Continuous Time Signal Analysis: the Fourier Transform

Lathi Chapter 4

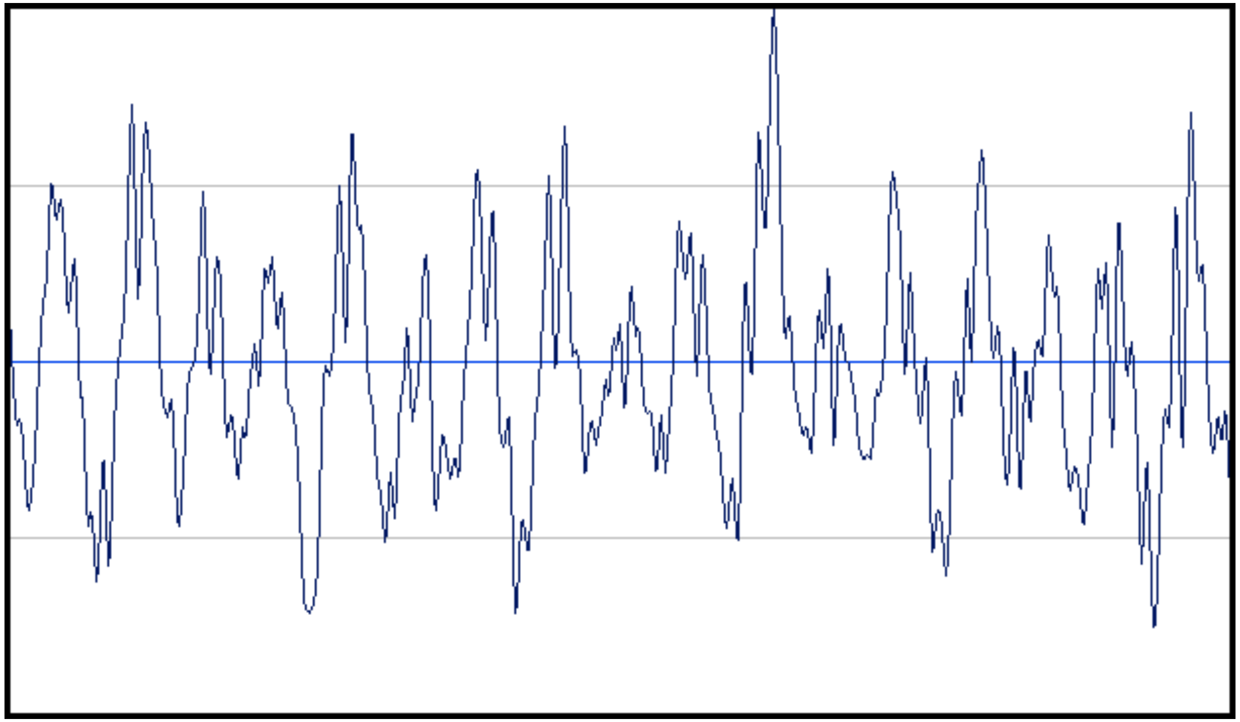
Topics

- Aperiodic signal representation by the Fourier integral (CTFT)
  - Continuous-time Fourier transform
- Transforms of some useful functions
- Properties of the Fourier Transform

- Filtering
- Modulation

Aperiodic signal representation by the Fourier Integral

- What happens if the signal is NOT periodic? How can we represent it over the whole temporal axis?



Towards the CTFT

- Starting from Fourier series, we will derive the CTFT by a "trick"
  - First, we will build a periodic signal starting from time-limited f(t)
  - Then, we will represent a NON periodic signal as a limit case of a periodic signal when the period goes to infinity
  - Finally, we will derive the CTFT and analyze the link with FS

Towards CTFT: steps 1 and 2

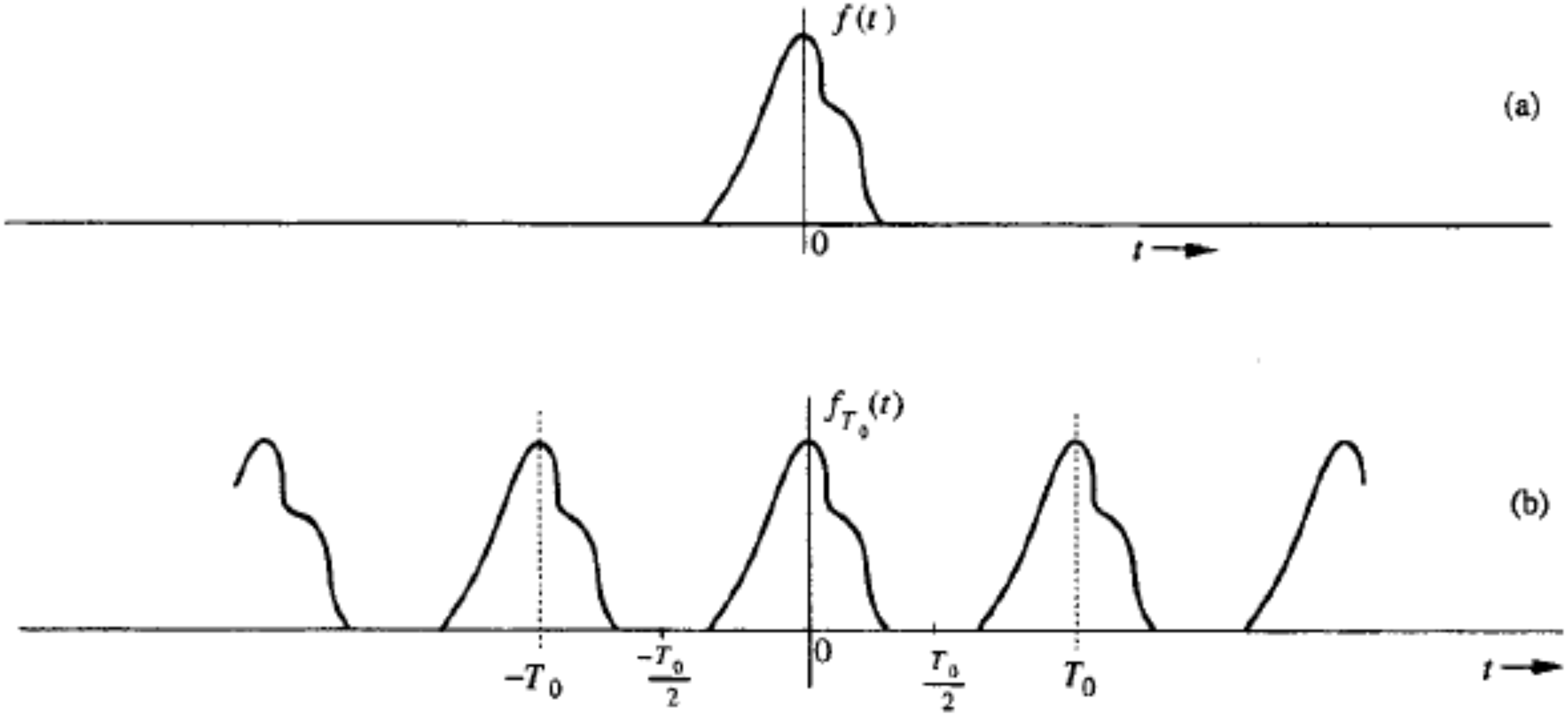


Fig. 4.1 Construction of a periodic signal by periodic extension of $f(t)$.

$$\lim_{T_0 \to \infty} f_{T_0}(t) = f(t)$$

Towards CTFT

- Since $f_{T0}(t)$ is periodic, we can represent it by the exponential FS

$$f_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \tag{4.1}$$

where

$$D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f_{T_0}(t) e^{-jn\omega_0 t}\, dt \tag{4.2a}$$

and

$$\omega_0 = \frac{2\pi}{T_0} \tag{4.2b}$$

- But:

$$\int_{-T_0/2}^{T_0/2} f_{T_0}(t)dt = \int_{-\infty}^{+\infty} f(t)dt$$

Towards CTFT

- Thus

$$D_n = \frac{1}{T_0} \int_{-\infty}^{+\infty} f(t) e^{-jn\omega_0 t} dt$$

- Let's then define the continuous function F(ω)

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt$$

$$\Downarrow$$

$$D_n = \frac{1}{T_0} F(n\omega_0)$$

The Fourier series coefficients D_n are $(1/T_0)$ times the coefficients of F(ω) uniformly distributed over the frequency axis with spacing ω_0

CFTF vs FS

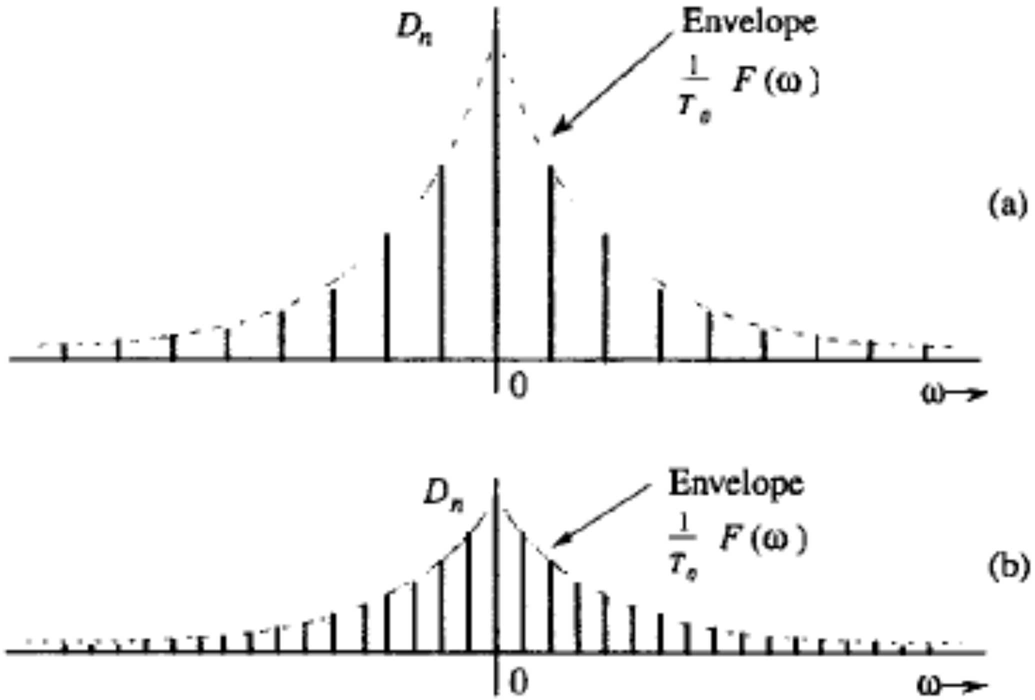


Fig. 4.2 Change in the Fourier spectrum when the period T_0 in Fig. 4.1 is doubled.

CTFT is the envelop of the coefficients of the FS or, alternatively, the FS can be seen as a sampled version of the CTFT

Approaching the limit

- When T0 goes to infinity
  - we recover the non-period signal f(t)
  - the DS samples get progressively closer to each other
  - The relative shape of the spectrum with respect to F(ω) stays unchanged
- However
  - Each single spectral component has an amplitude that decreases to zero

$$T_0 \to \infty$$
$$\omega_0 \to 0$$
$$D_n \to 0$$

We have nothing of everything, yet we have something

Solving the paradox

- Even though each *individual* component has vanishing amplitude, we can *group* components across intervals of infinitesimal width and make them *working in team*

$$f_{T_0}(t) = \sum_{n=-\infty}^{+\infty} \frac{F(n\omega_0)}{T_0} e^{jn\omega_0 t}$$

$$\omega_0 \to 0 \text{ different notation } \to \Delta\omega$$

$$\Delta\omega = \frac{2\pi}{T_0} \to T_0 = \frac{2\pi}{\Delta\omega}$$

$$f_{T_0}(t) = \sum_{n=-\infty}^{+\infty} \frac{F(n\Delta\omega)\Delta\omega}{2\pi} e^{jn\Delta\omega t}$$

Weight of the component of frequency

Getting to the CTFT

- When T_0 goes to infinity, the sum can be replaced by an integral leading to the CTFT

$$f(t) = \lim_{T_0 \to \infty} f_{T_0}(t) = \lim_{\Delta\omega \to 0} \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} F(n\Delta\omega) e^{jn\Delta\omega t} \Delta\omega$$

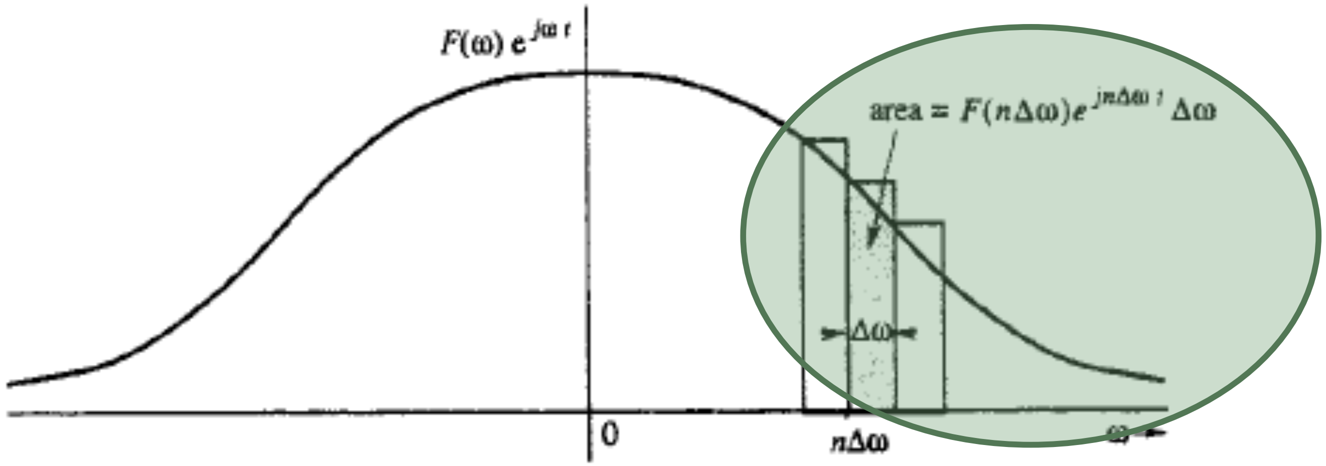


Fig. 4.3 The Fourier series becomes the Fourier integral in the limit as $T_0 \to \infty$.

CTFT qui

- Fourier integral

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{j\omega t} d\omega$$

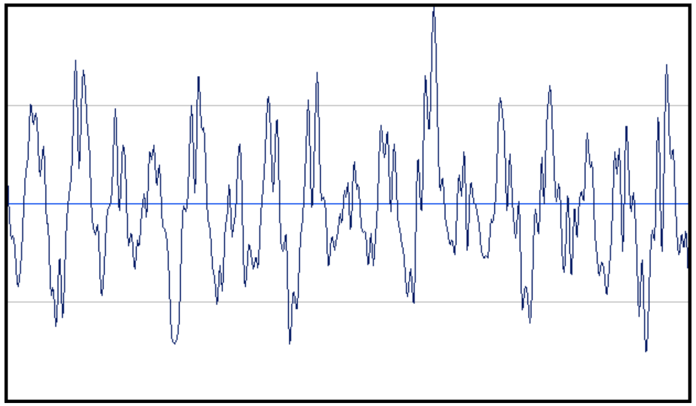
Analysis formula

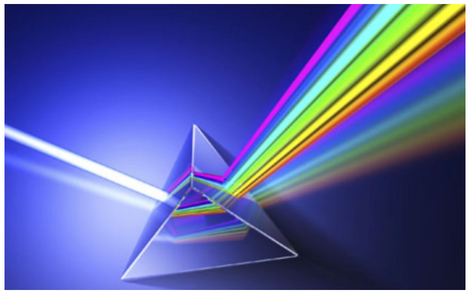
$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt$$

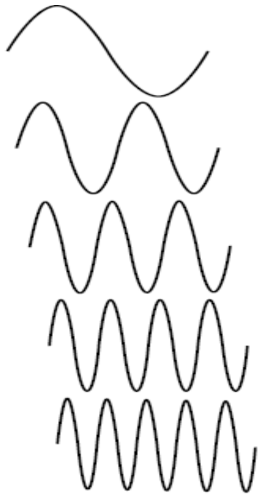
Synthesis formula

Fourier domain

Signal domain







Amplitude and Phase spectra

$$F(\omega) = |F(\omega)|e^{j\angle F(\omega)}$$

For real signals

$$F(-\omega) = \int_{-\infty}^{\infty} f(t)e^{j\omega t}\,dt \longrightarrow \begin{aligned} |F(-\omega)| &= |F(\omega)| \\ \angle F(-\omega) &= -\angle F(\omega) \end{aligned}$$

Fig. 4.4 $e^{-at}u(t)$ and its Fourier spectra.

Existence of the FT

- Dirichlet conditions (sufficient)

$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty \tag{4.13}$$

Because $|e^{-j\omega t}| = 1$, from Eq. (4.8a), we obtain

$$|F(\omega)| \leq \int_{-\infty}^{\infty} |f(t)|\, dt$$

- Not necessary
  - There exist functions not satisfying condition (4.13) still having a Fourier transform, like the sinc(t)
- The CTFT always exists for real (measured) signals

Linearity of the CTFT

The Fourier transform is linear; that is, if

$$f_1(t) \Longleftrightarrow F_1(\omega) \qquad \text{and} \qquad f_2(t) \Longleftrightarrow F_2(\omega)$$

then

$$a_1 f_1(t) + a_2 f_2(t) \Longleftrightarrow a_1 F_1(\omega) + a_2 F_2(\omega) \tag{4.14}$$

The proof is trivial and follows directly from Eq. (4.8a). This result can be extended to any finite number of terms.

LTIC systems response using the CTFT

- Back to the response to the everlasting exponential

$$e^{j\omega t} \Longrightarrow H(\omega)e^{j\omega t}$$

Therefore

$$e^{j(n\Delta\omega)t} \Longrightarrow H(n\Delta\omega)e^{j(n\Delta\omega)t}$$

and

$$\left[\frac{F(n\Delta\omega)\Delta\omega}{2\pi}\right]e^{(jn\Delta\omega)t} \Longrightarrow \left[\frac{F(n\Delta\omega)H(n\Delta\omega)\Delta\omega}{2\pi}\right]e^{j(n\Delta\omega)t}$$

Using the linearity property

$$\underbrace{\lim_{\Delta\omega\to 0}\sum_{n=-\infty}^{\infty}\left[\frac{F(n\Delta\omega)\Delta\omega}{2\pi}\right]e^{(jn\Delta\omega)t}}_{\text{input } f(t)} \Longrightarrow \underbrace{\lim_{\Delta\omega\to 0}\sum_{n=-\infty}^{\infty}\left[\frac{F(n\Delta\omega)H(n\Delta\omega)\Delta\omega}{2\pi}\right]e^{j(n\Delta\omega)t}}_{\text{output } y(t)}$$

LTIC systems response using the CTFT

The left-hand side is the input $f(t)$ [see Eqs. (4.6a) and (4.6b)], and the right-hand side is the response $y(t)$. Thus

$$
\begin{aligned}
y(t) &= \frac{1}{2\pi} \lim_{\Delta\omega \to 0} \sum_{n=-\infty}^{\infty} F(n\Delta\omega) H(n\Delta\omega) e^{j(n\Delta\omega)t} \Delta\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) H(\omega) e^{j\omega t} \, d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega) e^{j\omega t} \, d\omega \qquad (4.18)
\end{aligned}
$$

where $Y(\omega)$, the Fourier transform of $y(t)$, is given by†

$$
Y(\omega) = F(\omega) H(\omega)
$$

Time vs Frequency domain

1 For the time-domain case

$\delta(t) \implies h(t)$ the impulse response of the system is $h(t)$

$f(t) = \int_{-\infty}^{\infty} f(x)\delta(t-x)\,dx$ expresses $f(t)$ as a sum of impulse components

and

$y(t) = \int_{-\infty}^{\infty} f(x)h(t-x)\,dx$ expresses $y(t)$ as a sum of responses to impulse components

2 For the frequency-domain case

$e^{j\omega t} \implies H(\omega)e^{j\omega t}$ the system response to $e^{j\omega t}$ is $H(\omega)e^{j\omega t}$

$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega$ shows $f(t)$ as a sum of everlasting exponential components

and

$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)H(\omega)e^{j\omega t}\,d\omega$ $y(t)$ is a sum of responses to exponential components

CTFT of some useful functions

- Rectangular function

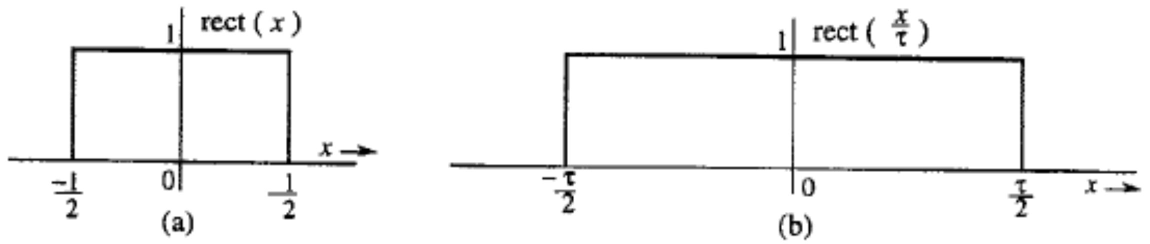
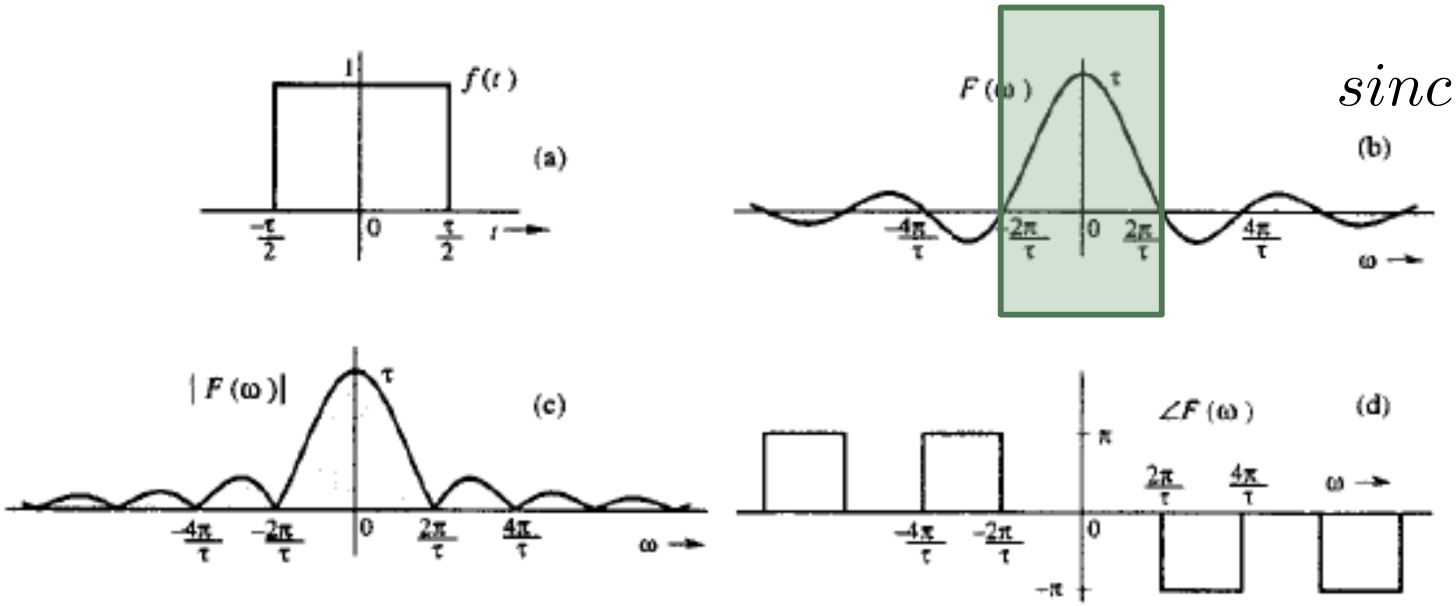


Fig. 4.7 A gate pulse.

Bandwidth



$$sinc(\frac{\omega\tau}{2})$$

CTFT of some useful functions

Example 4.2

Find the Fourier transform of $f(t) = \text{rect}\left(\frac{t}{\tau}\right)$ (Fig. 4.10a).

$$F(\omega) = \int_{-\infty}^{\infty} \text{rect}\left(\frac{t}{\tau}\right) e^{-j\omega t} dt$$

Since $\text{rect}\left(\frac{t}{\tau}\right) = 1$ for $|t| < \frac{\tau}{2}$, and since it is zero for $|t| > \frac{\tau}{2}$,

$$F(\omega) = \int_{-\tau/2}^{\tau/2} e^{-j\omega t} dt$$

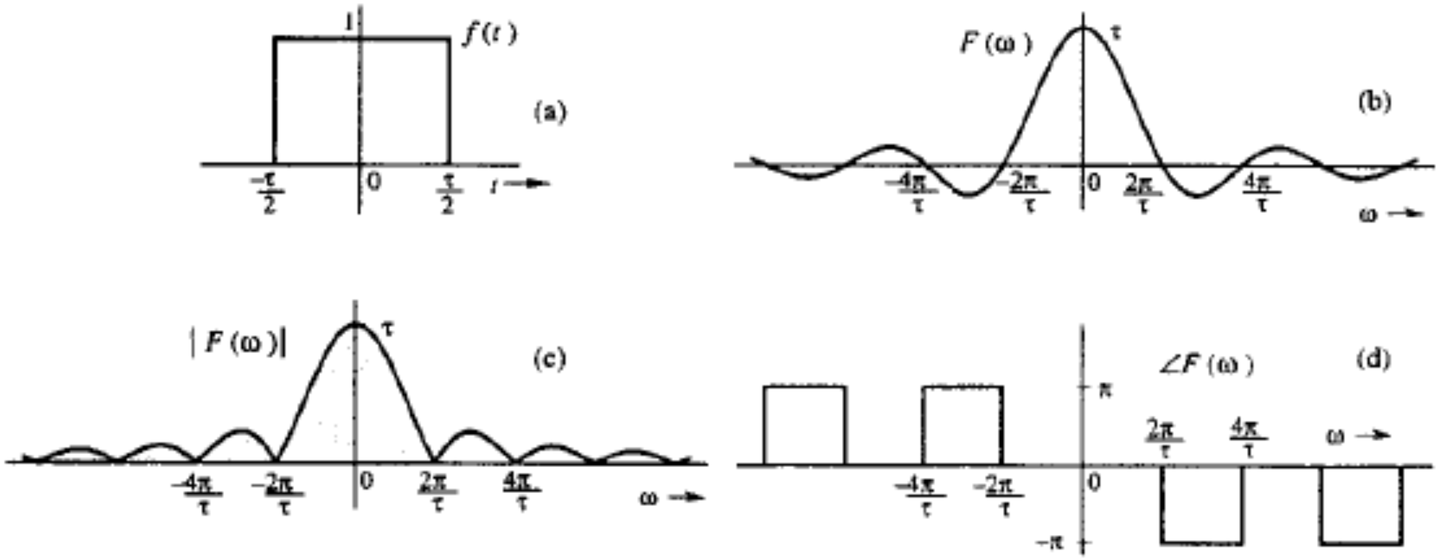
$$= -\frac{1}{j\omega}\left(e^{-j\omega\tau/2} - e^{j\omega\tau/2}\right) = \frac{2\sin\left(\frac{\omega\tau}{2}\right)}{\omega}$$

$$= \tau \frac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)} = \tau \text{ sinc}\left(\frac{\omega\tau}{2}\right)$$

Therefore

$$\text{rect}\left(\frac{t}{\tau}\right) \Longleftrightarrow \tau \text{ sinc}\left(\frac{\omega\tau}{2}\right) \tag{4.23}$$

Example: rect(t/τ)



$$\sin(\frac{\omega\tau}{2}) = 0 \qquad \frac{\omega\tau}{2} = k\pi \rightarrow \omega = \frac{2k\pi}{\tau}$$

$$\sin(\frac{\omega\tau}{2}) = 1 \qquad \frac{\omega\tau}{2} = k\frac{\pi}{2} \rightarrow \omega = \frac{k\pi}{\tau}$$

$$\sin(\frac{\omega\tau}{2}) = -1 \qquad \frac{\omega\tau}{2} = k\frac{3\pi}{2} \rightarrow \omega = \frac{k3\pi}{\tau}$$

CTFT of some useful functions

- Impulse

■ **Example 4.3**

Find the Fourier transform of the unit impulse $\delta(t)$.

Using the sampling property of the impulse [Eq. (1.24)], we obtain

$$\mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1 \tag{4.24a}$$

or

$$\delta(t) \Longleftrightarrow 1 \tag{4.24b}$$

Figure 4.11 shows $\delta(t)$ and its spectrum. ■

Fig. 4.11 Unit impulse and its Fourier spectrum.

CTFT of some useful functions

- Constant function
  - Derived as the inverse CTFT of the delta in Fourier domain

Find the inverse Fourier transform of $\delta(\omega)$.
On the basis of Eq. (4.8b) and the sampling property of the impulse function,

$$\mathcal{F}^{-1}[\delta(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t}\, d\omega = \frac{1}{2\pi}$$

Therefore

$$\frac{1}{2\pi} \Longleftrightarrow \delta(\omega) \tag{4.25a}$$

or

$$1 \Longleftrightarrow 2\pi\delta(\omega) \tag{4.25b}$$

CTFT of some useful functions

■ **Example 4.5**

Find the inverse Fourier transform of $\delta(\omega - \omega_0)$.

Using the sampling property of the impulse function, we obtain

$$\mathcal{F}^{-1}[\delta(\omega - \omega_0)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t}\, d\omega = \frac{1}{2\pi} e^{j\omega_0 t}$$

Therefore

$$\frac{1}{2\pi} e^{j\omega_0 t} \Longleftrightarrow \delta(\omega - \omega_0)$$

or

$$e^{j\omega_0 t} \Longleftrightarrow 2\pi\delta(\omega - \omega_0) \qquad (4.26a)$$

This result shows that the spectrum of an everlasting exponential $e^{j\omega_0 t}$ is a single impulse at $\omega = \omega_0$. We reach the same conclusion by qualitative reasoning. To represent the everlasting exponential $e^{j\omega_0 t}$, we need a single everlasting exponential $e^{j\omega t}$ with $\omega = \omega_0$. Therefore, the spectrum consists of a single component at frequency $\omega = \omega_0$.

From Eq. (4.26a) it follows that

$$e^{-j\omega_0 t} \Longleftrightarrow 2\pi\delta(\omega + \omega_0) \qquad (4.26b)$$

CTFT of some useful functions

- Everlasting sinusoid

$$\cos \omega_0 t = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$$ Eulero's formula

$$2\pi\delta(\omega - \omega_0) \qquad 2\pi\delta(\omega + \omega_0)$$

$$cos(\omega_0 t) \rightarrow \frac{1}{2} 2\pi(\delta(\omega + \omega_0) + \delta(\omega - \omega_0)) = \pi \left\{\delta(\omega + \omega_0) + \delta(\omega - \omega_0)\right\}$$

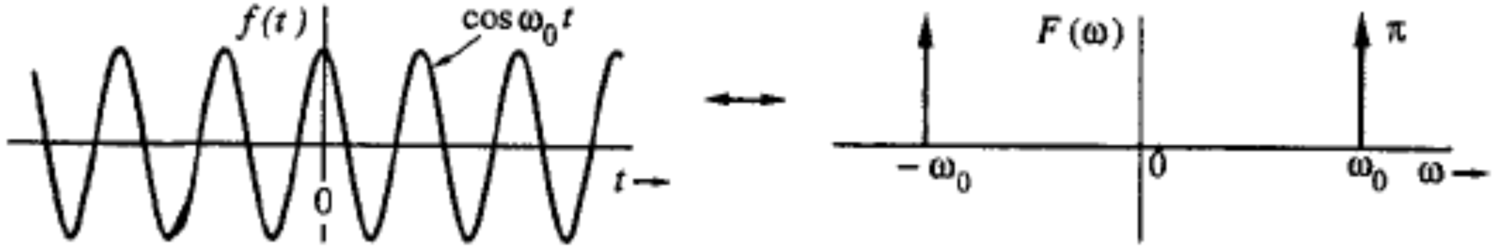


Fig. 4.13 A cosine signal and its Fourier spectrum.

Symmetry of the CTFT

$$\int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt$$

$f(t)$ **Time-frequency duality** $F(\omega)$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

Symmetry properties

$$f(t - t_0) \Longleftrightarrow F(\omega)e^{-j\omega t_0}$$

$$f(t)e^{j\omega_0 t} \Longleftrightarrow F(\omega - \omega_0)$$

Time-shift -> phase change in frequency domain

Frequency shift (modulation) -> phase change in time domain

Delaying a signal by t_0 seconds does not change the amplitude spectrum but changes the phase spectrum by $-\omega t_0$

Physical explanation of the time—shifting property

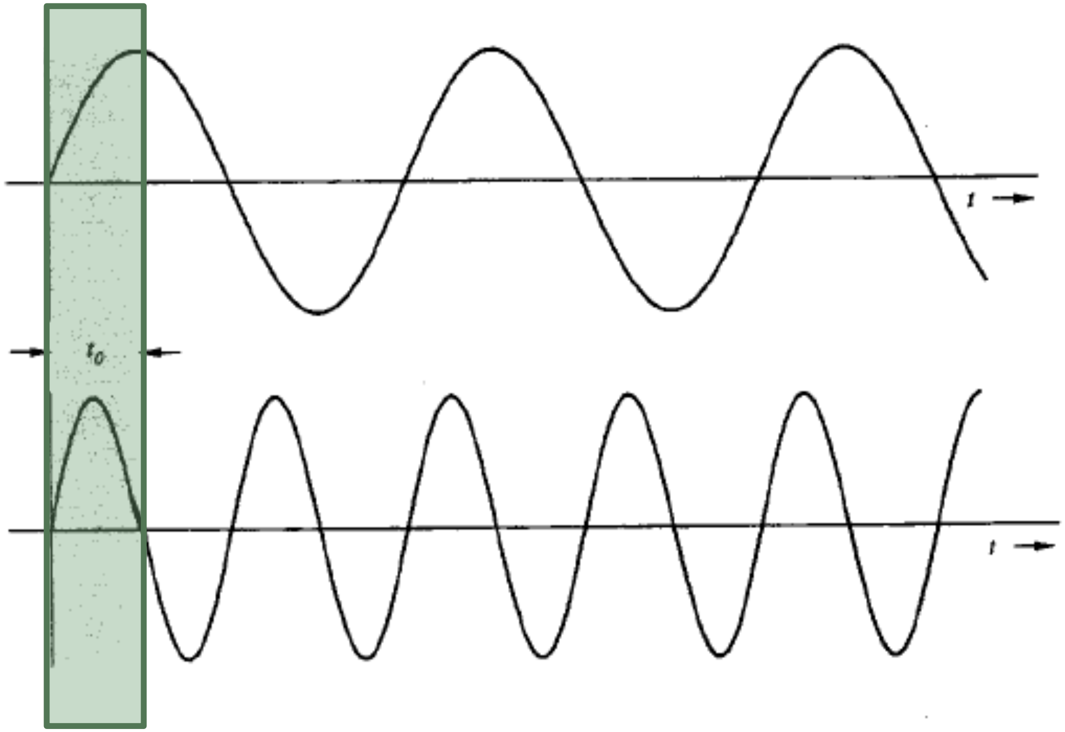


Fig. 4.20 Physical explanation of the time-shifting property.

To keep the signal spectrum unchanged after time-shifting, higher frequencies must go through larger phase shift. More precisely, the phase shift must be linear

Symmetry properties

4.3-2 Symmetry Property

This property states that if

$$f(t) \Longleftrightarrow F(\omega)$$

then

$$F(t) \Longleftrightarrow 2\pi f(-\omega) \qquad (4.$$

Proof: According to Eq. (4.8b)

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(x) e^{jxt}\, dx$$

Hence

$$2\pi f(-t) = \int_{-\infty}^{\infty} F(x) e^{-jxt}\, dx$$

Changing t to ω yields Eq. (4.31).

Symmetry properties

■ **Example 4.8**

In this example we apply the symmetry property [Eq. (4.31)] to the pair in Fig. 4.17a. From Eq. (4.23) we have

$$\underbrace{\mathrm{rect}\left(\frac{t}{\tau}\right)}_{f(t)} \Longleftrightarrow \underbrace{\tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2}\right)}_{F(\omega)} \tag{4.32}$$

Also, $F(t)$ is the same as $F(\omega)$ with ω replaced by t, and $f(-\omega)$ is the same as $f(t)$ with t replaced by $-\omega$. Therefore, the symmetry property (4.31) yields

$$\underbrace{\tau\,\mathrm{sinc}\left(\frac{\tau t}{2}\right)}_{F(t)} \Longleftrightarrow \underbrace{2\pi\,\mathrm{rect}\left(\frac{-\omega}{\tau}\right)}_{2\pi f(-\omega)} = 2\pi\,\mathrm{rect}\left(\frac{\omega}{\tau}\right) \tag{4.33}$$

$$\mathrm{rect}\,(-x) = \mathrm{rect}\,(x)$$

Symmetry properties

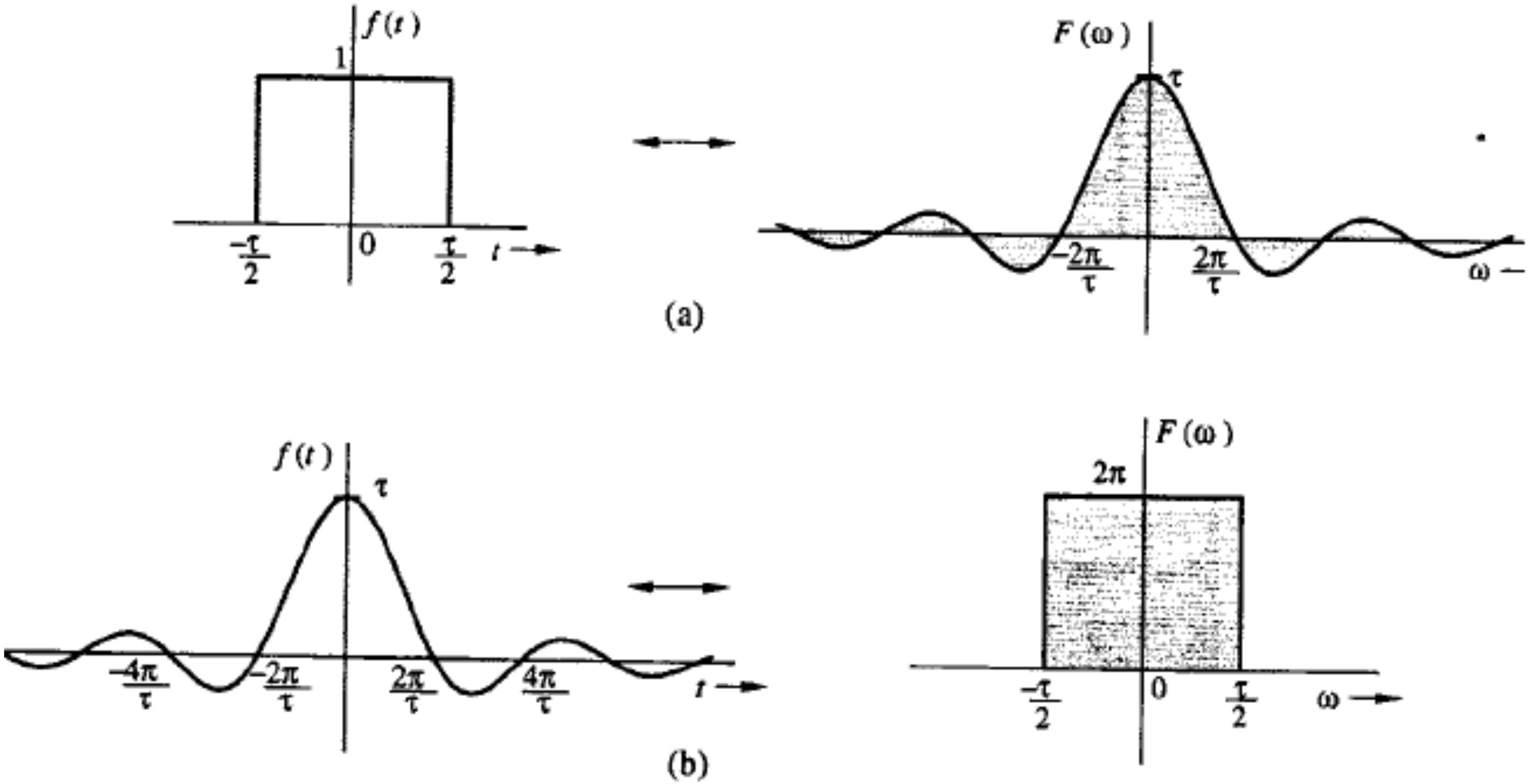


Fig. 4.17 Symmetry property of the Fourier transform.

Scaling

$$\mathcal{F}[f(at)] = \int_{-\infty}^{\infty} f(at)e^{-j\omega t}\,dt = \frac{1}{a}\int_{-\infty}^{\infty} f(x)e^{(-j\omega/a)x}\,dx = \frac{1}{a}F\left(\frac{\omega}{a}\right)$$

$$f(at) \Longleftrightarrow \frac{1}{|a|}F\left(\frac{\omega}{a}\right)$$

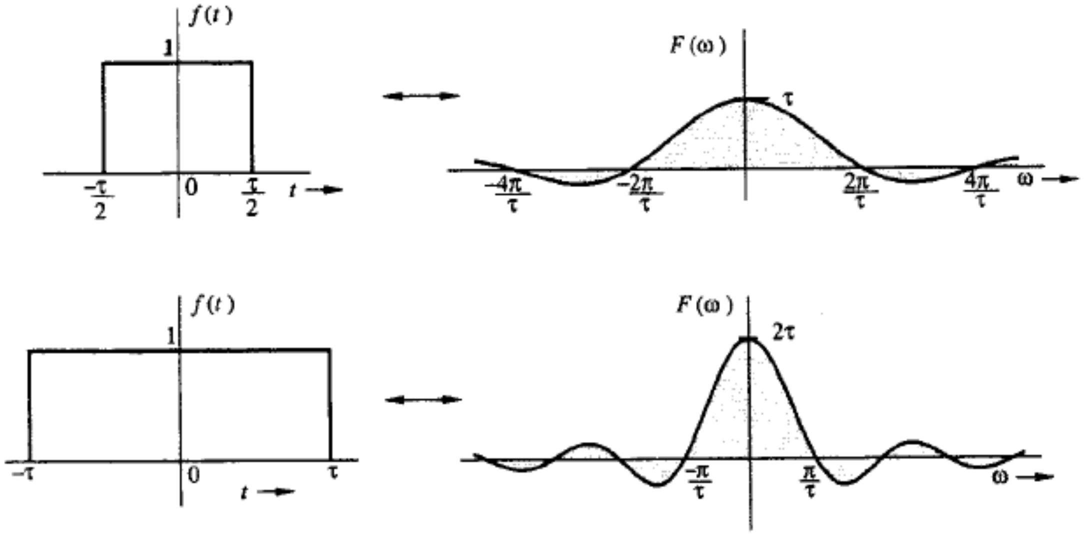


Fig. 4.18 Scaling property of the Fourier transform.

The time shrinking of a signal results in a spectral expansion and, viceversa, a spectral shirking results in a temporal expansion

Reciprocity of signal duration and bandwidth

Convolution

$$f_1(t) * f_2(t) \Longleftrightarrow F_1(\omega)F_2(\omega)$$

$$f_1(t)f_2(t) \Longleftrightarrow \frac{1}{2\pi}F_1(\omega) * F_2(\omega)$$

2π

Function to be transformed

Proof: By definition

$$\mathcal{F}|f_1(t) * f_2(t)| = \int_{-\infty}^{\infty} e^{-j\omega t}\left[\int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)d\tau\right]dt$$

$$= \int_{-\infty}^{\infty} f_1(\tau)\left[\int_{-\infty}^{\infty} e^{-j\omega t}f_2(t-\tau)dt\right]d\tau$$

Hence

$$F_2(\omega)e^{-j\omega\tau}.$$

$$\mathcal{F}[f_1(t)*f_2(t)] = \int_{-\infty}^{\infty} f_1(\tau)e^{-j\omega\tau}F_2(\omega)d\tau = F_2(\omega)\int_{-\infty}^{\infty} f_1(\tau)e^{-j\omega\tau}d\tau = F_1(\omega)F_2(\omega)$$

LTIC response

Impulse response

$$f(t) = \delta(t)$$

$$f(t) \rightarrow \boxed{h(t)} \rightarrow$$

$$y(t) = h(t)$$

Time domain

$$y(t) = f(t) \star h(t)$$

Transfer function

$$F(\omega) = e^{j\omega t}$$

$$F(\omega) \rightarrow \boxed{H(\omega)} \rightarrow$$

$$Y(\omega) = H(\omega)$$

Frequency domain

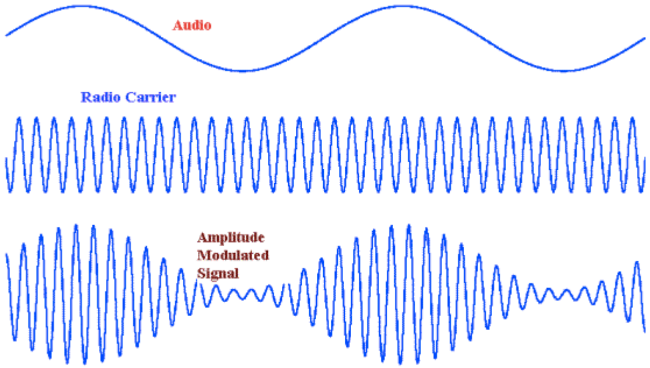
$$Y(\omega) = F(\omega)H(\omega)$$

$$|Y(\omega)|e^{j\angle Y(\omega)} = |F(\omega)||H(\omega)|e^{j[\angle F(\omega)+\angle H(j\omega)]}$$

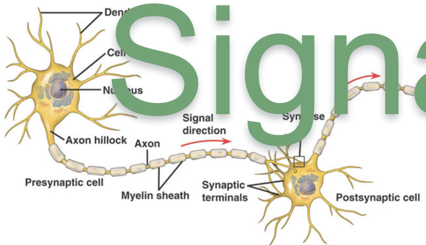
Fourier transform operations

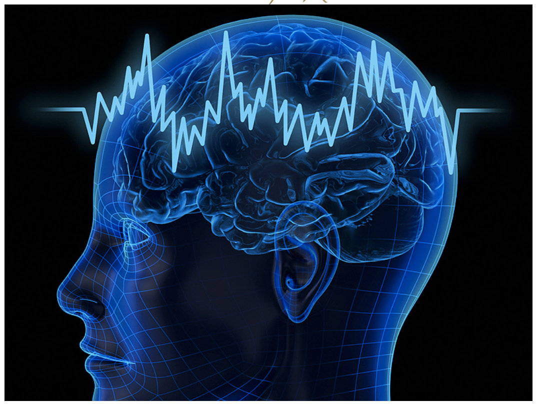
Operation	$f(t)$	$F(\omega)$
Addition	$f_1(t) + f_2(t)$	$F_1(\omega) + F_2(\omega)$
Scalar multiplication	$kf(t)$	$kF(\omega)$
Symmetry	$F(t)$	$2\pi f(-\omega)$
Scaling (a real)	$f(at)$	$\frac{1}{\lvert a\rvert} F\left(\frac{\omega}{a}\right)$
Time shift	$f(t - t_0)$	$F(\omega)e^{-j\omega t_0}$
Frequency shift (ω_0 real)	$f(t)e^{j\omega_0 t}$	$F(\omega - \omega_0)$
Time convolution	$f_1(t) * f_2(t)$	$F_1(\omega)F_2(\omega)$
Frequency convolution	$f_1(t)f_2(t)$	$\frac{1}{2\pi} F_1(\omega) * F_2(\omega)$
Time differentiation	$\frac{d^n f}{dt^n}$	$(j\omega)^n F(\omega)$
Time integration	$\int_{-\infty}^{t} f(x)\,dx$	$\frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$

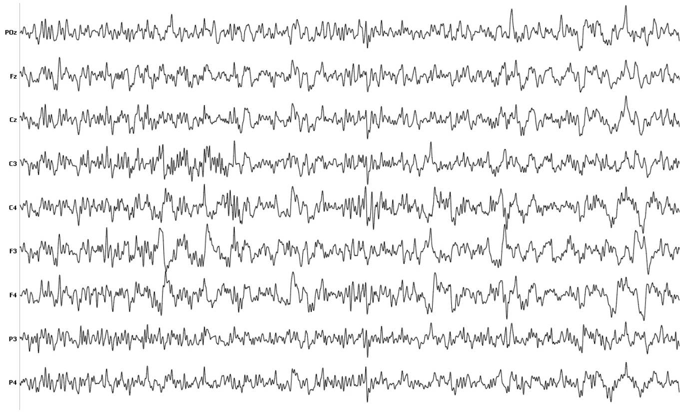
Signal transmission through LTIC systems

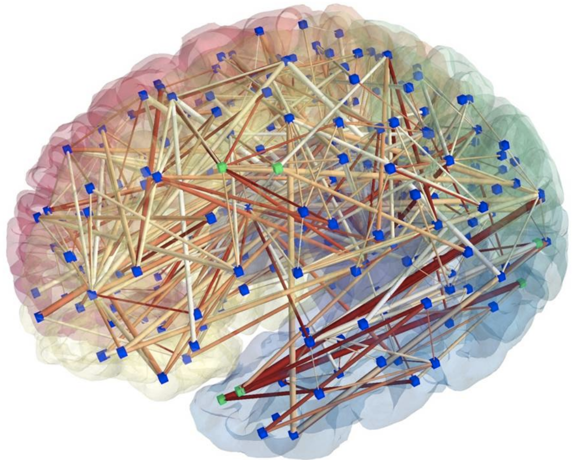


Signals are everywhere!









Distortionless transmission: linear phase

- Warning: transmission systems do not perform any filtering operations
- For the transmission to be distortionless, the output signal must be equal to the input signal except for a time-shift

$$y(t) = kf(t - t_d)$$

The Fourier transform of this equation yields

$$Y(\omega) = kF(\omega)e^{-j\omega t_d}$$

But

$$Y(\omega) = F(\omega)H(\omega)$$

Therefore

$$H(\omega) = k\,e^{-j\omega t_d}$$

This is the transfer function required for distortionless transmission. From this equation it follows that

$$|H(\omega)| = k \tag{4.58a}$$

$$\angle H(\omega) = -\omega t_d \tag{4.58b}$$

This result shows that for distortionless transmission, the amplitude response $|H(\omega)|$ must be a constant, and the phase response $\angle H(\omega)$ must be a linear function of ω with slope $-t_d$, where t_d is the delay of the output with respect to input (Fig. 4.26).

Linear phase systems

$|H(\omega)| = k$ The amplitude of the spectrum is constant

$\angle H(\omega) = -\omega t_d$ The phase is proportional to frequency

$t_d(\omega) = -\frac{d}{d\omega}\angle H(\omega)$ The time-delay is the slope of the phase spectrum

Ideal filters

- Ideal filters are distortionless transmission systems for certain frequencies and suppress the remaining ones
- Ideal *low-pass* filter

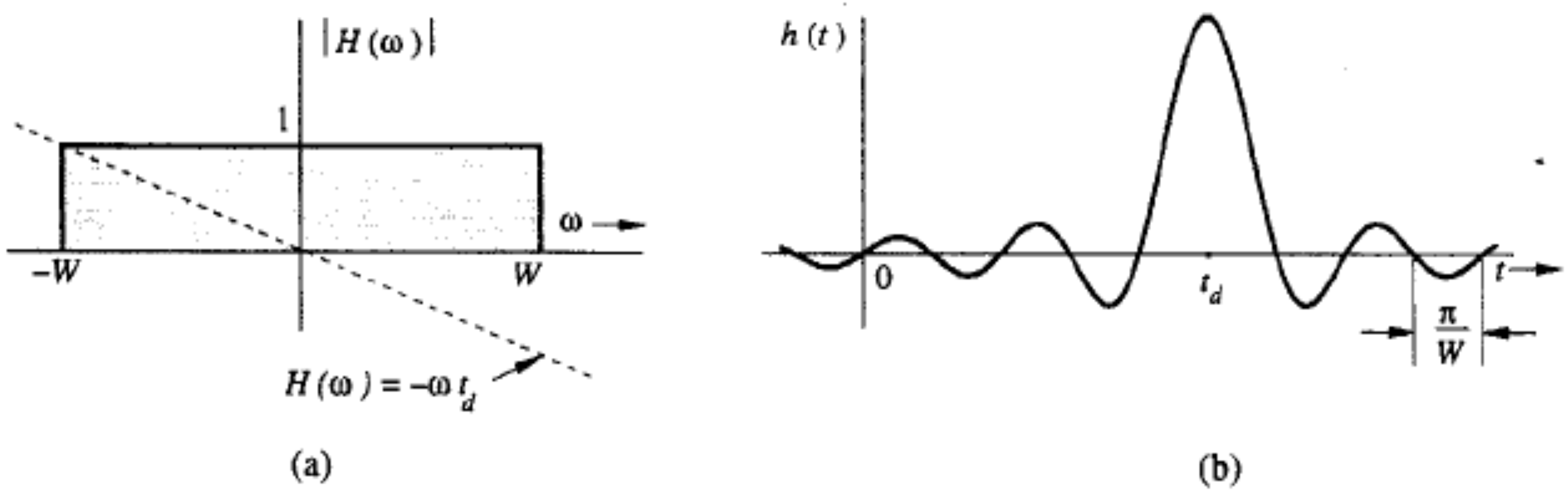


Fig. 4.27 Ideal lowpass filter: its frequency response and impulse response.

Ideal filters

- Ideal *high-pass* and *band-pass* filters

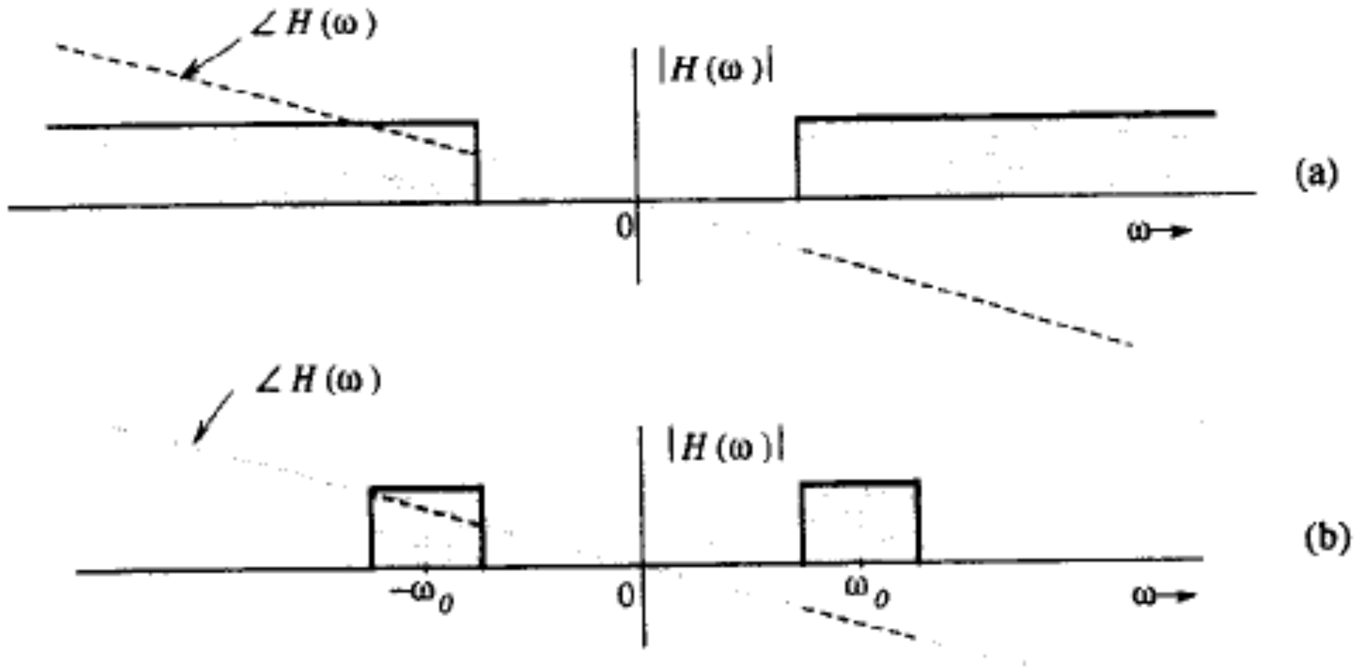
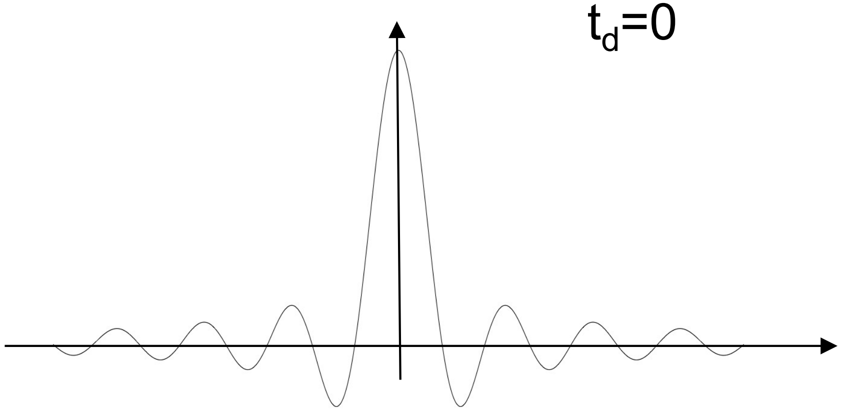


Fig. 4.28 Ideal highpass and bandpass filter frequency response.

Ideal filters

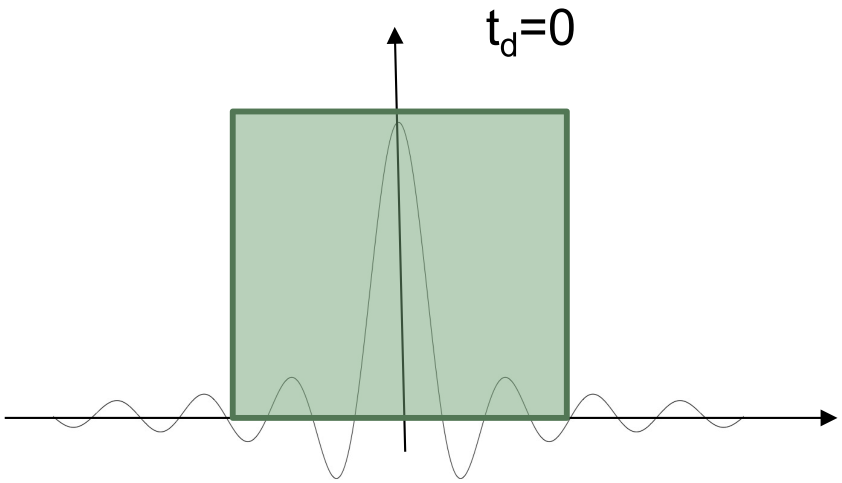
- Ideal filters are physically not realizable

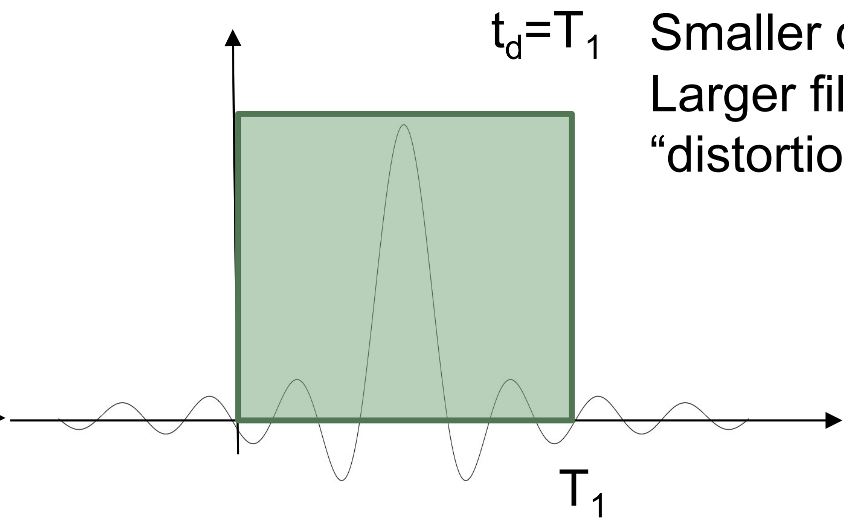


Non causal, infinite length tails

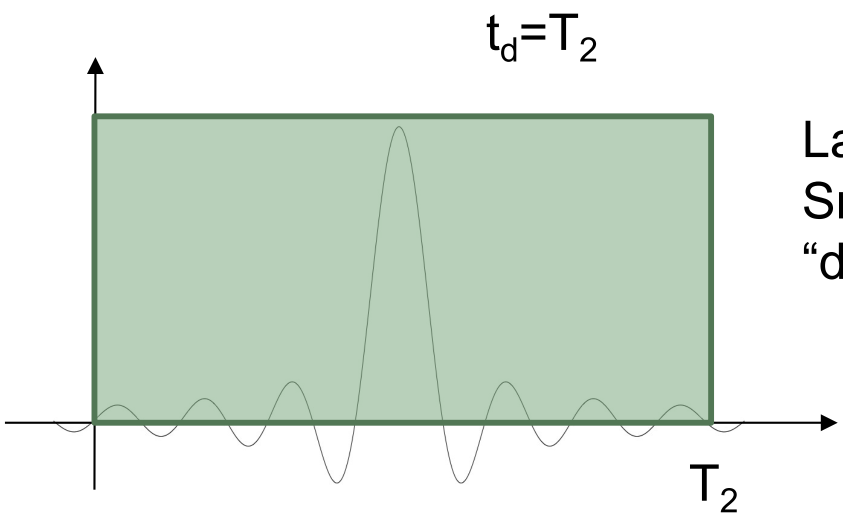
- To make it realizable we have to cut the tails, which implies
  - A time-delay
  - A smoothing of the response in the frequency domain

Ideal low pass filter





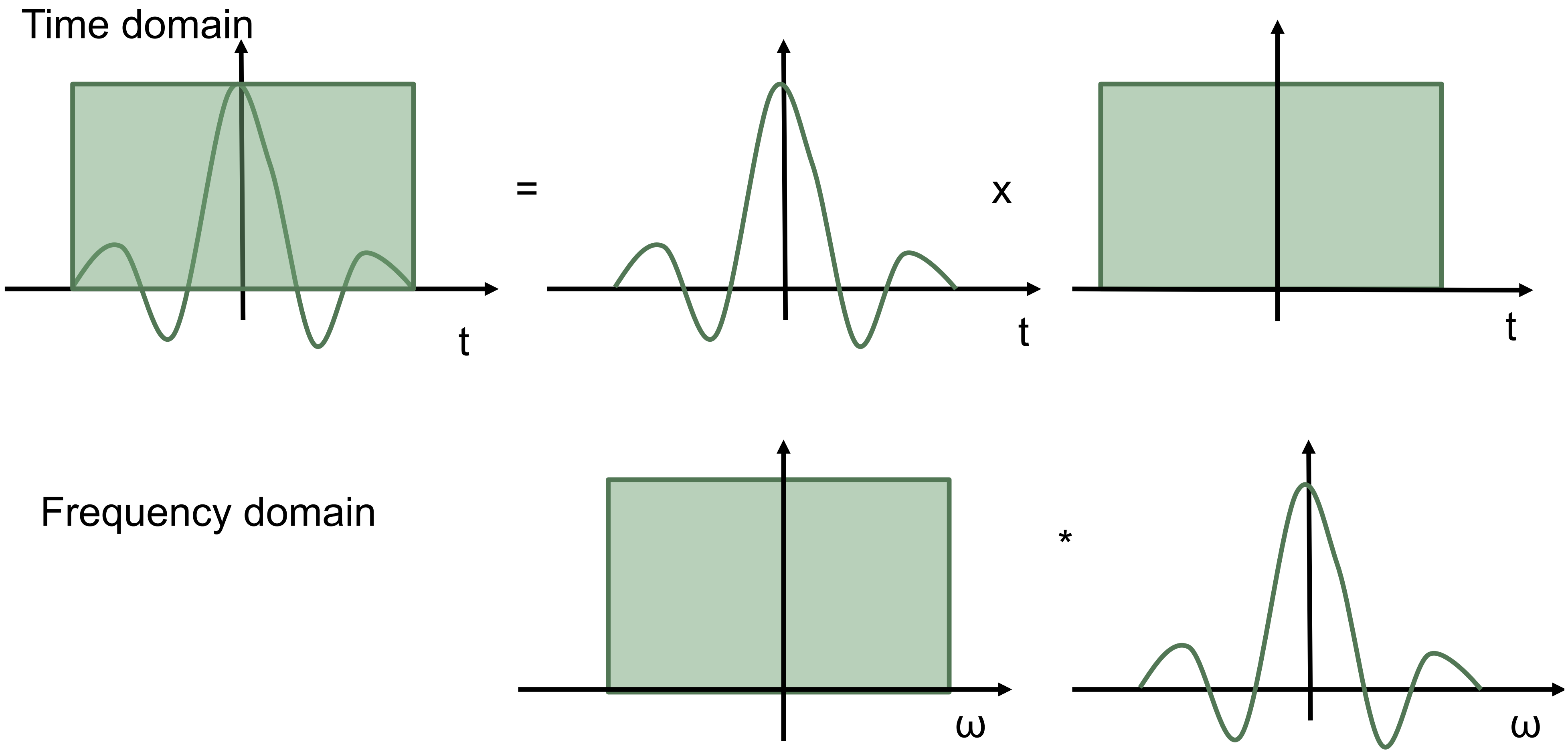
Smaller delay
Larger filter
“distortion”



Larger delay
Smaller filter
“distortion”

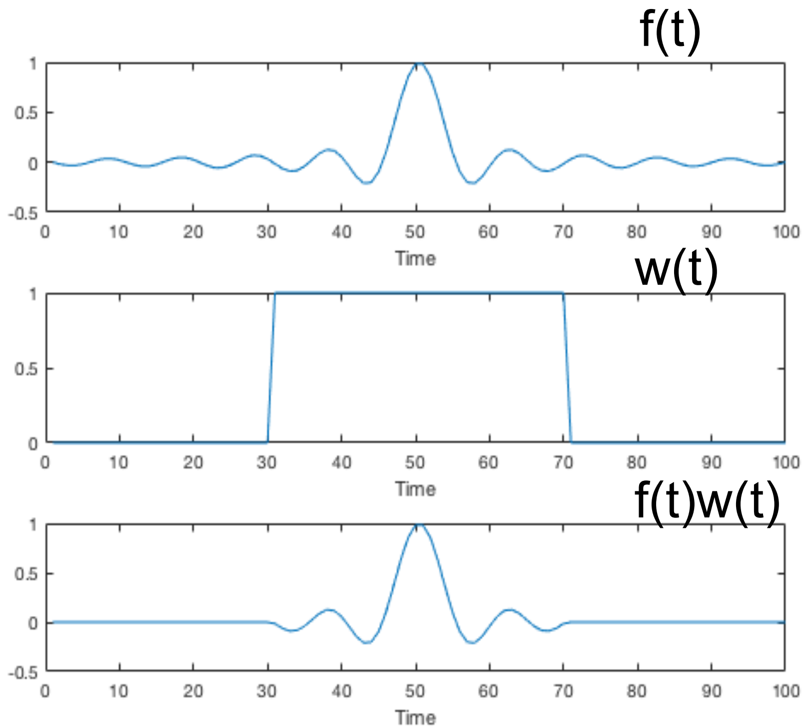
The filter impulse response is set to zero out of the window

Implications of windowing in time-domain

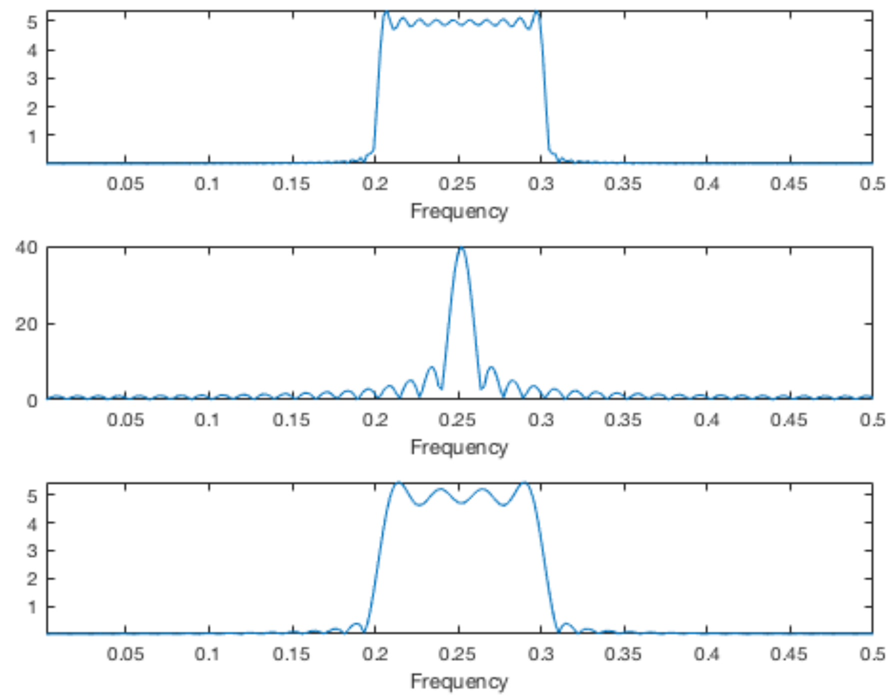


Implications of windowing in time domain

Time domain

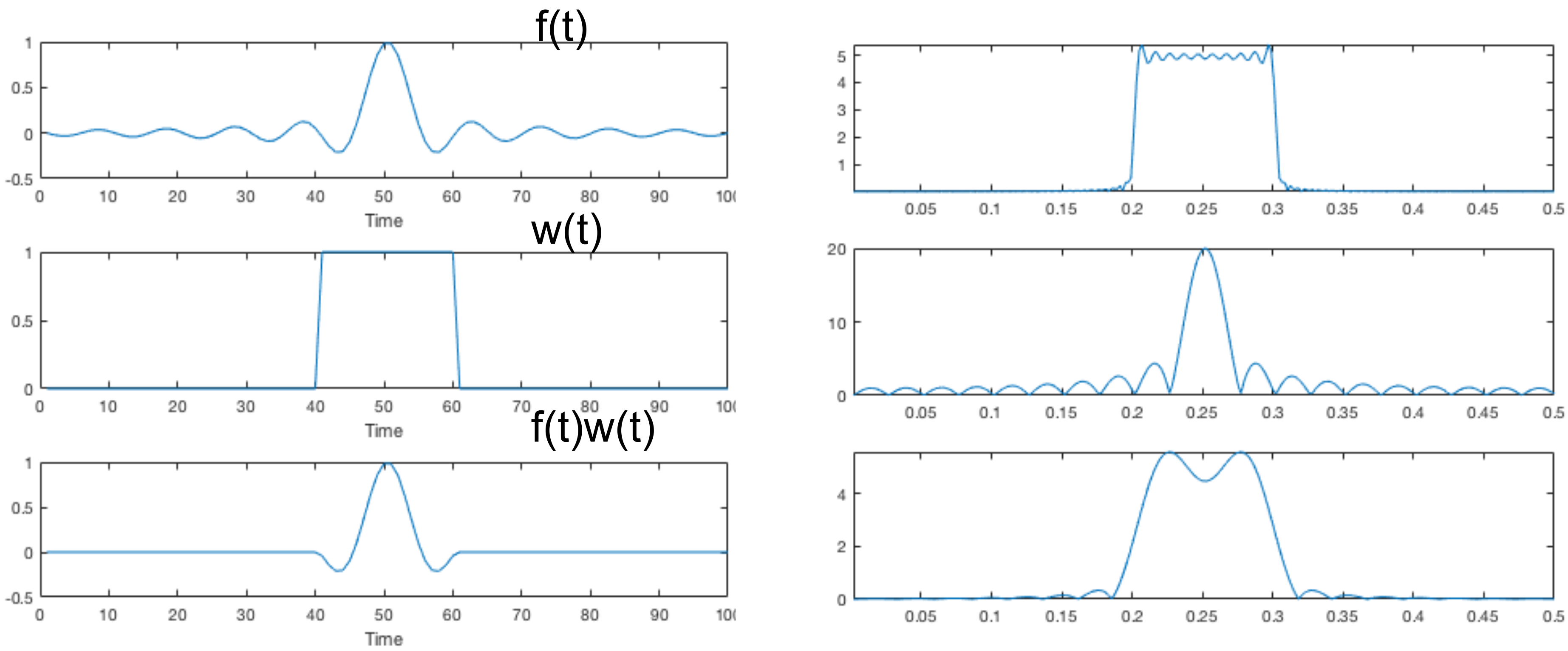


Fourier domain

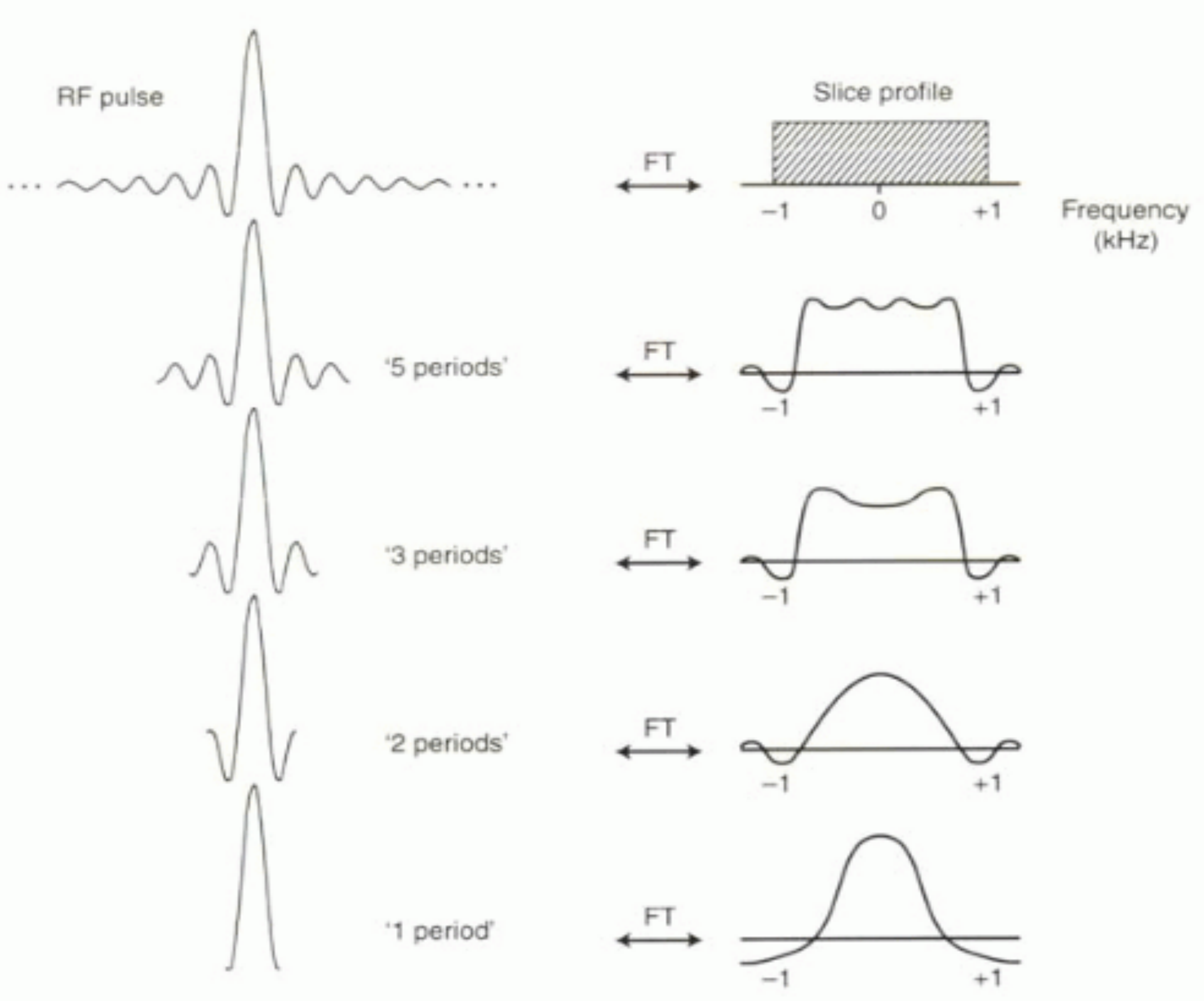


Implications of windowing in time domain

Time domain | Fourier domain



Effect of changing window length: example



Window functions

- Windows of different shapes can be used

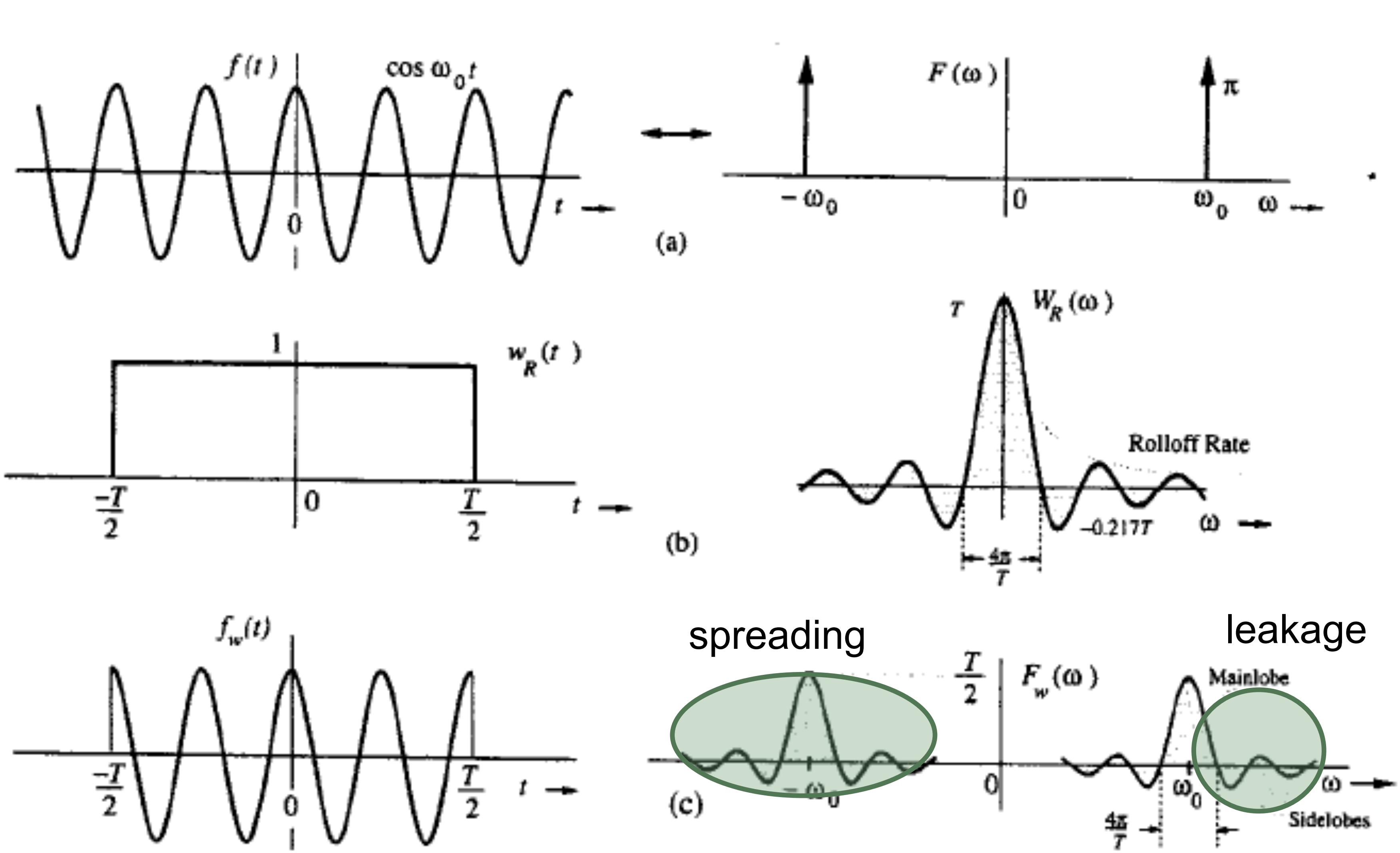
$$f_w(t) = f(t)w(t) \qquad \text{and} \qquad F_w(\omega) = \frac{1}{2\pi} F(\omega) * W(\omega)$$

Windowing in time domain → Spectral spreading; Spectral leakage

- Spreading (or smearing): the spectrum gets larger
- Leakage: the spectrum "leaks" towards frequency ranges where it is supposed to be zero

Windowing in frequency domain → Temporal spreading

The effect of windowing



Windowing: how can we remedy?

- Spectral spreading depends on window width → *increase window width*
- Spectral rate of decay depends on the regularity of the signal: the higher the number of contiunuos derivatives, the fastest the decay → *choose a smooth window*
- There is a *trade-off* between the two
  - The *rectangular* window has the *smallest spread* but the *largest leakage*. This can be mitigated by choosing large windows

- Take home message: to minimize the effect of windowing choose *smooth wide* windows

Signal energy

- By definition

$$E_f = \int_{-\infty}^{\infty} |f(t)|^2 \, dt$$

$f^*(t)$

$$E_f = \int_{-\infty}^{\infty} f(t) f^*(t) \, dt = \int_{-\infty}^{\infty} f(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega) e^{-j\omega t} \, d\omega \right] dt$$

- Interchanging the order of integration

$$E_f = \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega) \left[\int_{-\infty}^{\infty} f(t) e^{-j\omega t} \, dt \right] d\omega$$

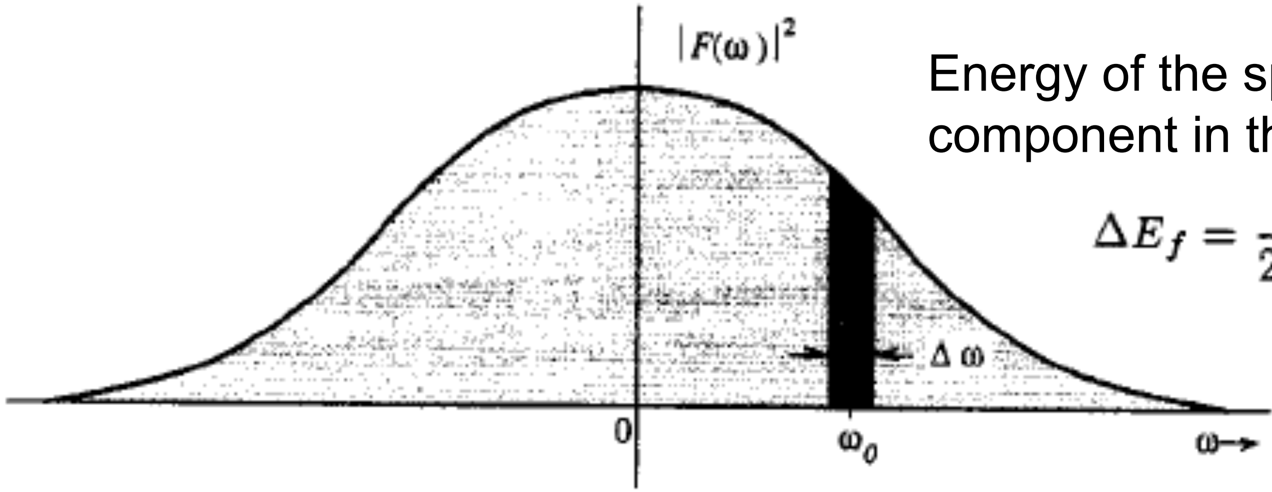
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) F^*(\omega) \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega$$

Parseval's theorem

$$\boxed{E_f = \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega}$$

Signal energy and energy spectral density



Energy of the spectral component in the band $\Delta\omega$

$$\Delta E_f = \frac{1}{2\pi} |F(\omega)|^2 \Delta\omega$$

Fig. 4.30 Interpretation of Energy spectral density of a signal.

$|F(\omega)|^2$ Spectral energy density (energy per unit bandwidth in Hz)

Autocorrelation function and energy spectral density

- Autocorrelation function

$$\Psi_f(\tau) = \int_{-\infty}^{+\infty} f(\tau) f(\tau - t) d\tau$$

$$g(\tau) = f(-\tau) \rightarrow$$

$$\Psi_f(t) = \int_{-\infty}^{+\infty} f(\tau) g(t - \tau) d\tau = f(t) * g(t) = f(t) * f(-t)$$

Relation to convolution

$$\Psi_f(t) = \int_{-\infty}^{+\infty} f(\tau) f(\tau - t) d\tau \rightarrow$$

$$\Psi_f(-t) = f(-t) * f(t) = f(t) * f(-t) = \Psi_f(t)$$

The autocorrelation function is even symmetric

Autocorrelation function and power spectral density

- For real signals

$$F\{f(-t)\} = \int_{-\infty}^{+\infty} f(-t)e^{j\omega t}dt = F(-\omega) = F^{\star}(\omega)$$

- Thus

$$F\{f(t) \star f(-t)\} = F(\omega)F^{\star}(\omega) = |F(\omega)|^2$$

- Hence

$$\Psi_f(t) \rightarrow |F(\omega)|^2$$

The power spectral density is the F-transform of the autocorrelation function