGY

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Lecture XVIII

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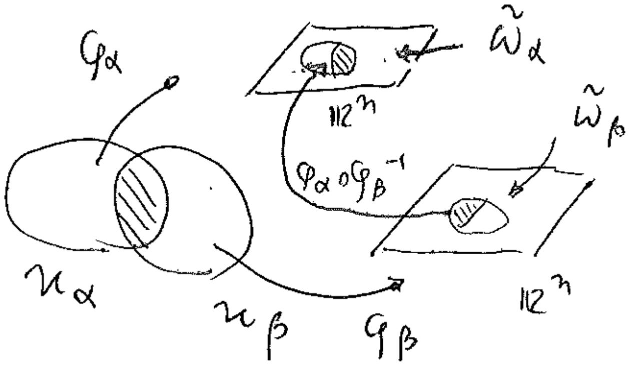
We wish to perform differential calculus on $\Lambda(M) = \bigoplus_{k=0}^{n} \Lambda^k(M)$ (Grassmann algebra of differential forms on a manifold M); specifically, we want to define operators d (exterior differential), $\imath_X$ (contraction with $X \in \mathfrak{X}(M)$) and $\mathcal{L}_X$ (Lie derivative), which will turn out to be related by Cartan's magic formula

$$\mathcal{L}_X = d\,\imath_X + \imath_X\, d$$

◇ ◇ ◇

Now, $\omega \in \Lambda^k(M)$ is given locally by forms $\tilde{\omega}_\alpha$ on $\mathbb{R}^n$ such that

$$\tilde{\omega}_\beta = (\varphi_\alpha \circ \varphi_\beta^{-1})^* \, \tilde{\omega}_\alpha \quad (\triangle!)$$



Exploiting our knowledge about forms on $\mathbb{R}^n$, we easily conclude that

we can define d <u>locally</u>, and its definition is <u>well posed</u>:

$$d\tilde{\omega}_\beta = d\left\{(\varphi_\alpha \circ \varphi_\beta^{-1})^* \tilde{\omega}_\alpha\right\} = (\varphi_\alpha \circ \varphi_\beta^{-1})^* \, d\,\tilde{\omega}_\alpha$$

(d commutes with pull-back)

and all properties thereof persist :

$$\boxed{d^2 = 0} \quad , \quad \boxed{\underset{\Lambda^k(M)}{d(\omega} \wedge \underset{\Lambda(M)}{\tau)} = d\omega \wedge \tau + (-1)^k \omega \wedge d\tau} .$$

Also observe that, using a partition of unity, a form defined on $U \subset M$ (open) can be extended to a form defined on M.

We wish to give an intrinsic formulation of $d\omega$, $\omega \in \Lambda^1(M)$

$$\boxed{d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])} .$$

Proof. We work locally, so it is enough to take $\omega = u\,dv$,

$u, v \in \mathcal{C}^\infty(M)$.

$\boxed{\text{l.h.s.}}$ $\quad d\omega = du \wedge dv$ and

$$d\omega(X,Y) = (du \wedge dv)(X,Y) = (du)(X)(dv)(Y) - (du)(Y)(dv)(X)$$
$$= X(u)Y(v) - X(v)Y(u) .$$

$\boxed{\text{r.h.s.}}$ $\quad X\omega(Y) = X[u\,dv(Y)] = X[u\,Y(v)]$
$$= X(u)\,Y(v) + u\,X(Y(v)) .$$

they add to $u \cdot [X,Y](v)$

$$-Y\omega(X) = -Y(u\,dv(X)) = -Y(u\,X(v))$$
$$= -Y(u)\,X(v) - u\,Y(X(v)) .$$

$$-\omega([X,Y]) = -u\,dv([X,Y]) = -u\,[X,Y](v)$$

Thus r.h.s. = ... l.h.s

In general, one shows that, for $\omega \in \Lambda^k(M)$

$$\boxed{d\omega(X_1 \dots X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} X_i\, \omega(X_1 \dots \hat{X}_i, \dots X_{k+1}) + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1 \dots \hat{X}_i \dots \hat{X}_j \dots X_{k+1})}$$

first argument

$\wedge$: omission

* Interior multiplication (or contraction)

$$i_X : \Lambda^k(M) \to \Lambda^{k-1}(M) \qquad \omega \mapsto i_X \omega$$

extended to $i : \Lambda \to \Lambda$

$X \in \mathfrak{X}(M)$

actually, i is a purely algebraic operation.

$$(i_X \omega)(X_1 \dots X_{k-1}) := \omega(X, X_1 \dots X_{k-1})$$

first slot

(other notation: $X \lrcorner \omega$

i_X is linear and, for $X, Y \in \mathfrak{X}(M)$, $\alpha, \beta \in \mathcal{C}^\infty(M)$

$$i_{\alpha X + \beta Y} \omega = \alpha i_X \omega + \beta i_Y \omega$$

($\diamond$) $\boxed{i_X^2 = 0}$ and i is an antiderivation

($\diamond\diamond$) $\boxed{i_X(\omega \wedge \eta) = i_X \omega \wedge \eta + (-1)^k \omega \wedge i_X \eta}$, $\omega \in \Lambda^k$

i behaves like d

Let us check that $i_X^2 = 0$ $\qquad \omega \in \Lambda^k$

$$i_X(i_X\omega)(X_1\dots X_{k-2}) =$$

$$= i_X\omega(X, X_1\dots X_{k-2}) = \omega(X, X, X_1\dots X_{k-2}) = 0$$

As for (◇◇), it is enough to check the formula

$$\underbrace{i_X(\omega^1\wedge\dots\wedge\omega^k)}_{\text{1-forms}} = i_X\omega^1\wedge\omega^2\wedge\dots\wedge\omega^k$$
$$- i_X\omega^2\wedge\omega^1\wedge\dots\wedge\omega^k$$
$$+ i_X\omega^3\wedge\omega^1\wedge\dots\wedge\omega^k$$

and this is true by Laplace's formula:

if $X_1 = X$,

$$(\omega^1\wedge\omega^2\dots\wedge\omega^k)(\overset{X}{\overset{\shortparallel}{X_1}}, X_2\dots X_k) = \det\big(\overset{\mathbb{X}}{(\omega^i(X_j))}\big)$$

$$= \sum_{i=1}^{k} (-1)^{i-1}\,\omega^i(\underset{\underset{X}{\shortparallel}}{X_1})\,\underbrace{(\omega^1\wedge\dots\hat{\omega}^i\wedge\dots\wedge\omega^k)(X_2,\dots X_k)}_{\det \mathbb{X}_1^i}$$

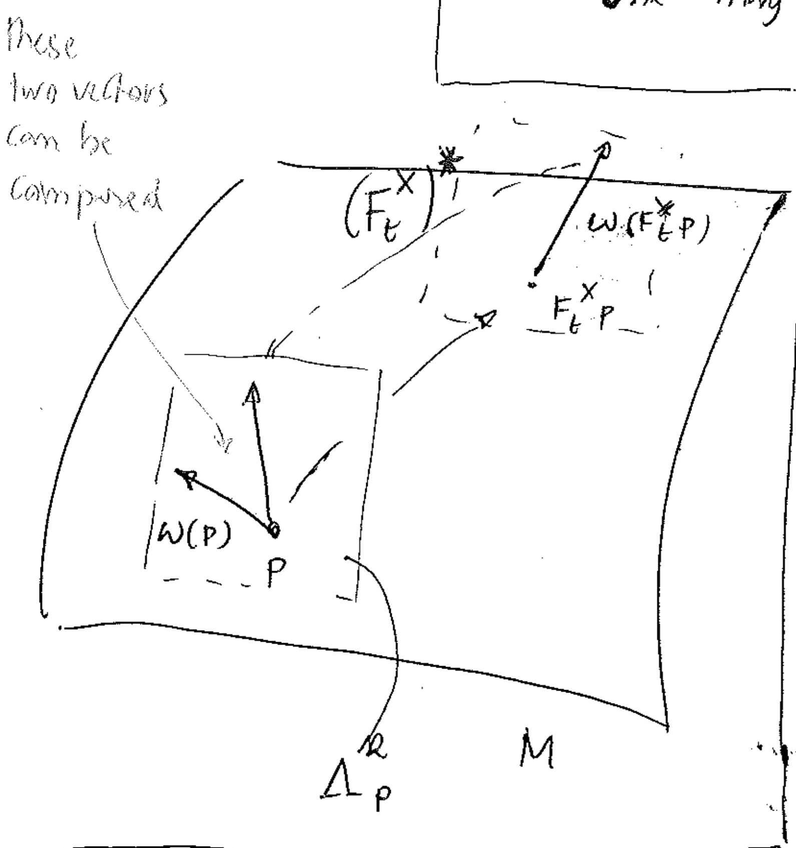
* Lie derivative of a k-form

Let $X \in \mathcal{X}(M)$, $\omega \in \Lambda^k(M)$

The $\boxed{\text{Lie derivative of } \omega \text{ along } X}$ $\mathcal{L}_X \omega \in \Lambda^k(M)$ is defined as:

$F^X_\cdot$: flow of X

$$(\mathcal{L}_X \omega)(p) = \frac{d}{dt}\left[(F_t^X)^* \omega\right]\Big|_{t=0} = \lim_{t\to 0} \frac{(F_t^X)^* \omega(F_t^X \cdot p) - \omega(p)}{t}$$



One may prove that $\mathcal{L}_X(\omega \wedge \tau) = \mathcal{L}_X \omega \wedge \tau + \omega \wedge \mathcal{L}_X \tau$ and, more generally

$$\boxed{\mathcal{L}_X(T \otimes S) = \mathcal{L}_X T \otimes S + T \otimes \mathcal{L}_X S} \quad (\star)$$

for any tensor fields (generalized Leibniz rule)

Also, for a $(0,k)$-tensor one has:

$$(\mathcal{L}_X \sigma)(Y_1 \ldots Y_k) = X(\sigma(Y_1 \ldots Y_k)) - \sigma([X, Y_1], Y_2 \ldots Y_k) - \cdots - \sigma(Y_1 \ldots Y_{k-1}, [X, Y_k]),$$

This, in turn, coming from

$$\underbrace{X(\sigma(Y_1 \ldots Y_k))}_{\mathcal{L}_X(\sigma(Y_1 \ldots Y_k))} = (\mathcal{L}_X \sigma)(Y_1 \ldots Y_k) + \sigma(\overset{[X, Y_1]}{\mathcal{L}_X Y_1}, Y_2 \ldots Y_k) + \cdots + \sigma(Y_1 \ldots \mathcal{L}_X Y_k)$$

Leibniz rule: all arguments are varied one at a time

Let us check (☆) in the case of a tensor product of covariant tensors, just to pin point the (quite simple) basic idea.

$$\mathcal{L}_X(\sigma\otimes\tau)(p) = \lim_{t\to 0} \frac{(F_t^X)^*\left[(\sigma\otimes\tau)(F_t^X(p))\right] - (\sigma\otimes\tau)(p)}{t}$$

$$= \lim_{t\to 0} \frac{\overbrace{(F_t^X)^*\sigma(F_t^X(p)) \otimes \overbrace{(F_t^X)^*\tau(F_t^X(p))}^{C}}^{A} - \overbrace{\sigma(p)\otimes\tau(p)}^{B}}{t}$$

$$= \lim_{t\to 0}\left[\frac{A - \sigma(p)\otimes\overbrace{(F_t^X)^*\tau(F_t^X(p))}^{C}}{t} + \frac{C - B}{t}\right]$$

[This is the same idea one uses to prove $(fg)' = f'g + fg'$ in calculus]

$$= \lim_{t\to 0} \underbrace{\frac{(F_t^X)^*\sigma(F_t(p)) - \sigma(p)}{t}}_{\to\, \mathcal{L}_X\sigma} \otimes \underbrace{(F_t^X)^*\tau(F_t^X(p))}_{\xrightarrow{t\to 0}\ \tau(p)}$$

$$+ \quad \sigma(p)\otimes \overbrace{\lim_{t\to 0}\frac{(F_t^X)^*\tau(F_t^X(p)) - \tau(p)}{t}}^{\mathcal{L}_X\tau}$$

$$= \mathcal{L}_X\sigma\otimes\tau + \sigma\otimes\mathcal{L}_X\tau$$

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also recall that in general if $S = \sigma^I_J$, $T = \tau^K_L$

$S\otimes T = \sigma^I_J \cdot \tau^K_L$ ← no summation, just take all products

($\cdot$ ↑ numerical product)

Theorem Let $\omega \in \Lambda(M)$, $X \in \mathfrak{X}(M)$.

One has
$$\mathcal{L}_X \omega = d\, i_X \omega + i_X d\omega$$

(Cartan's magic formula)

⚠ It holds for forms, not for general tensors (d is not defined...)

Pf. We prove it for $\omega = u\, dv$. The general case follows by induction.

First of all, notice that
$$(\mathcal{L}_X dv)(Y) \underset{\substack{\text{general}\\ \text{formula}}}{=} X[dv(Y)] - dv([X,Y])$$
$$= X[Y(v)] - [X,Y](v) = (XY - XY + YX)(v) = Y \underbrace{X(v)}_{\mathcal{L}_X v}$$
$$= d(\mathcal{L}_X v)(Y)$$

namely $\boxed{\mathcal{L}_X dv = d\mathcal{L}_X v}$

As a corollary, if $(x^1 \ldots x^n)$ are local coordinates and $X = \frac{\partial}{\partial x^1}$, just to fix ideas, then $\mathcal{L}_X (dx^{i_1} \wedge \ldots dx^{i_k}) = 0$,

this following from $\mathcal{L}_X dx^j = d\, \mathcal{L}_X x^j = d\left(\frac{\partial x^j}{\partial x^1}\right)$
and from the general Leibniz rule. $= d(\delta_{j1}) = 0$

Now compute:

$$\mathcal{L}_X(u\,dv)(Y) = (\mathcal{L}_X u \cdot dv + u\,\mathcal{L}_X dv)(Y)$$
$$= (\mathcal{L}_X u\, dv + u\, d\mathcal{L}_X v)(Y)$$
$$= (X(u)\,dv + u\,d(X(v)))(Y)$$
$$= X(u)Y(v) + uY(X(v)) \qquad \mathrm{I}$$

Compute II =

$$= (d\,i_X + i_X d)(u\,dv)(Y) =$$
$$= d(i_X(u\,dv))(Y) + i_X\,d(u\,dv)(Y)$$
$$= d(\underbrace{i_X u}_{=0}\,dv + u\,\underbrace{i_X dv}_{X(v)})(Y) + \underbrace{i_X(du\wedge dv)(Y)}_{(du\wedge dv)(X,Y)}$$
$$= d(u\,X(v))(Y) + X(u)Y(v) - X(v)Y(u)$$
$$= (du\,X(v) + u\,d(X(v)))(Y) + X(u)Y(v) - X(v)Y(u)$$
$$= Y(u)X(v) + u\,Y(X(v)) + X(u)Y(v) - X(v)Y(u)$$
$$= X(u)Y(v) + u\,Y(X(v)) = \mathrm{I}$$

□

Notice that, in general, for forms, one has:

$$\boxed{\mathcal{L}_X d = d\,\mathcal{L}_X}$$

Indeed: $\mathcal{L}_X d = (d\,i_X + i_X d)\,d = d\,i_X d + \underbrace{i_X d^2}_{=0} = d\,i_X d$

whereas
$$d\,\mathcal{L}_X = d(d\,i_X + i_X d) = \underbrace{d^2 i_X}_{=0} + d\,i_X d = d\,i_X d$$ □