Summer School on Formal Methods for Cyber-Physical Systems

Edition 2019: Numerical and Symbolic Methods for Reachability Analysis of Hybrid Systems

DIPARTIMENTO DI INGEGNERIA INFORMATICA
AUTOMATICA E GESTIONALE ANTONIO RUBERTI

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PRESENTATION STRUCTURE

An Overview of Models for dynamical phenomena.

Modeling techniques.

- LTI systems. 😊
- Automata, FSM. 😊
- Hybrid Systems. 😡

Hybrid Systems in Automotive Electronics Design:

- Engine and Power Train Hybrid Model

Applications:

- Cut-off control
- Actual engaged gear identification

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Modeling dynamical phenomena

Model:*"something that represents another thing, either as a physical object that is usually smaller than the real object, or as a simple description"*

a plastic model aircraft

***Mathematical model**: a representation in mathematical terms of the behavior of real devices and objects*

Why do we do Mathematical Modeling?

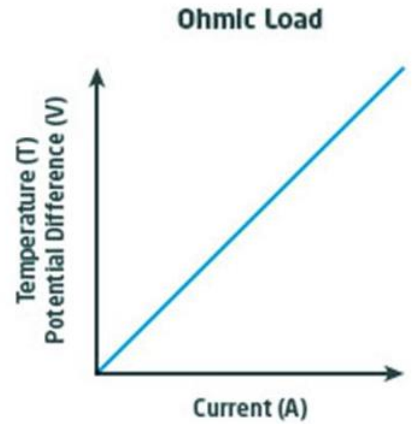
- to predict
- to understand and explain
- to control

phenomena and behaviors

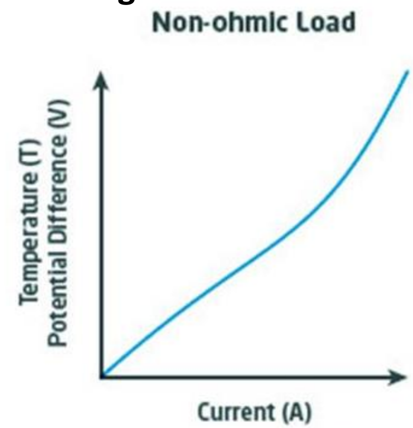
Electrical resistor

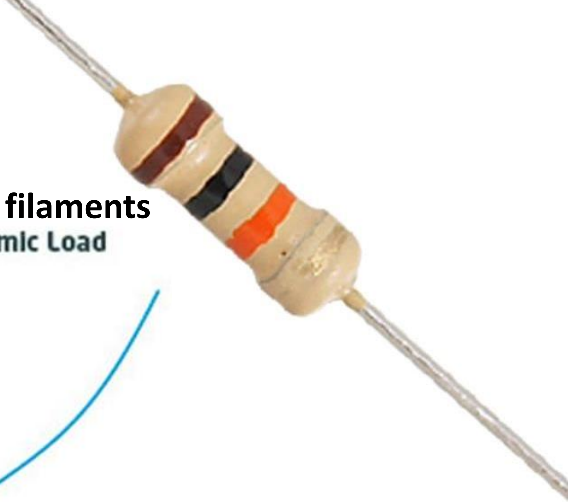
Current–voltage relation

$$V(t) = R\,I(t)$$



Light bulb filaments

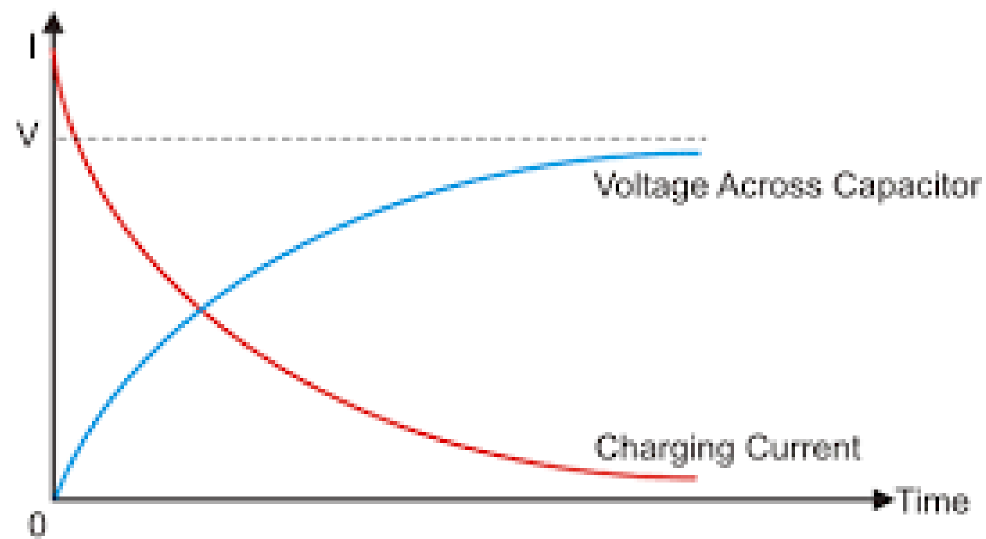


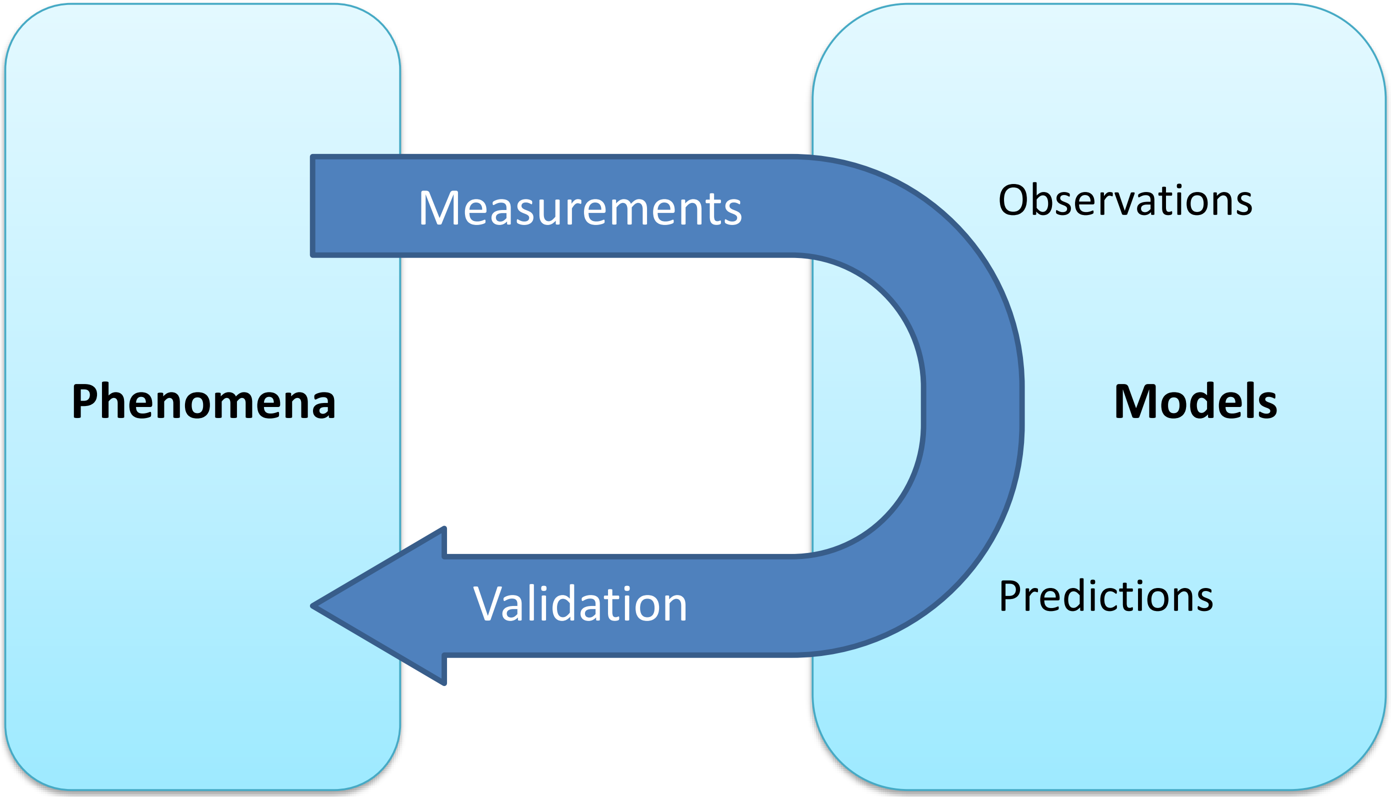


Electrical capacitor

Current–voltage relation

$$I(t) = C\frac{\mathrm{d}V(t)}{\mathrm{d}t}$$





The modeling part is concerned with analyzing and rationalizing the above observations in order to construct:
models that allow to **predict** future behaviors;
models that **describe** the behavior or results observed;
models that **explain** why that behavior and results occurred as they did.

*Which kind of **mathematical tools** can be used?*

Time Series Analysis

Ordinary differential equation (ODEs)

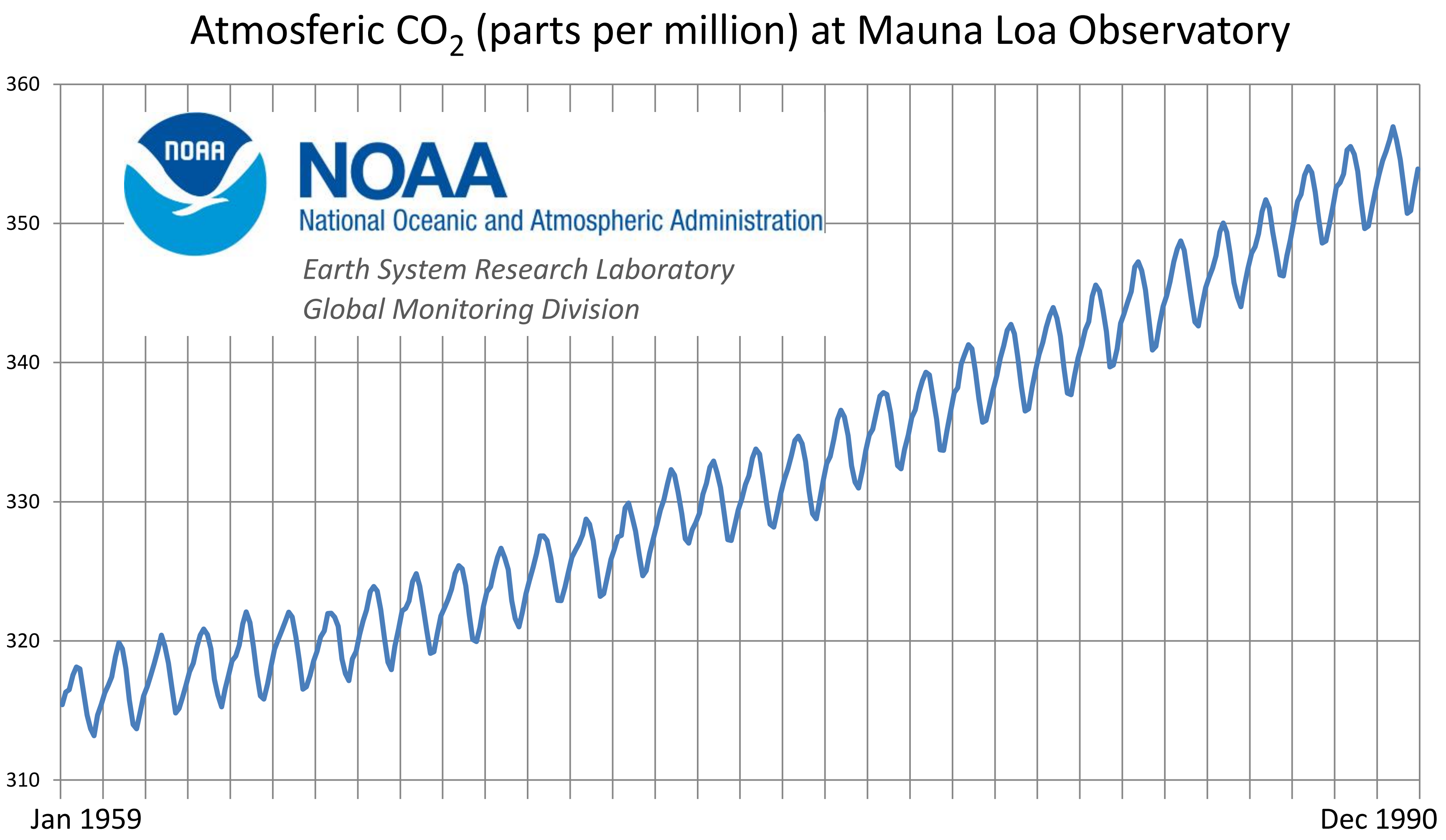
Difference equations or recurrence relations

Partial differential equations (PDEs)

Finite state machines (FSMs)

Automata

Predictive model

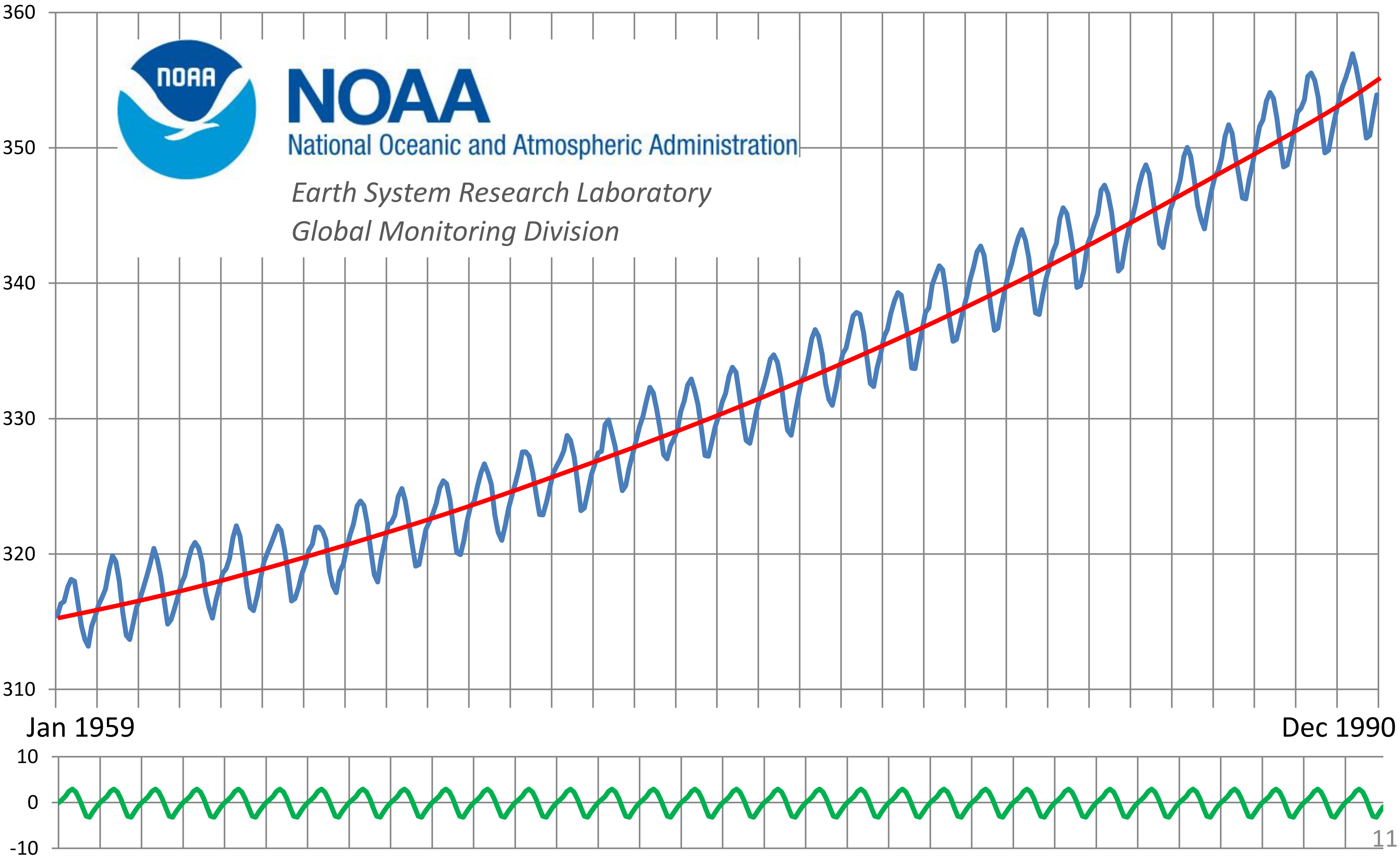


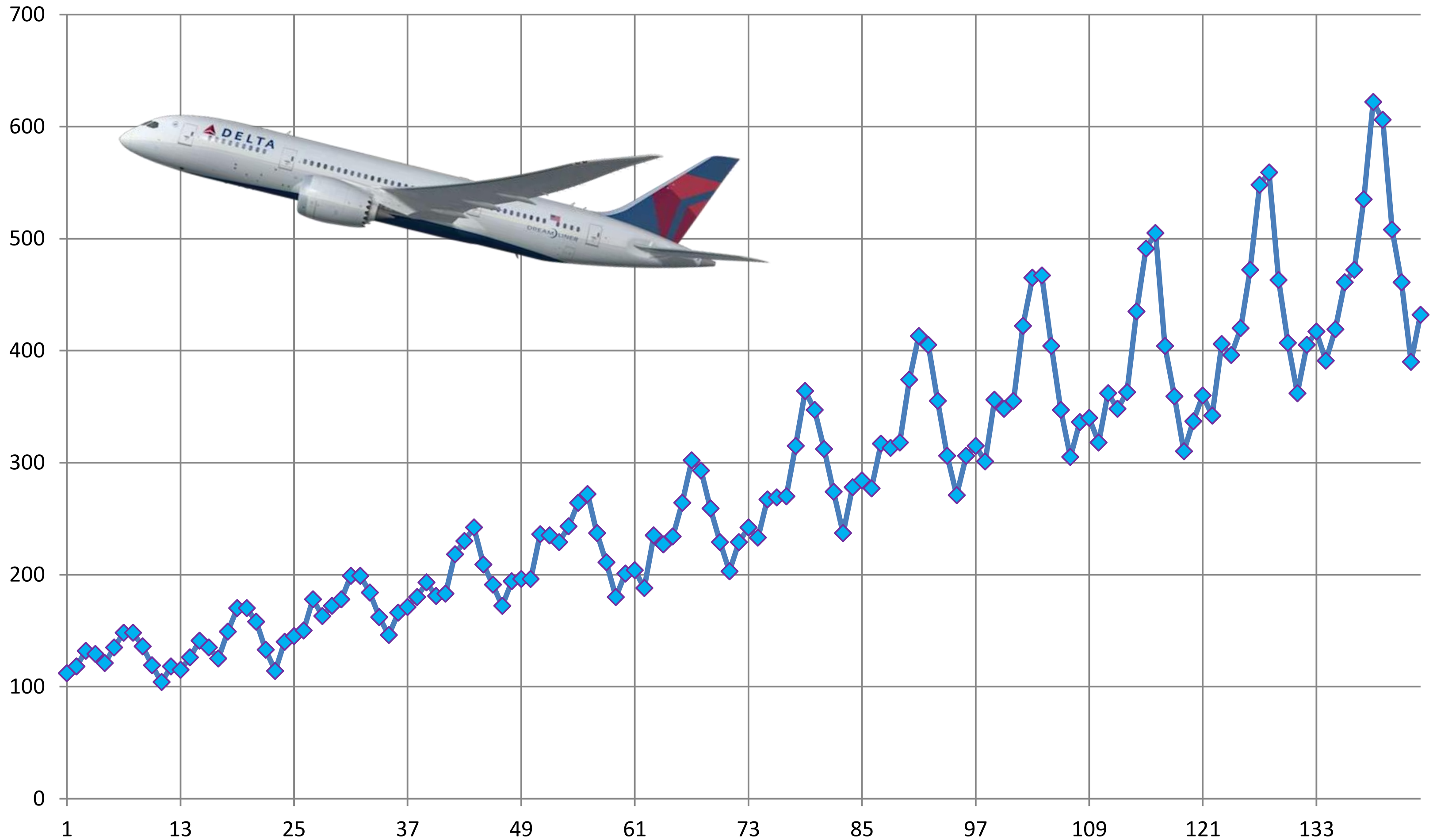
Predictive model

Additive time series decomposition:

$$y_t = T_t + S_t + noise$$

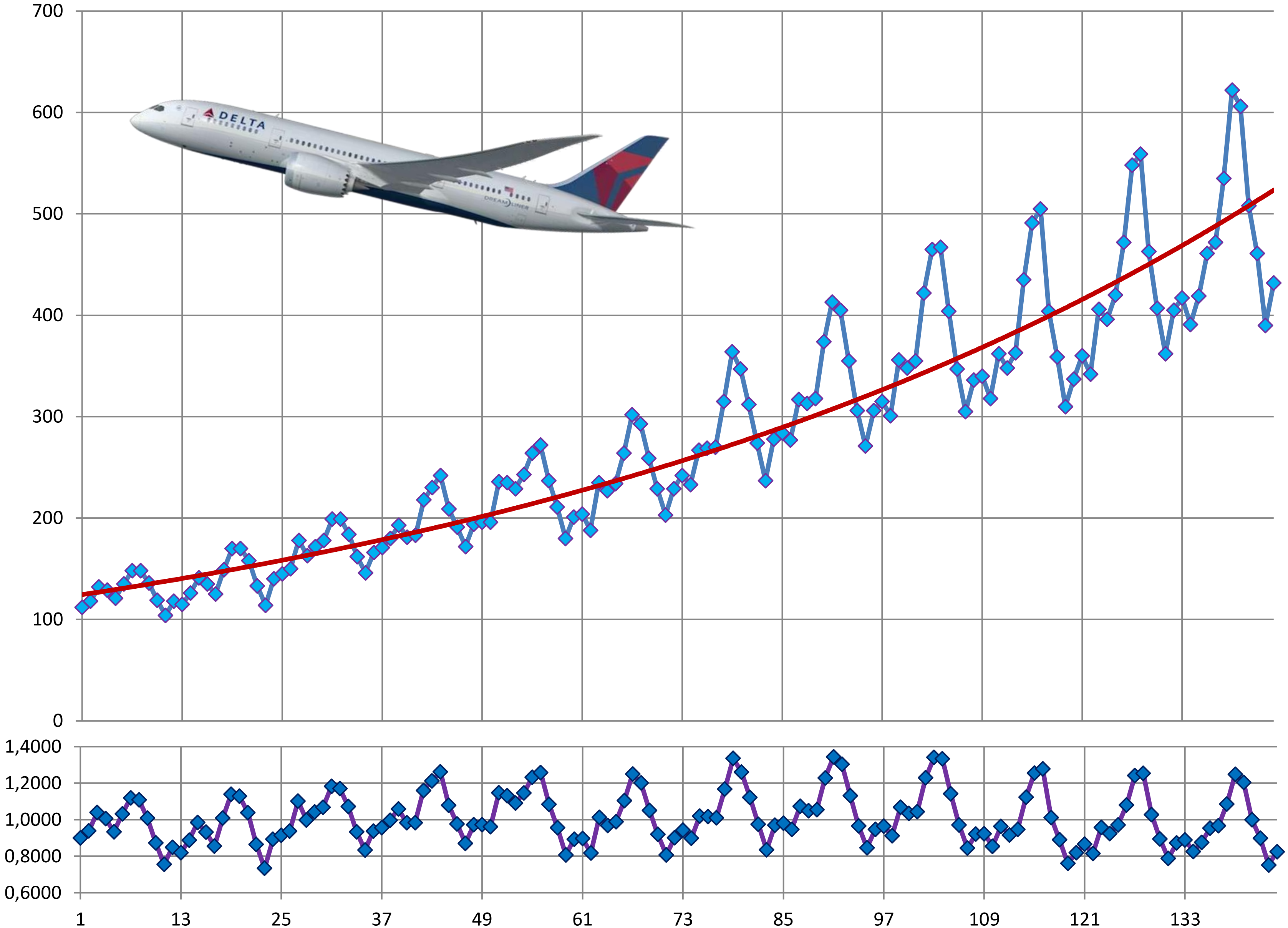
Atmosferic CO_2 (parts per million) at Mauna Loa Observatory





Monthly international airline US passenger totals,
Jan 1949 – Dec 1960 (thousands)

Multiplicative time series decomposition: $y_t = T_t \cdot S_t \cdot noise$



Interpretative model

Thomas Malthus's table of population growth in England 1780-1810, from his *An Essay on the Principle of Population*, 1826

Ch. ix. *in England (continued).* 435

Table, calculated from the births alone, in the Preliminary Observations to the Population Abstracts, printed in 1811.

Table, calculated from the excess of the births above the deaths, after an allowance made for the omissions in the registers, and the deaths abroad.

$$\frac{P(1785) - P(1780)}{P(1780)}$$

Population in		Population in		Percentage increase	Predicted population	Prediction error
1780	7,953,000	1780	7,721,000			
1785	8,016,000	1785	7,998,000	0.036	7,998,000	
1790	8,675,000	1790	8,415,000	0.052	8,421,894	0.1 %
1795	9,055,000	1795	8,831,000	0.049	8,868,254	0.3 %
1800	9,168,000	1800	9,287,000	0.052	9,338,272	0.1 %
1805	9,828,000	1805	9,837,000	0.059	9,833,200	-0.6 %
1810	10,488,000	1810	10,488,000	0.066	10,354,360	-1.2 %

In the first table, or table calculated from the births alone, the additions made to the population in each period of five years are as follow ;—

From 1785 to 1805, the five years percentage increase is constant and equal to about 0.053

$$P(k+1) = P(k) + 0{,}053\ P(k)$$

Interpretative model

Establishing cause-and-effect relationship

Cause : The reason why something happened

Effect : What happened because of the cause

Cause : The excess of births above the deaths is constant (in percentage) in each period of five years

Effect : The population is increasing as a geometric sequence

$$P(k+1) = 1{,}053\ P(k)$$

Interpretative model

Establishing cause-and-effect relationship

Cause : The reason why something happened

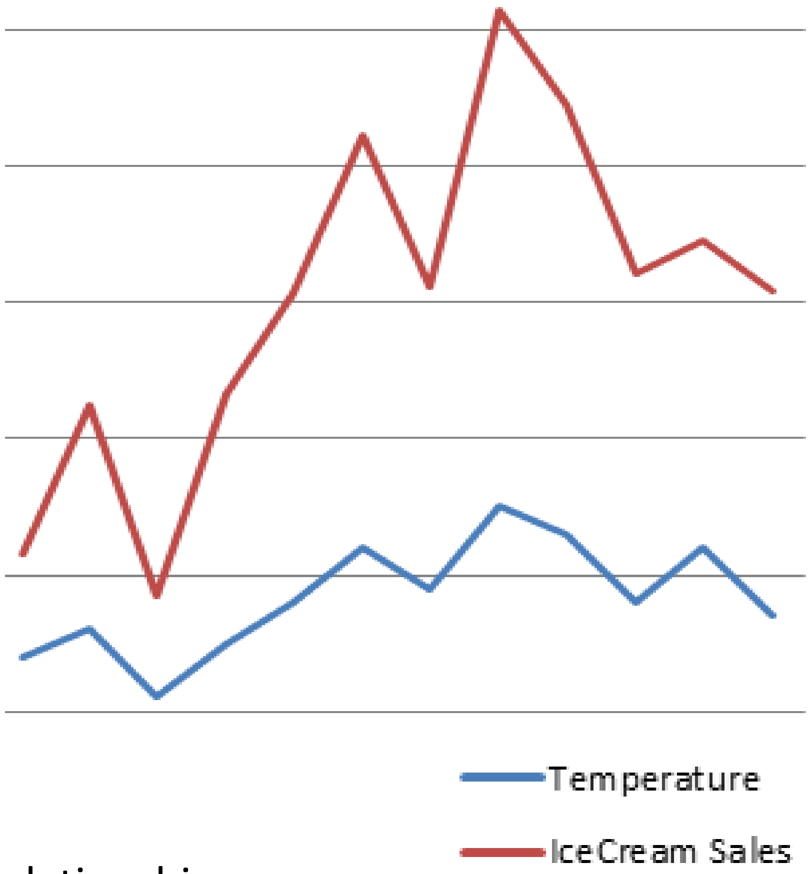
Effect : What happened because of the cause

1) The effects happen always after the cause ?

2) Which is the cause
and
which the effect?

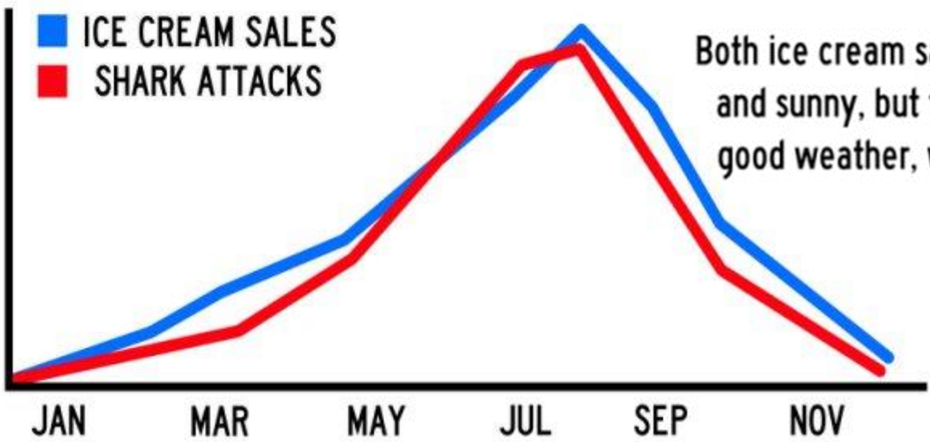
Correlation

$$r = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}$$

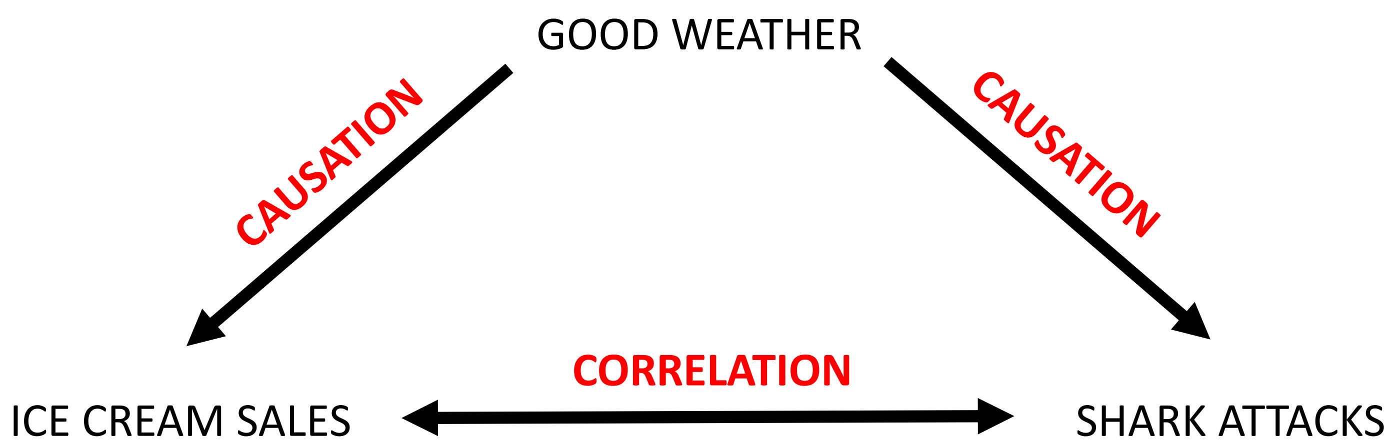


- Correlation does not imply causation!
- We cannot establish cause and effect relationships
- We cannot say that one variable causes an increase (or decrease) in the other variable
- We can say that one variable is associated (linearly) with another variable

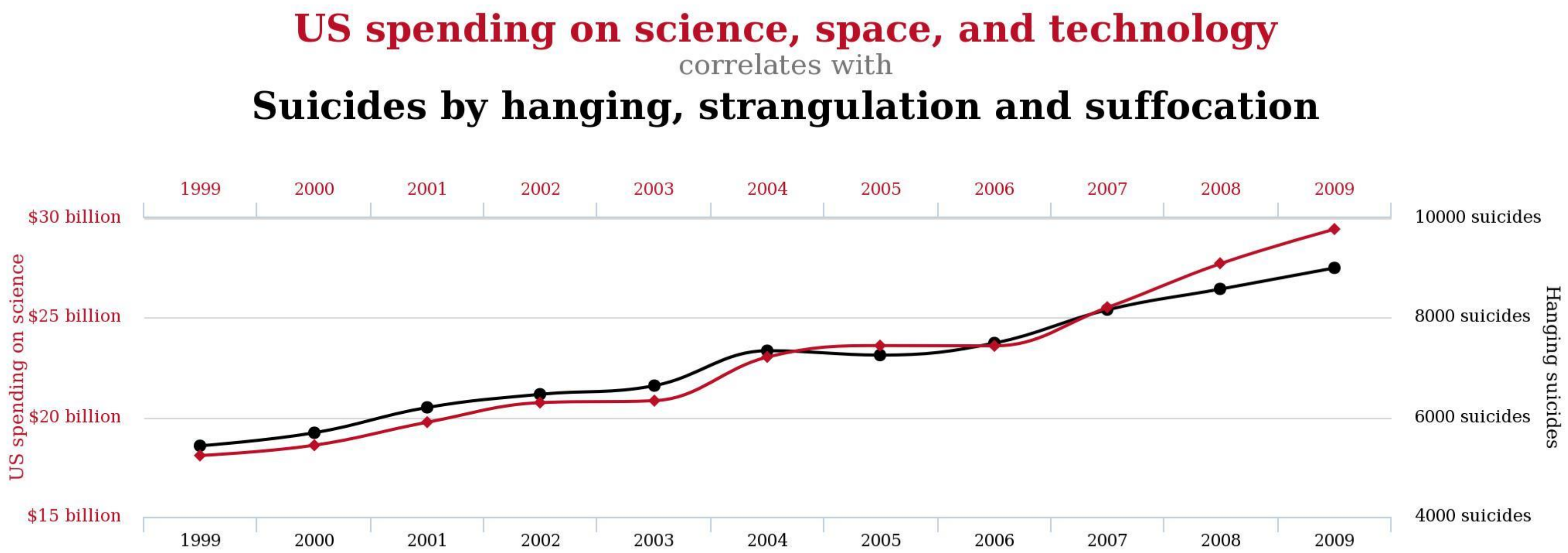
CORRELATION IS NOT CAUSATION!

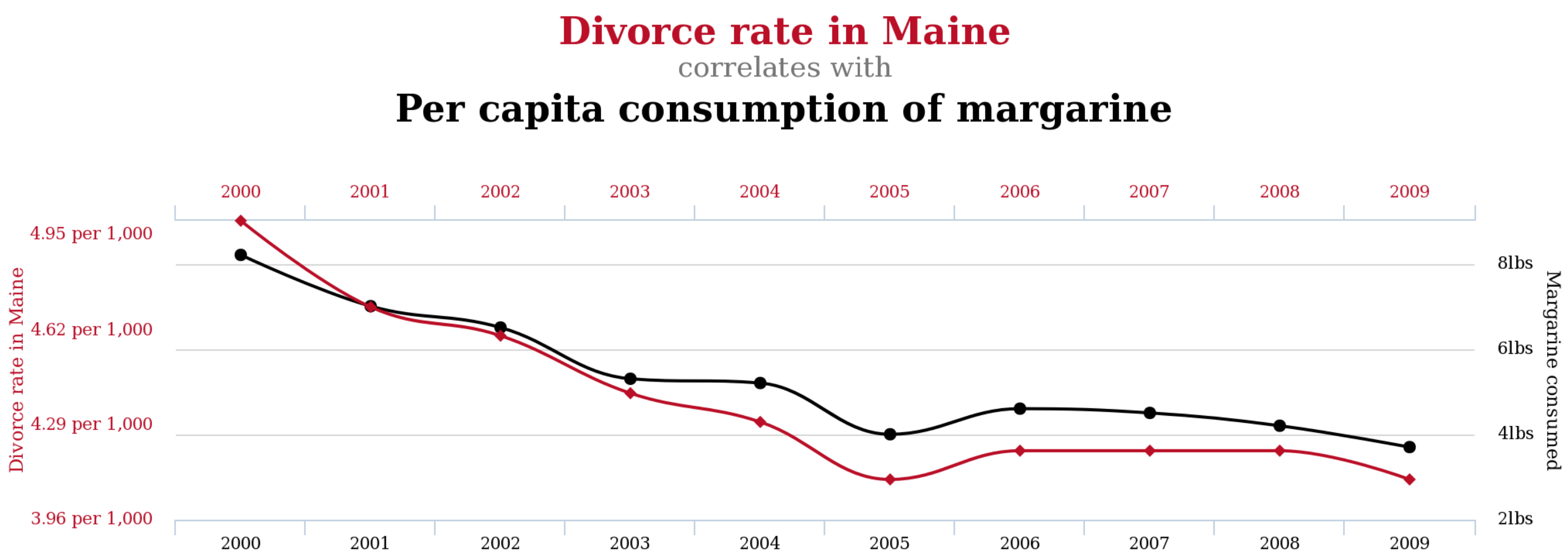


Both ice cream sales and shark attacks increase when the weather is hot and sunny, but they are not caused by each other (they are caused by good weather, with lots of people at the beach, both eating ice cream and having a swim in the sea)

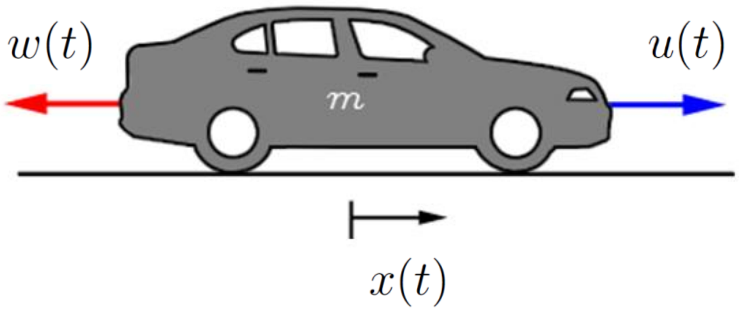


Spurious correlations





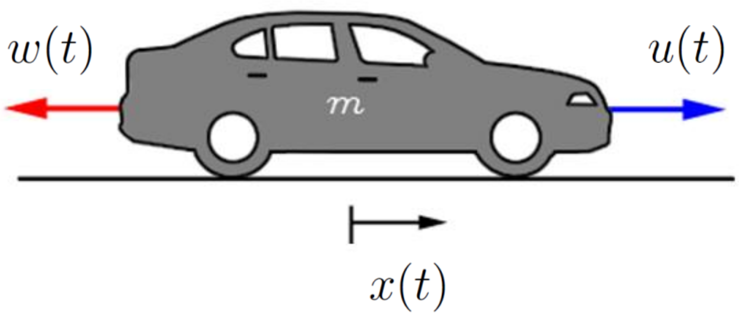
Modeling: the art of neglecting



$$m\,\ddot{x}(t) = u(t) - w(t)$$

$u(t)$ traction force

Modeling: the art of neglecting



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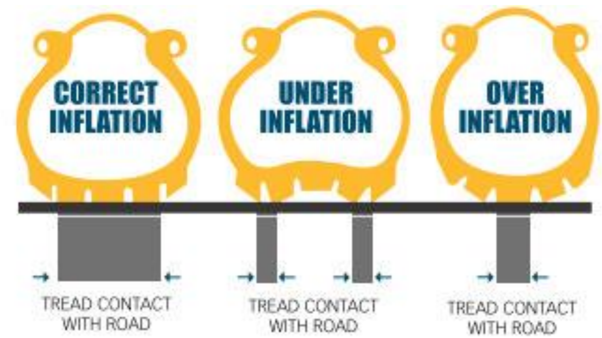
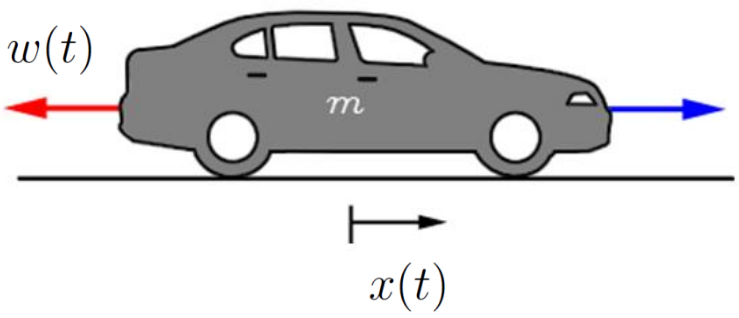


coefficient of friction between the driving wheels and surface

kind of tires
tires pressure

weight applied to the driving wheels

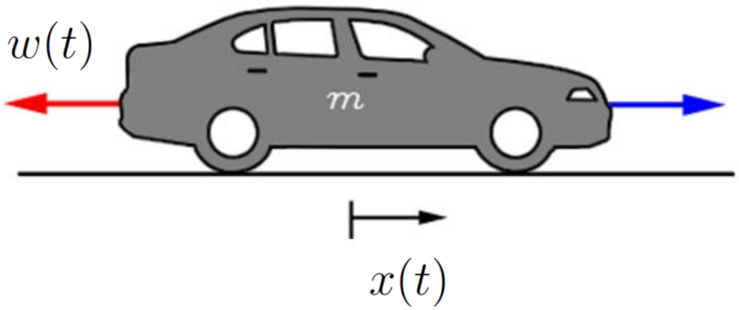
Modeling: the art of neglecting



$$m\,\ddot{x}(t) = u(t) - w(t)$$

$w(t)$ wind drag

Modeling: the art of neglecting



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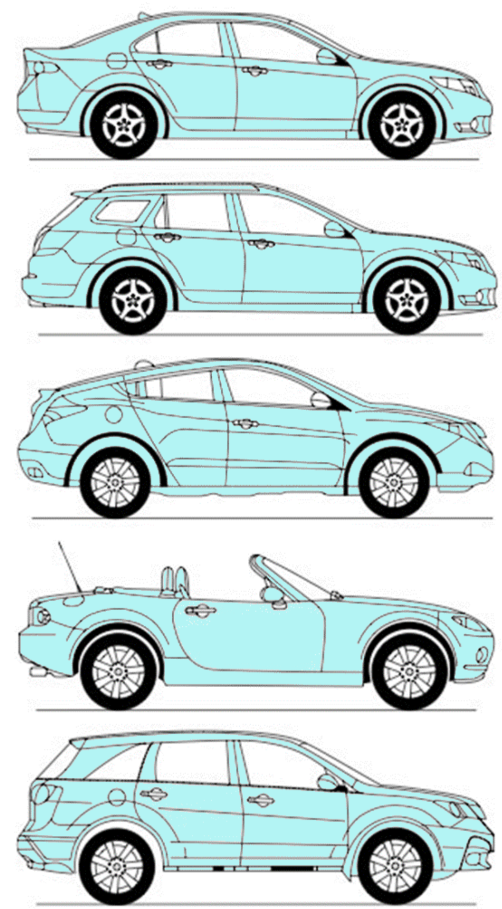
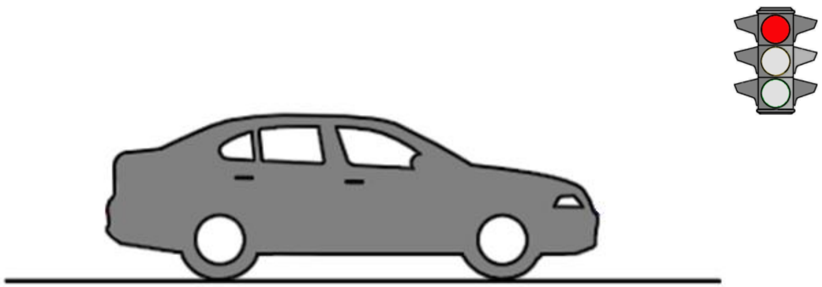


	Drag coefficient
Sedan	0.32
Wagon	0.30
Hatchback	0.31
Convertible	0.40
SUV	0.40 - 0.50

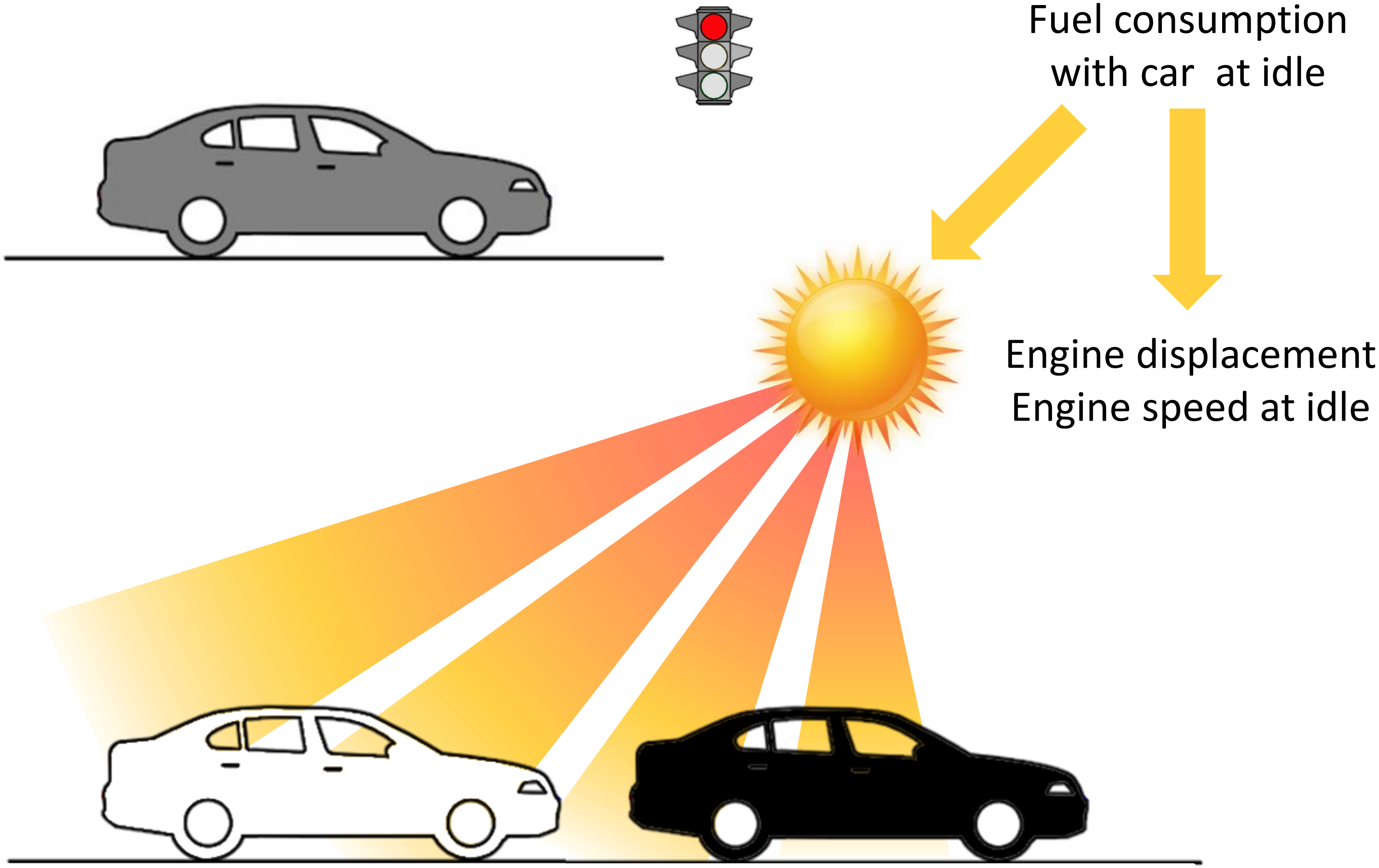
Modeling: the art of neglecting



Fuel consumption
with car at idle

Engine displacement
Engine speed at idle

Modeling: the art of neglecting



In a modern automobile, the A/C system uses around 4 horsepower (3 kW) of the engine's power, thus increasing fuel consumption of the vehicle

Modeling: the art of neglecting

De corporibus fluitantibus (On floating bodies)

Archimede

Indeed, in the first book Archimedes derives the fact that ***the surface of the oceans is spherical***.

In the second book, ***the surface of the liquid is implicitly assumed to be flat*** from the start:
Archimedes does not spend a single word in justifying this assumption as an approximation of the "true" spherical shape.

Obviously we are dealing with two different models, appropriate for phenomena at different scales.

On Exactitude In Science

Jorge Luis Borges

. . . In that Empire, the Art of Cartography attained such Perfection that the map of a single Province occupied the entirety of a City, and the map of the Empire, the entirety of a Province.

In time, those Unconscionable Maps no longer satisfied, and the Cartographers Guilds struck a Map of the Empire whose size was that of the Empire, and which coincided point for point with it.

The following generations, who were not so fond of the Study of Cartography as their forebears had been, saw that that vast map was Useless, and not without some pitilessness was it, that they delivered it up to the inclemencies of Sun and winters. In the Deserts of the West, still today, there are Tattered Ruins of that Map, inhabited by animals and beggars; in all the Land there is no other relic of the disciplines of Geography.

—Suarez Miranda, Viajes de varones prudentes, Libro IV, Cap. XLV, Lerida, 1658

“All models are wrong

but some are useful.”

G. Box, *Robustness in the strategy of scientific model building, 1979*

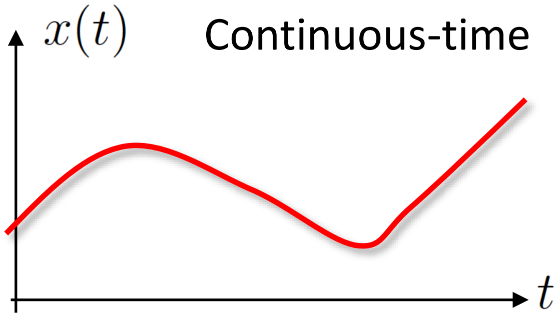
A classification of dynamical models

Time-Driven and Event-Driven systems

Time-driven systems change state in response to a uniformly progressing physical time

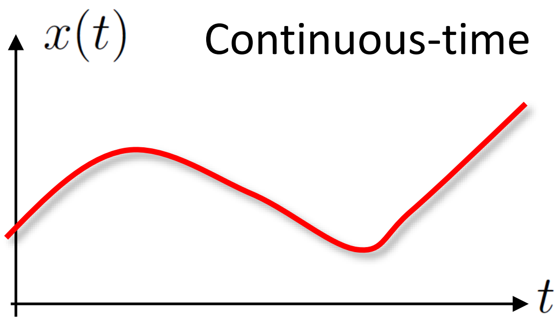
Time-Driven and Event-Driven systems

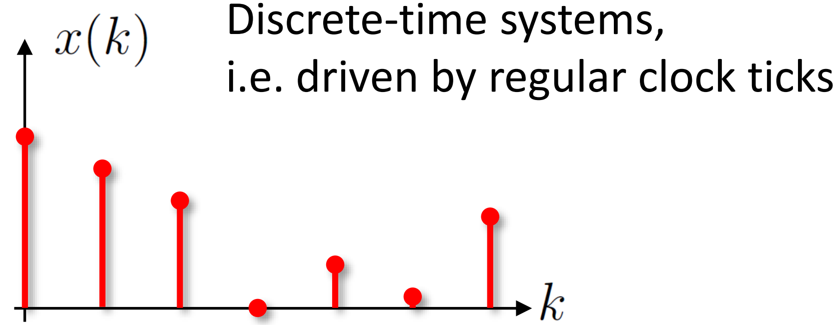
Time-driven systems change state in response to a uniformly progressing physical time



Time-Driven and Event-Driven systems

Time-driven systems change state in response to a uniformly progressing physical time

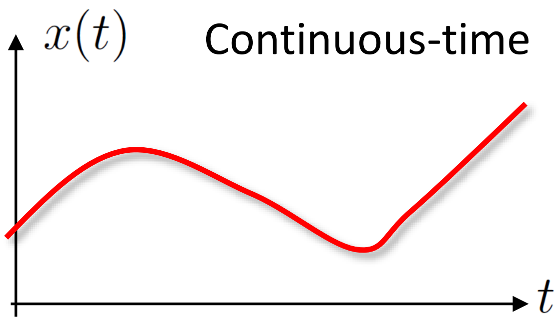




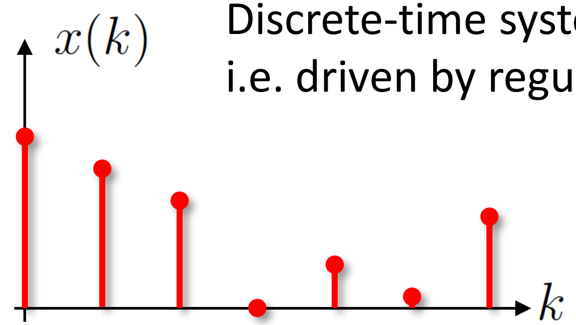
Time-Driven and Event-Driven systems

Time-driven systems change state in response to a uniformly progressing physical time

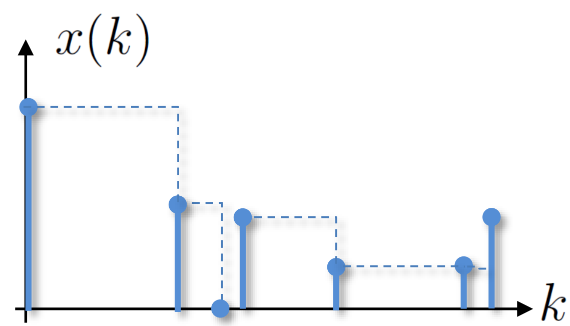
Continuous-time



Discrete-time systems,
i.e. driven by regular clock ticks

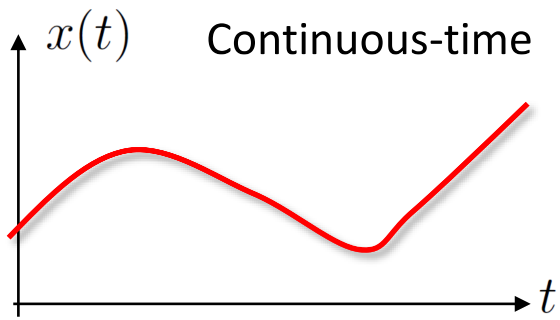


Event-driven systems change state in response to the occurrence of *asynchronous* discrete events that result in instantaneous state transitions

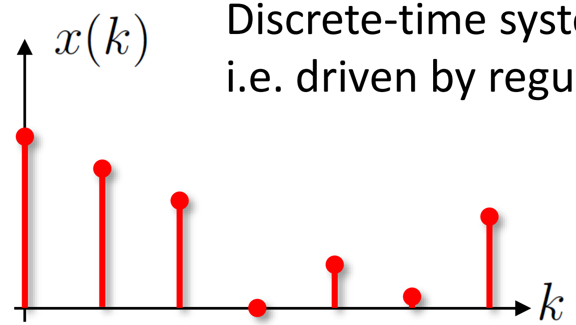


Time-Driven and Event-Driven systems

Time-driven systems change state in response to a uniformly progressing physical time



Continuous-time



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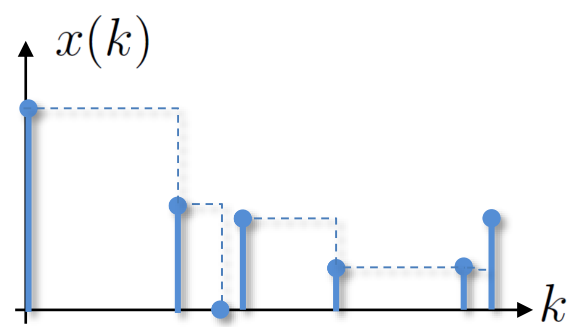


Continuous-state and discrete-state systems

Continuous-state system: the state variables are real

Discrete-state system: the state variables takes values in a discrete (finite/infinite) set

Discrete-time continuous-state systems

1) Mortgage repaying

$x(k)$	residual mortgage after *k* months
$x(0)$	principal amount borrowed
$u(k)$	regular monthly payment
r	monthly interest rate

$$x(k+1) = x(k) + rx(k) - u(k) = (1+r)x(k) - u(k)$$

The residual mortgage changes only at the end of every month

Discrete-time continuous-state systems

2) Population growth

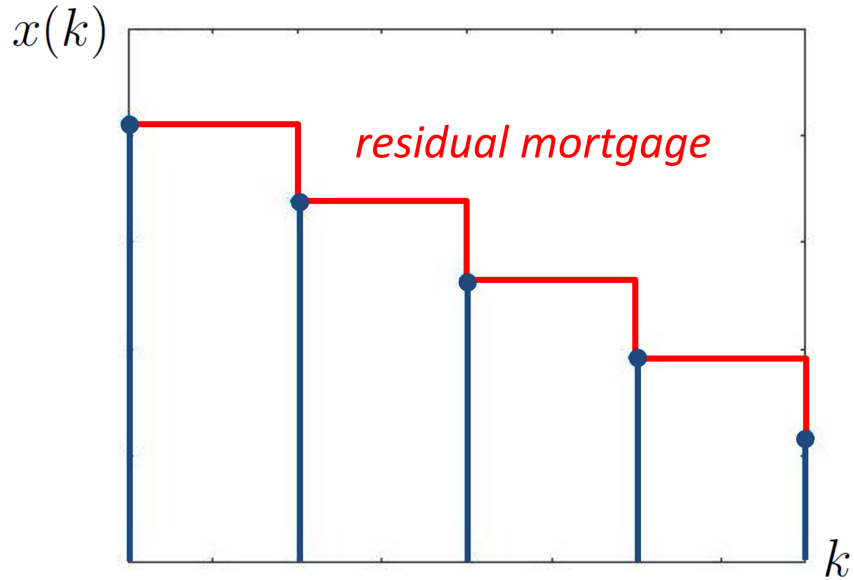
$x(k)$ population size after *k* years

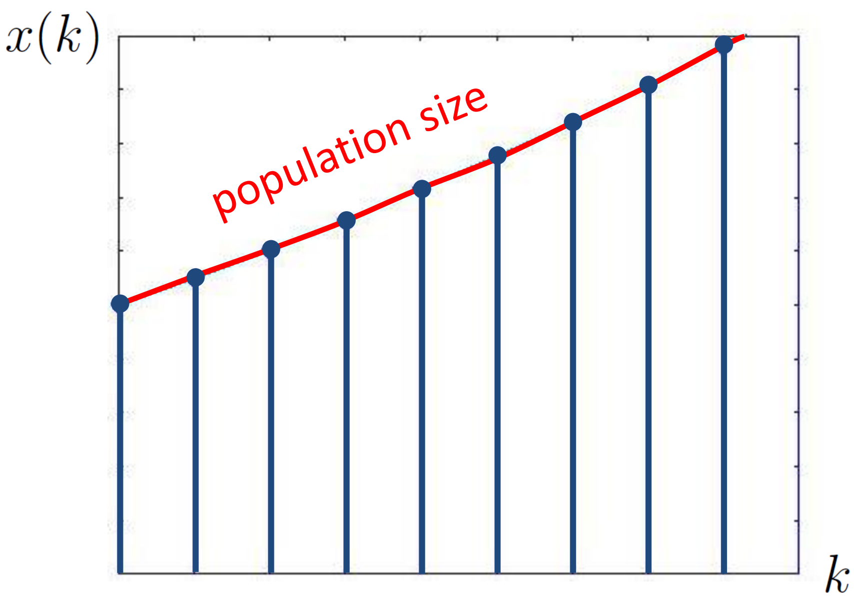
b birth (births/population) rate

m mortality (deaths/population) rate

$r = b - m$ population growth rate

$$x(k+1) = x(k) + bx(k) - mx(k) = (1+r)x(k)$$

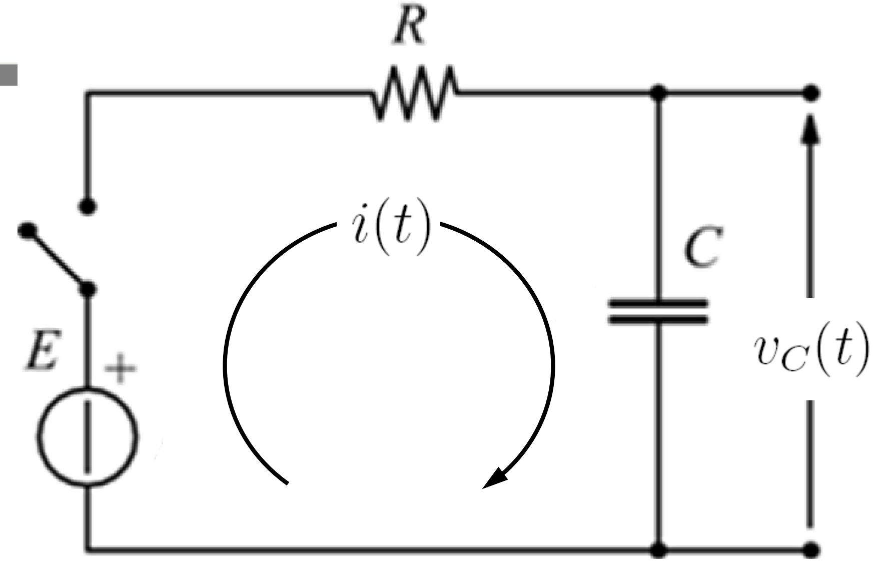




The size of the population changes continuously
but it is observed at regular intervals of time

Continuous-time continuous-state systems

Capacitor charging



$x(t) = v_C(t)$ voltage across the capacitor

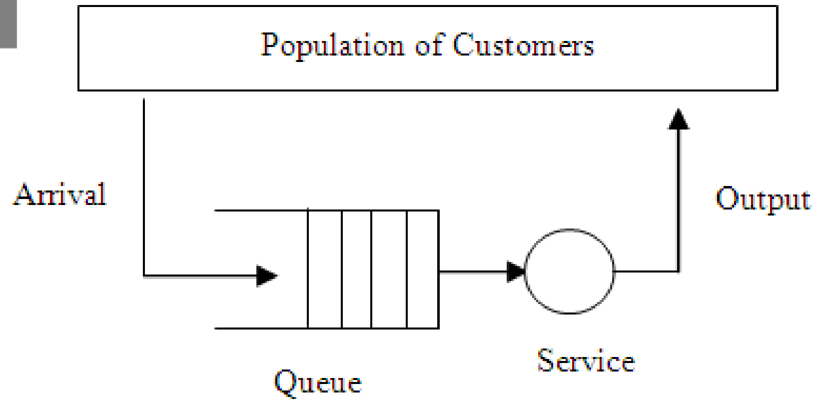
$u(t) = E$ DC source of voltage

$$\left.\begin{aligned} i(t) &= C\frac{dv_C(t)}{dt} \\ v_R(t) &= Ri(t) \\ E &= v_R(t) + v_C(t) \end{aligned}\right\} \quad \dot{x}(t) = -\frac{1}{RC}x(t) + \frac{1}{RC}u(t)$$

Event-driven discrete-state systems

Queueing system

with FIFO service

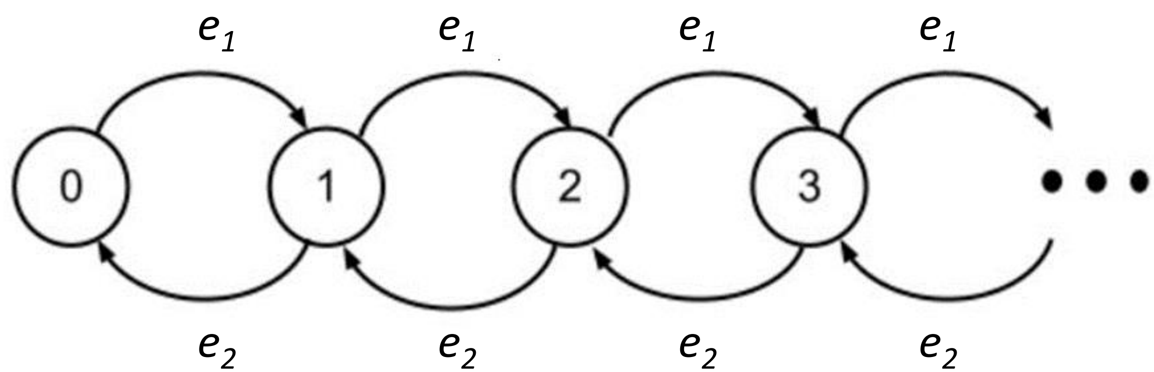


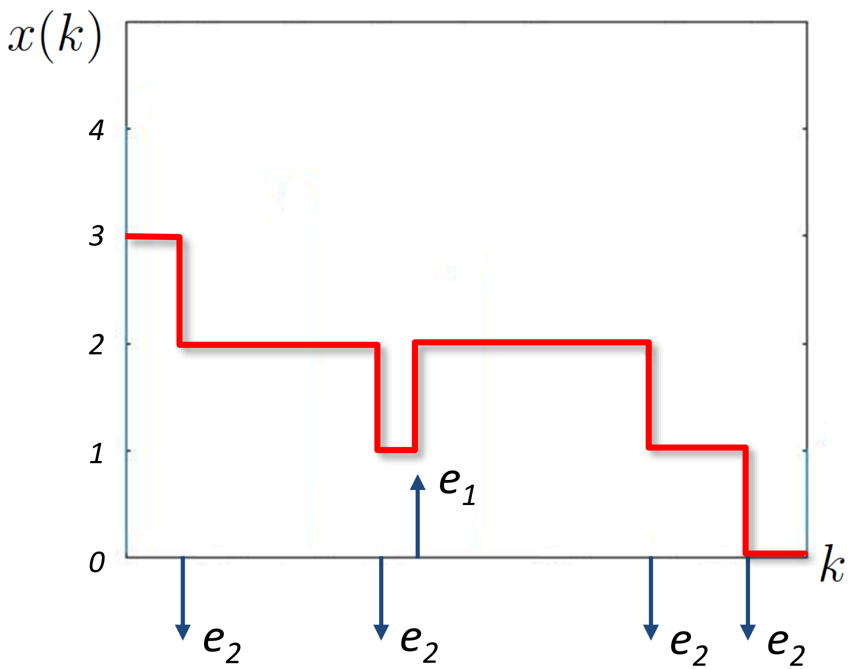
$x(k)$ Number of customers in the queue after the *k*-th event

The state of the system remains unchanged except at the following **events**

- e_1 : arrival times *t* of customers: $x(k+1) = x(k) + 1$
- e_2 : departure times t of customers: $x(k+1) = x(k) - 1$

State transitions are asynchronous, NOT synchronized by a clock



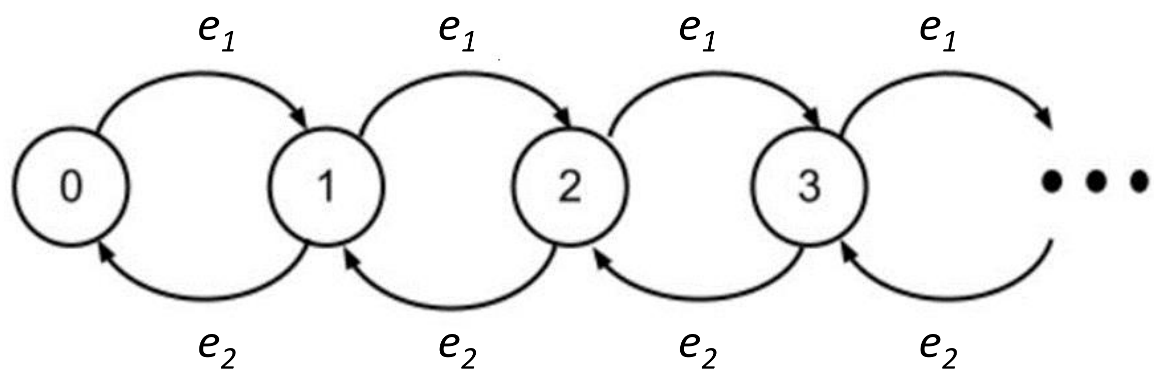


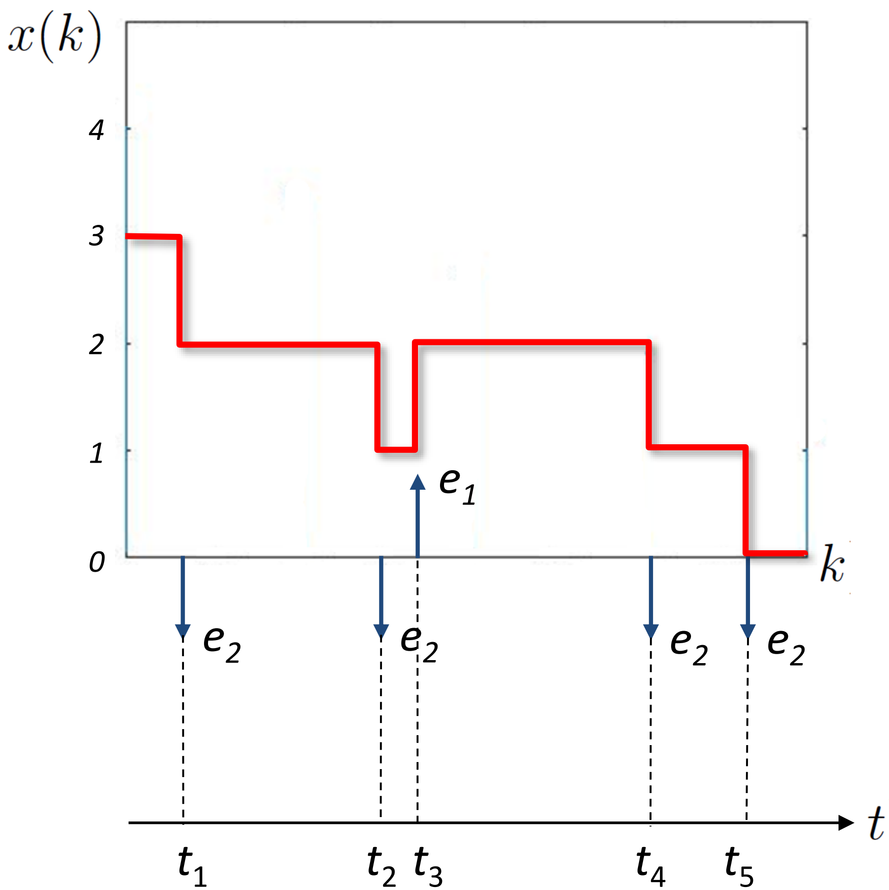
Untimed models (logical behavior)
The event sequence

$\{e_2, e_2, e_1, \ldots\}$

(without information about the occurrence times), determines the sequence of states

$\{3, 2, 1, \ldots\}$





Untimed models (logical behavior)
The event sequence

$$\{e_2, e_2, e_1, \ldots\}$$

(without information about the occurrence times), determines the sequence of states

$$\{3, 2, 1, \ldots\}$$

Timed models (quantitative behavior)
The timed event sequence

$$\{(e_2, t_1), \{(e_2, t_2), , \ldots\}$$

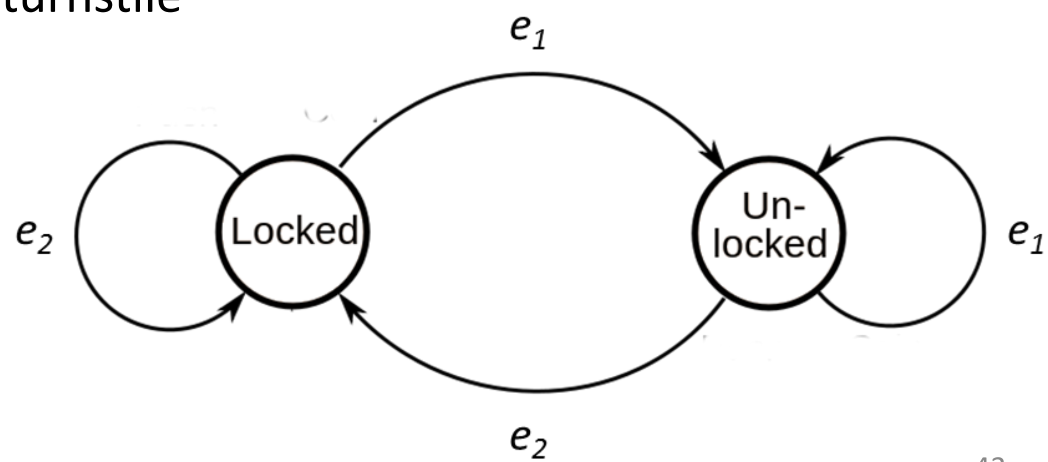
determines the entire state path over time.

Event-driven finite-state systems

Coin-operated turnstile

The state of the system can assume only two values (Locked and Unlocked) and remains unchanged except at the following **events**

- e_1 : putting a coin in the slot
- e_2 : pushing the arm of the turnstile



Finite automata

FSM (finite state machine)

A finite automaton is a mathematical model of an event-driven finite-state system:

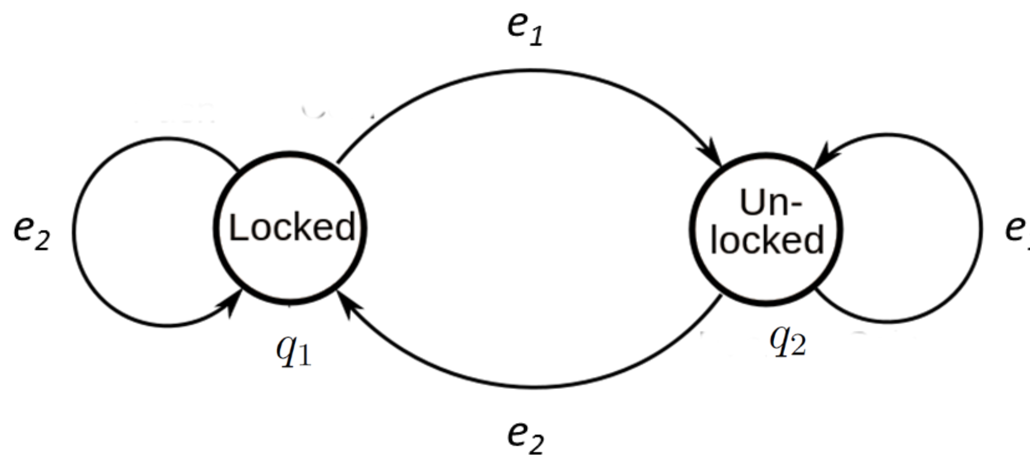
$$M = (Q, \Sigma, \delta)$$

- $Q = \{q_1, q_2, \ldots\}$ is a set of **discrete states**
- $\Sigma = \{e_1, e_2, \ldots, e_m\}$ finite set (alphabet) of input symbols (**events**)
- $\delta : Q \times \Sigma \to 2^Q$ transition (set-valued) function

$\delta(q, e) \subseteq Q$ is the set of all states the system can transit to from state *q* under the input symbol *e*

2^Q denotes the power set of Q, i.e. the set of all subsets of Q

$$M = (Q, \Sigma, \delta)$$



$$Q = \{q_1, q_2\}$$

$$\Sigma = \{e_1, e_2\}$$

$$\delta : Q \times \Sigma \to 2^Q$$

	e_1	e_2
q_1	q_2	q_1
q_2	q_2	q_1

Finite automata

$$M = (Q, \Sigma, \delta)$$

Given an initial set of states $Init \subseteq Q$

M *accepts an input sequence* $\{\sigma_0, \sigma_1, \sigma_2, \ldots\}$ with $\sigma_k \in \Sigma$, if

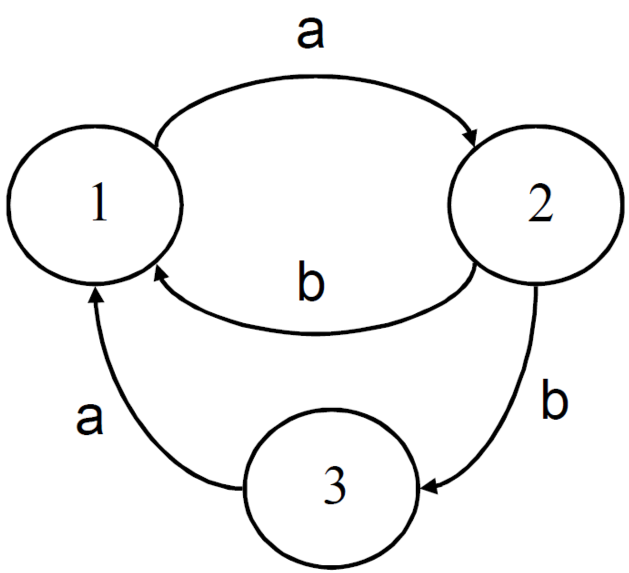
there exists a sequence of states $\{s_0, s_1, s_2, \ldots\}$ wit $s_k \in Q$ such that:

- $s_0 \in Init$
- $s_{k+1} \in \delta(s_k, \sigma_k)$

$\{s_0, s_1, s_2, \ldots\}$ in the definition above is called an execution of M under the input sequence $\{\sigma_0, \sigma_1, \sigma_2, \ldots\}$

Finite automata

$$Init = \{1\}$$



$\{1, 2, 1\}$ is an execution under input $\{a, b\}$

$\{1, 2, 3, 1, 2\}$ under input $\{a, b, a, a\}$

$\{a, b, b\}$ is not accepted as an input

Given $Init \subseteq Q$, the **language** $\mathcal{L}(M)$ accepted by M

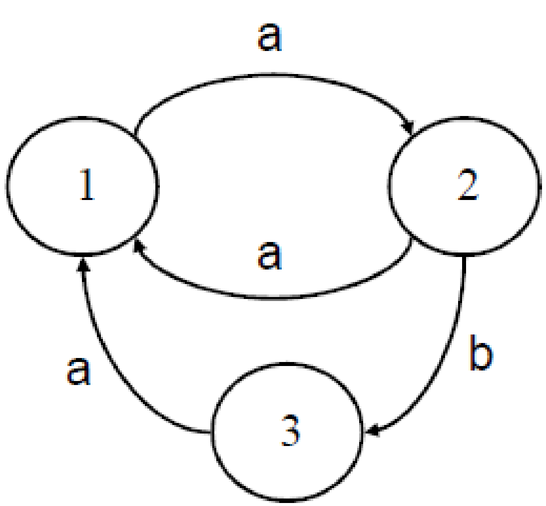
is the set of all sequences of input symbols that are accepted by M with

$Init$ as initial set of states

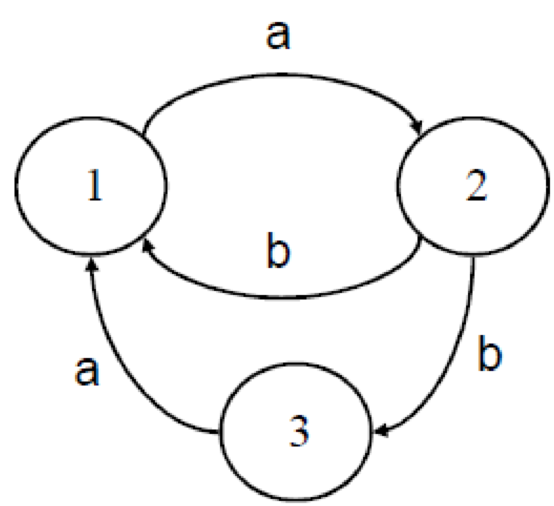
Finite automata

A finite automaton M is **deterministic** if there is at most one execution for any initial state and any input string

Otherwise, it is **nondeterministic**



deterministic

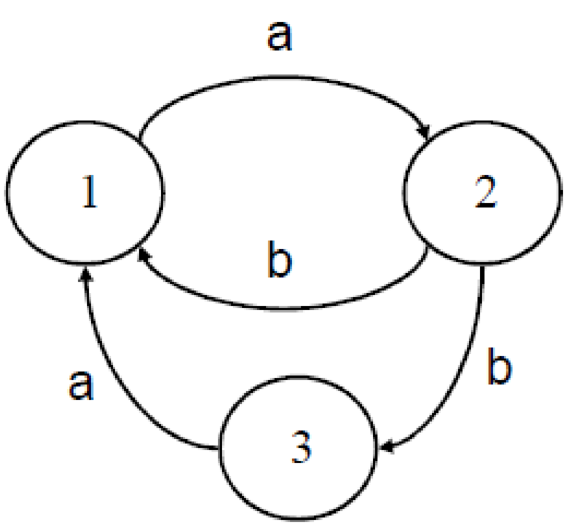


nondeterministic

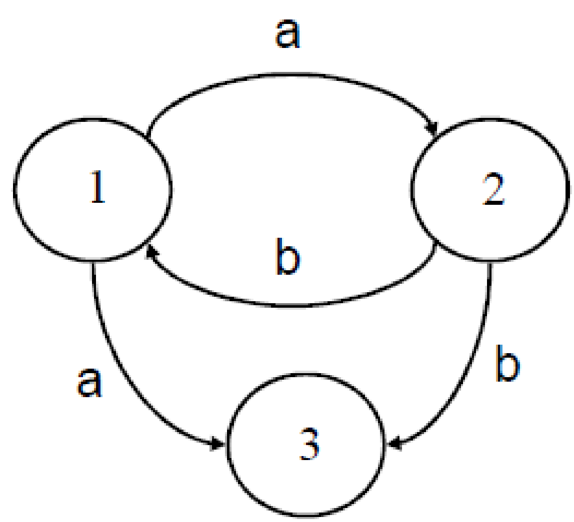
Finite automata

A finite automaton $M = (Q, \Sigma, \delta)$ has a **blocking state** $q \in Q$ if

$\delta(q, e) = \emptyset$, for all $e \in \Sigma$



no blocking state



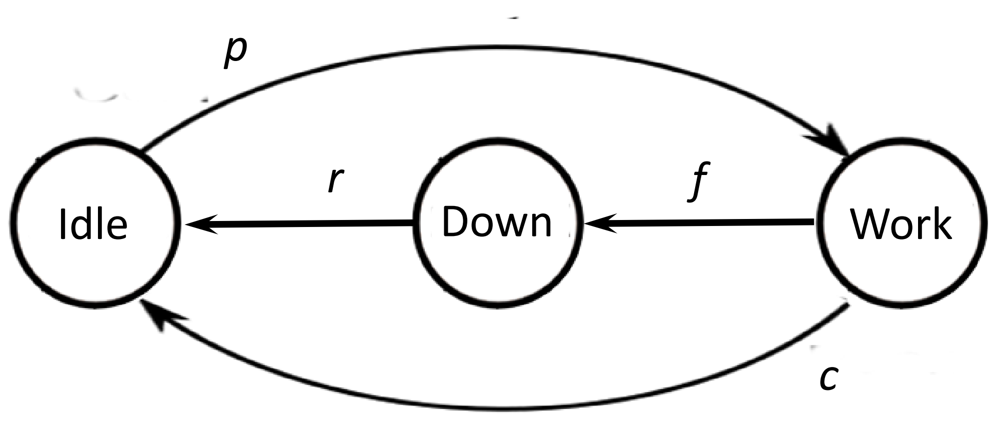
blocking state: q = 3

Event-driven finite-state systems

Manufacturing machine

The machine can be: *Idle*, *Working* or *Down*
Transitions between states depend on the following **events**:

- *p* : if the machine is *Idle* and a part arrives, it will start *Working the part*
- *c* : while the machine is *Working* it may complete the part and become *Idle*
- *f* : while the machine is *Working* it may break *Down*
- *r* : while the machine is *Down* it may repaired and become *Idle*

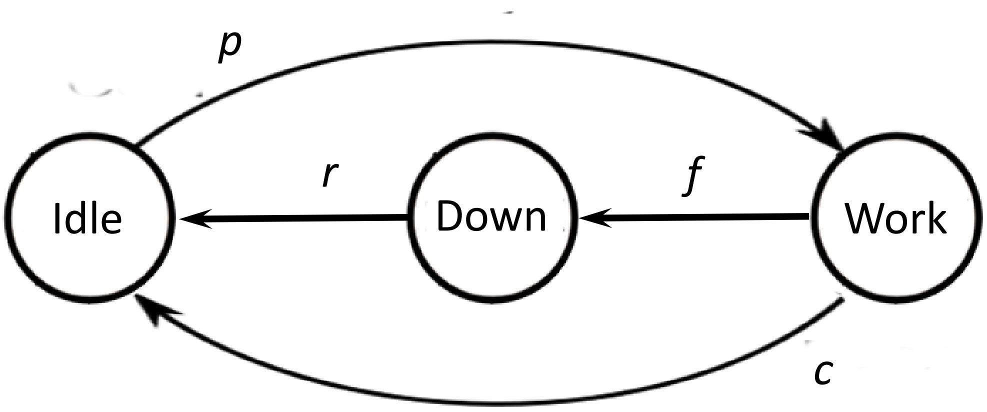


Some **events** are not accepted in some states:

- the machine cannot process a part when it is *Down*
- the machine cannot process a new part when it is *Working*
- …

Not all the sequences of **events** are acceptable by the machine

The finite automaton is **deterministic** and **nonblocking**

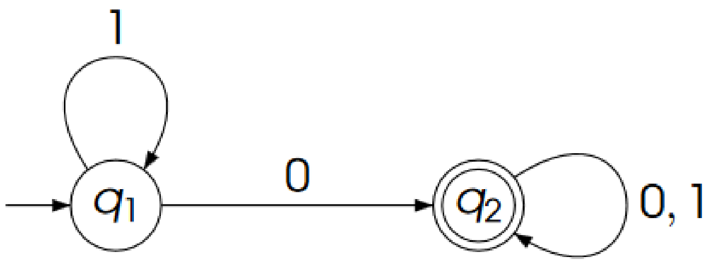


When the initial state is *Idle, the language of the machine is (**regular expression**)*

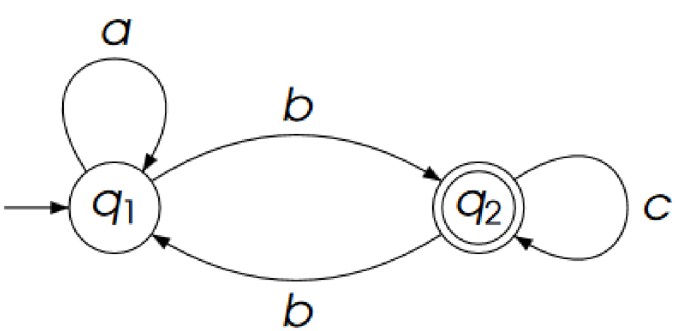
$$(pc + pfr)^*(1 + p + pf)$$

Time to exercise

Compute the **regular expressions** of the following machines



$1^*0(0+1)^*$



$(a+bc^*b)^*bc^*$

Linear time-invariant systems (time-driven)

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad\qquad x \in I\!R^n \qquad u \in I\!R^p$$

$$x(k+1) = Ax(k) + Bu(k)$$

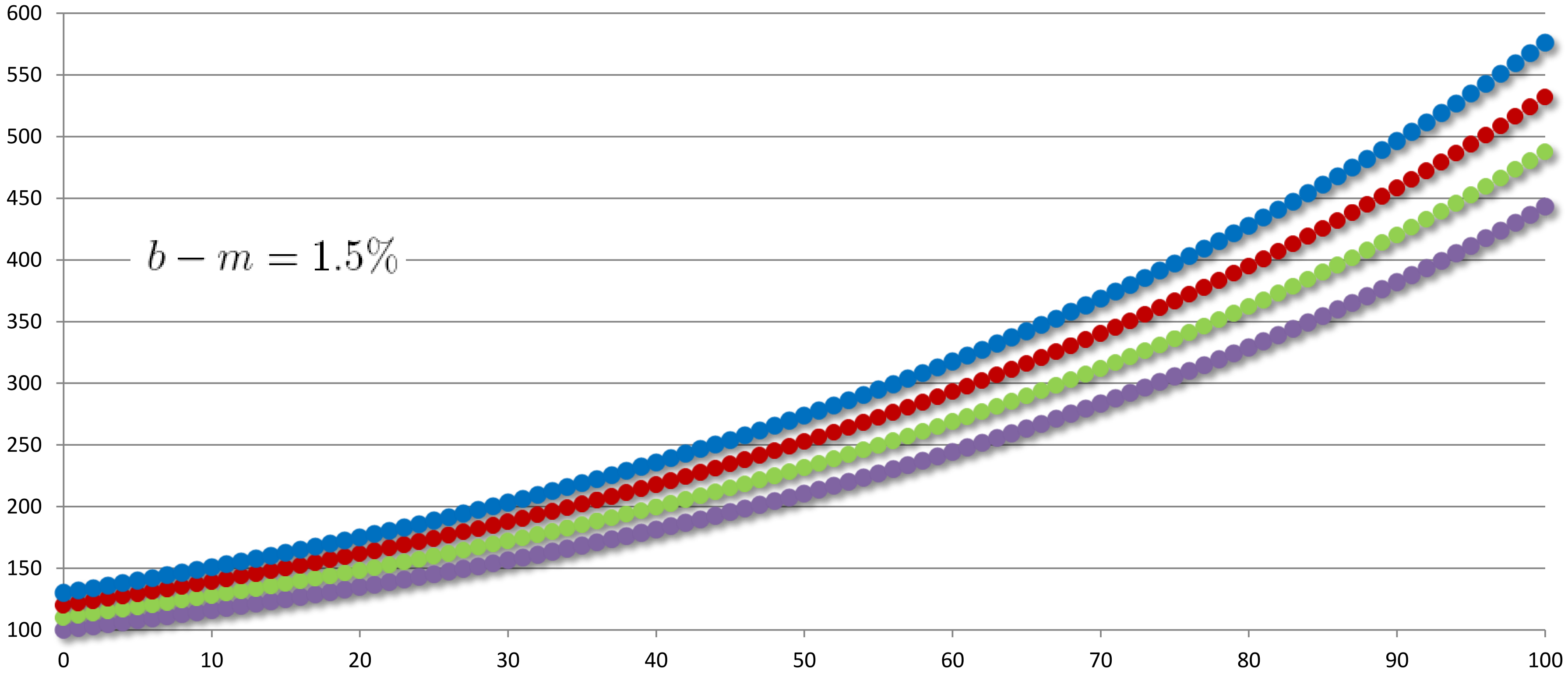
Global existence and uniqueness

$\forall x_0$ there exists a single solution with $x(0) = x_0$ defined on $[0, \ \infty)$.

The solutions depend continuously from the initial conditions x_0 .

Continuous dependence of the solutions from the initial conditions

$$x(k+1) = (1 + b - m)x(k)$$



Exponential population growth

Superposition principle

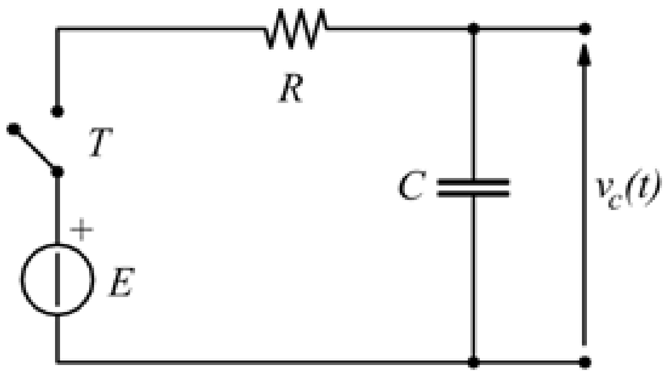
a *linear combination* of solutions is again a *solution*

$$x(t) = x_f(t) + x_F(t)$$

$x(0)$ $\quad$ $u(t)$

$$x(k) = x_f(k) + x_F(k)$$

$x(0)$ $\quad$ $u(k)$



$$\dot{x}(t) = -\frac{1}{RC}x(t) + \frac{1}{RC}u(t)$$

$u(t) = E$ DC source of voltage

$$x(t) = x(0)e^{-\frac{t}{RC}} + E\left(1 - e^{-\frac{t}{RC}}\right)$$

$$x(k+1) = (1+r)x(k) - u(k)$$

$u(k) = P$ fixed monthly payment

$$x(k) = x(0)(1+r)^k + \frac{P}{r}\left(1 - (1+r)^k\right)$$

Superposition principle on the free evolution

$$\dot{x}(t) = Ax(t) \qquad\qquad x(t) = \sum_{i=1}^{n} c_i \; e^{\lambda_i t} v_i$$

$$x(k+1) = Ax(k) \qquad\qquad x(k) = \sum_{i=1}^{n} c_i \; \lambda_i^k v_i$$

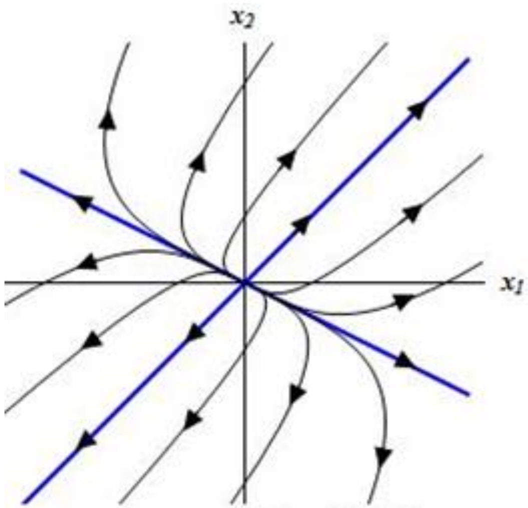
Nice properties:

- Unique and constant equilibrium
- Stability directly given by the eigenvalues of *A*
- Local properties = global properties (like stability, etc.)

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$

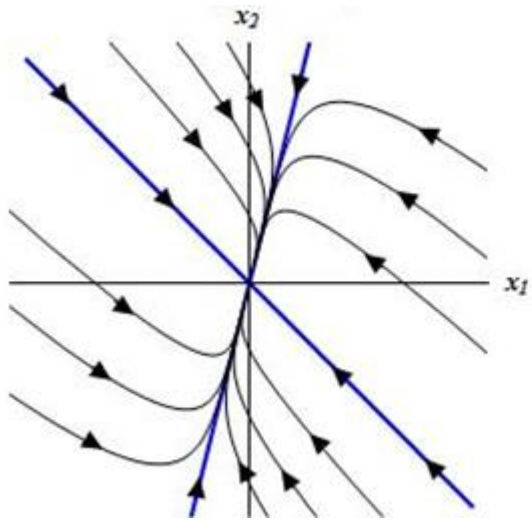
$$(A - \lambda_1 I)\, v_1 = 0$$

$$(A - \lambda_2 I)\, v_2 = 0$$



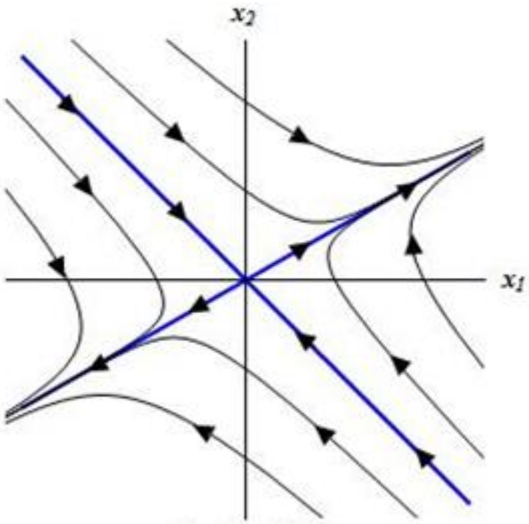
$\lambda_1, \lambda_2 > 0$

Unstable node



$\lambda_1, \lambda_2 < 0$

Asymptotically stable node



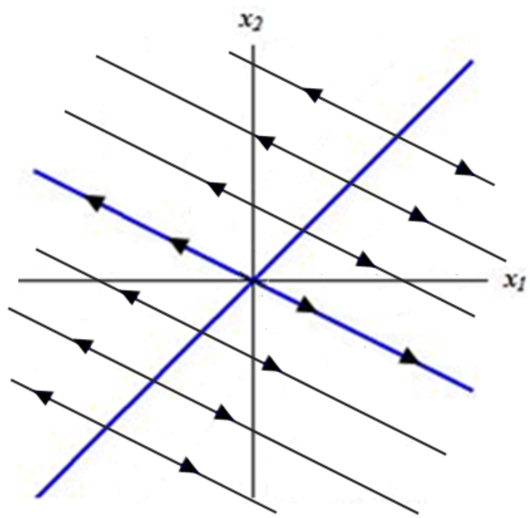
$\lambda_1 < 0 < \lambda_2$

Unstable saddle point

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$

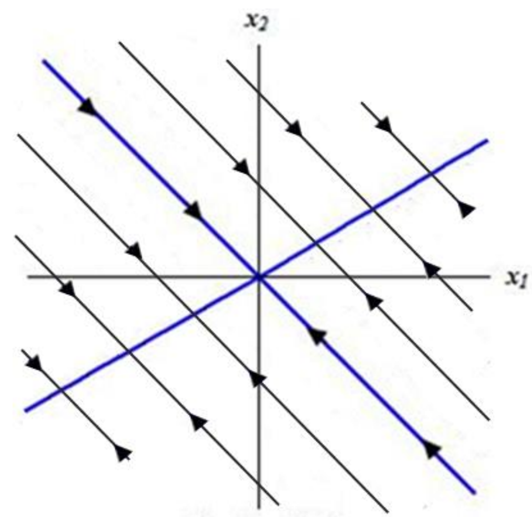
$$(A - \lambda_1 I)\, v_1 = 0$$

$$(A - \lambda_2 I)\, v_2 = 0$$



$$\lambda_1 > \lambda_2 = 0$$

Unstable linear subspace



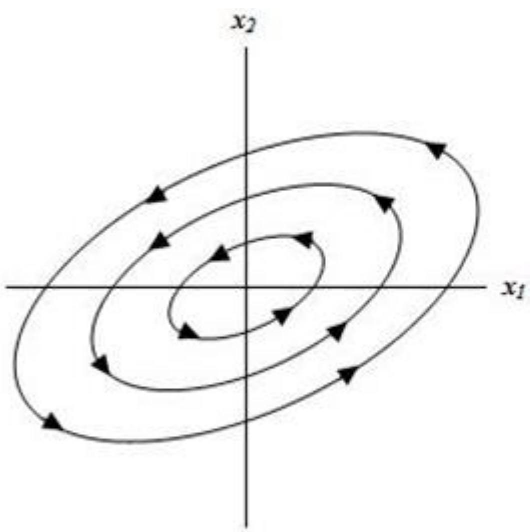
$$\lambda_1 < \lambda_2 = 0$$

Stable linear subspace

$$x(t) = me^{\alpha t}\left(\sin(\omega t + \varphi)v_R + \cos(\omega t + \varphi)v_I\right)$$

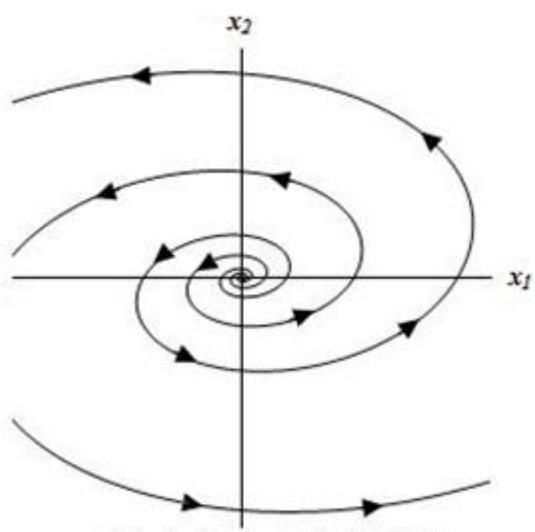
$$\alpha \pm j\omega$$

$$v_R \pm jv_I$$



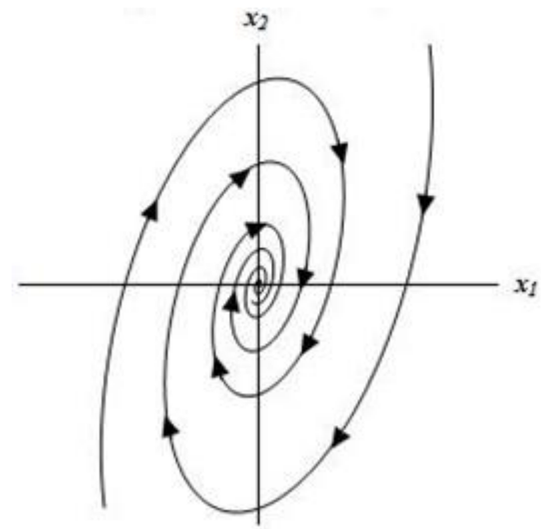
$\alpha = 0$

Stable center



$\alpha > 0$

Unstable spiral



$\alpha < 0$

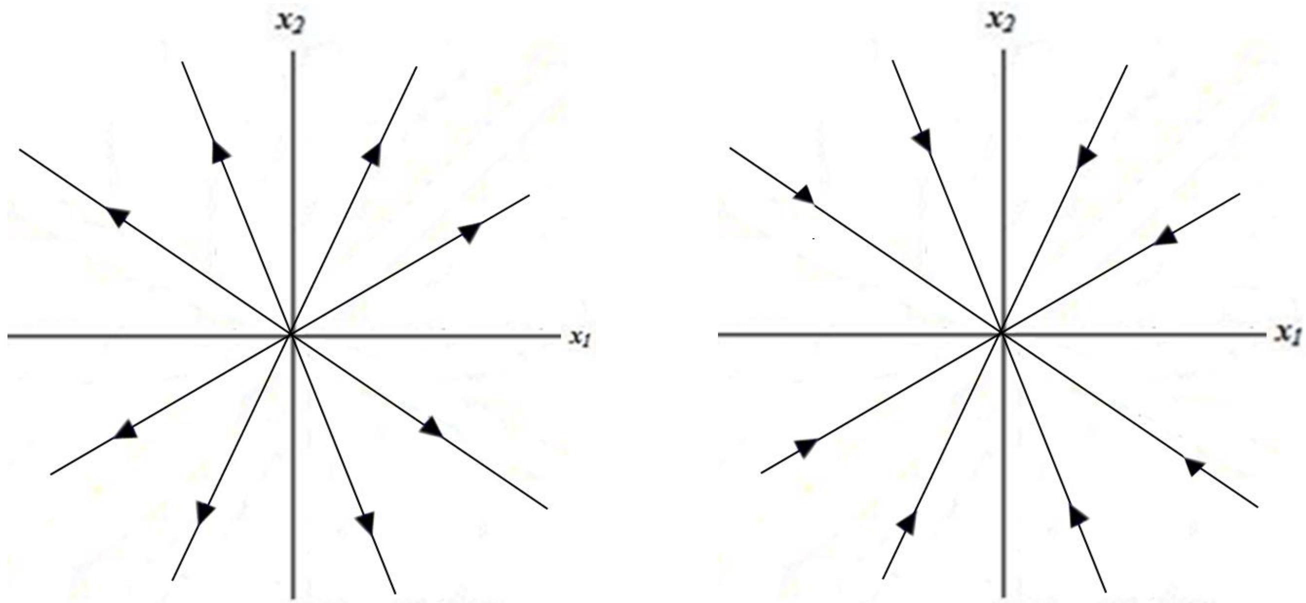
Asymptotically stable spiral

Repeated eigenvalues

$$(A - \lambda I)\, v_1 = 0$$

$$(A - \lambda I)\, v_2 = 0$$

algebraic multiplicity = **geometric** multiplicity



$\lambda > 0$ $\lambda < 0$

geometric multiplicity of an eigenvalue =
number of linearly independent eigenvectors for the eigenvalue.

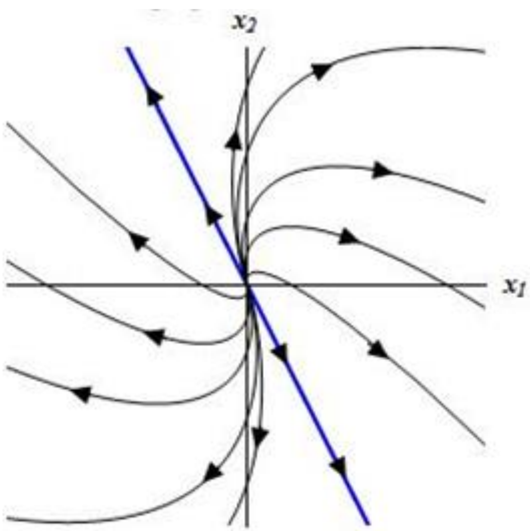
Repeated eigenvalues

$$x(t) = c_1 e^{\lambda t} v_1 + c_2 \left(t e^{\lambda t} v_1 + e^{\lambda t} v_2 \right)$$

$$(A - \lambda I)\, v_1 = 0$$

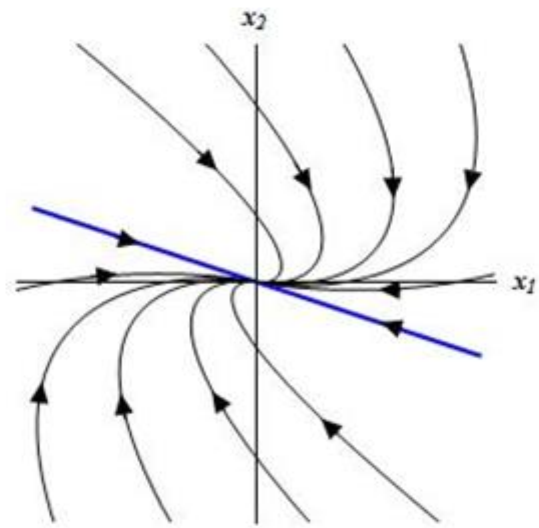
$$(A - \lambda I)\, v_2 = v_1$$

algebraic multiplicity > **geometric** multiplicity



$\lambda > 0$

Unstable improper node



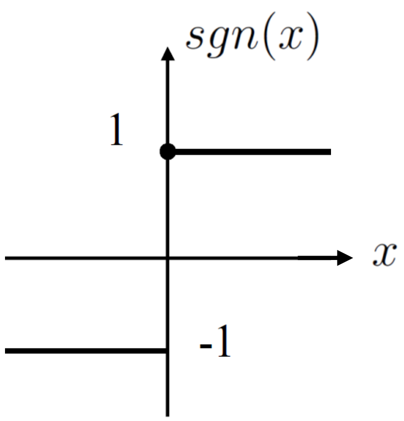
$\lambda < 0$

Asymptotically stable improper node

Linear vs nonlinear

Existence of solutions

A solution may not exist



$$\dot{x}(t) = -sgn(x(t)) \qquad x(0) = 0$$

on any interval $[0,\ \epsilon)$, $x(t)$ cannot remain zero, become positive, or become negative

Theorem [local existence]

$$\dot{x}(t) = f(x(t))$$

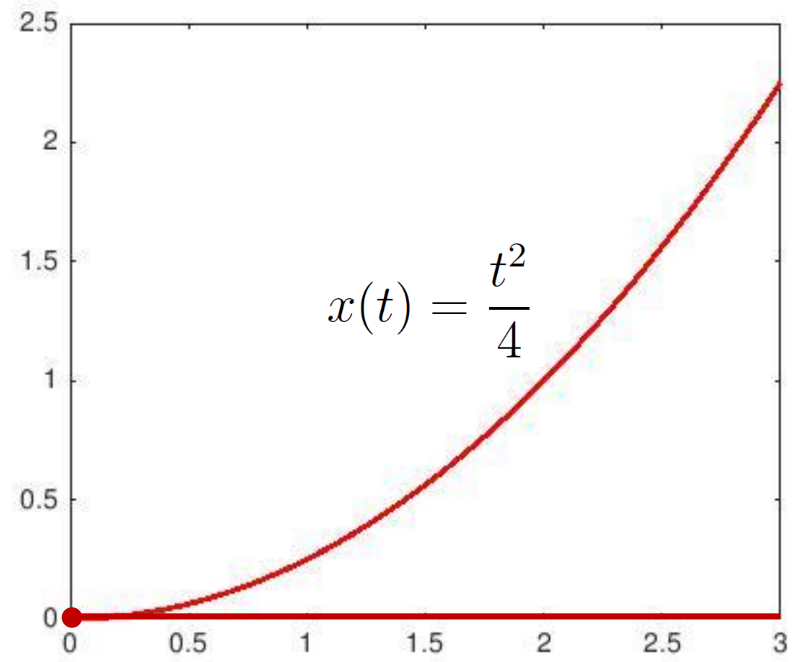
If $f : I\!R^n \to I\!R^n$ is continuous, then $\forall x_0$ there exists at least a solution with $x(0) = x_0$ defined on some $[0,\ \epsilon)$.

Linear vs nonlinear

Multiple solutions may exist

$$\dot{x}(t) = \sqrt{x(t)} \qquad x(0) = 0$$

Uniqueness of solutions

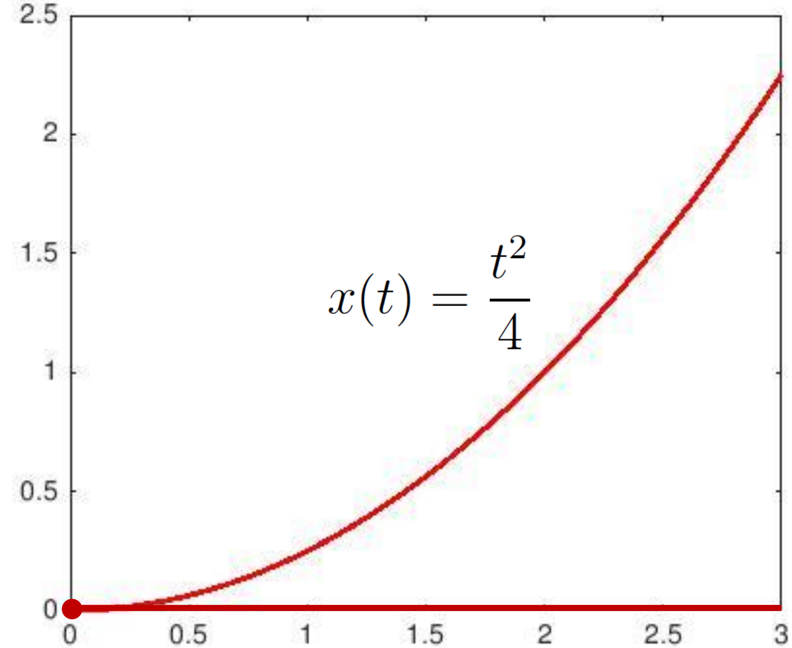


Linear vs nonlinear

Uniqueness of solutions

Multiple solutions may exist

$$\dot{x}(t) = \sqrt{x(t)} \qquad x(0) = 0$$



Theorem [local existence and uniqueness]

If $f : I\!R^n \rightarrow I\!R^n$ is Lipschitz continuous, then $\forall x_0$ there exists a single solution with $x(0) = x_0$ defined on some $[0, \ \epsilon)$.

$$\|f(x_1) - f(x_2)\| \leq L \|x_1 - x_2\| \, , \quad \forall x_1, x_2 \in \text{any bounded set of } I\!R^n$$

L depending on the set

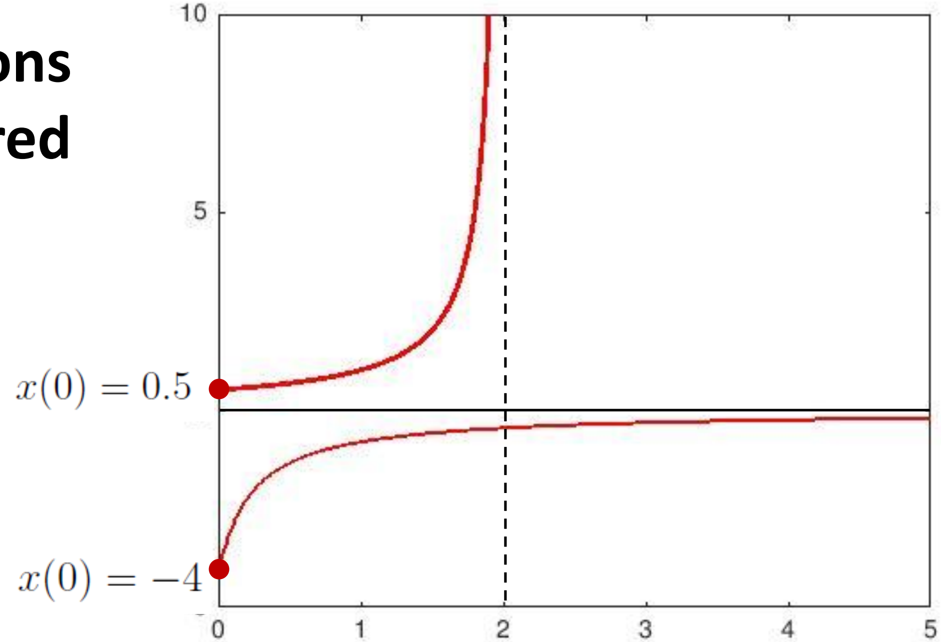
Linear vs nonlinear

Global Existence and uniqueness of solutions

Global existence of solutions is not ensured

$$\dot{x}(t) = x(t)^2$$

$$x(t) = \frac{x(0)^2}{1 - x(0)t}$$



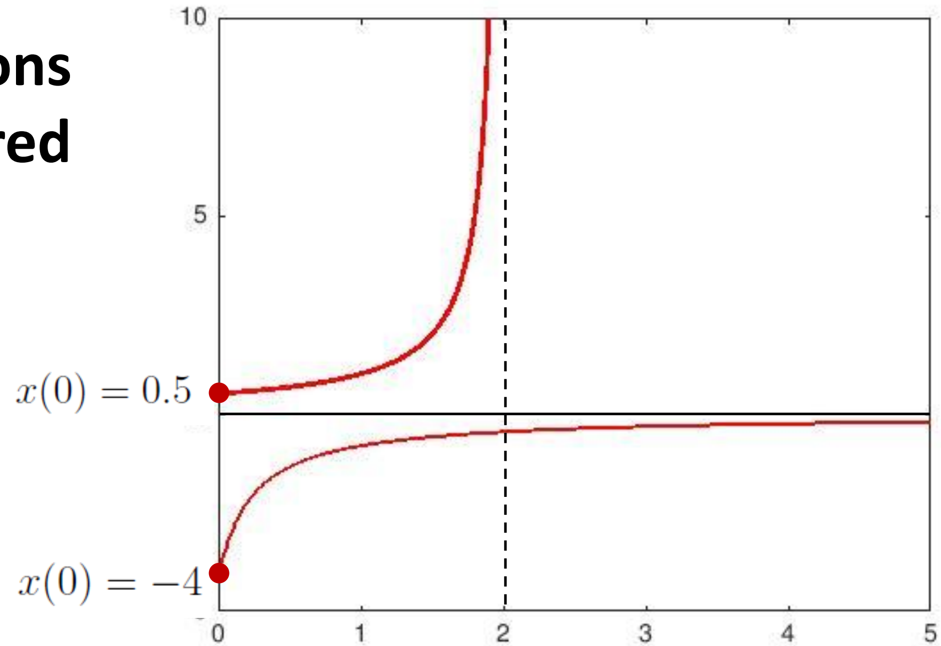
Linear vs nonlinear

Global Existence and uniqueness of solutions

Global existence of solutions is not ensured

$$\dot{x}(t) = x(t)^2$$

$$x(t) = \frac{x(0)^2}{1 - x(0)t}$$



Theorem [global existence and uniqueness]

If $f : I\!R^n \to I\!R^n$ is globally Lipschitz continuous, then $\forall x_0$ there exists a single solution with $x(0) = x_0$ defined on $[0,\ \infty)$.

$$\|f(x_1) - f(x_2)\| \leq L\,\|x_1 - x_2\|\,, \quad \forall x_1, x_2 \in I\!R^n$$

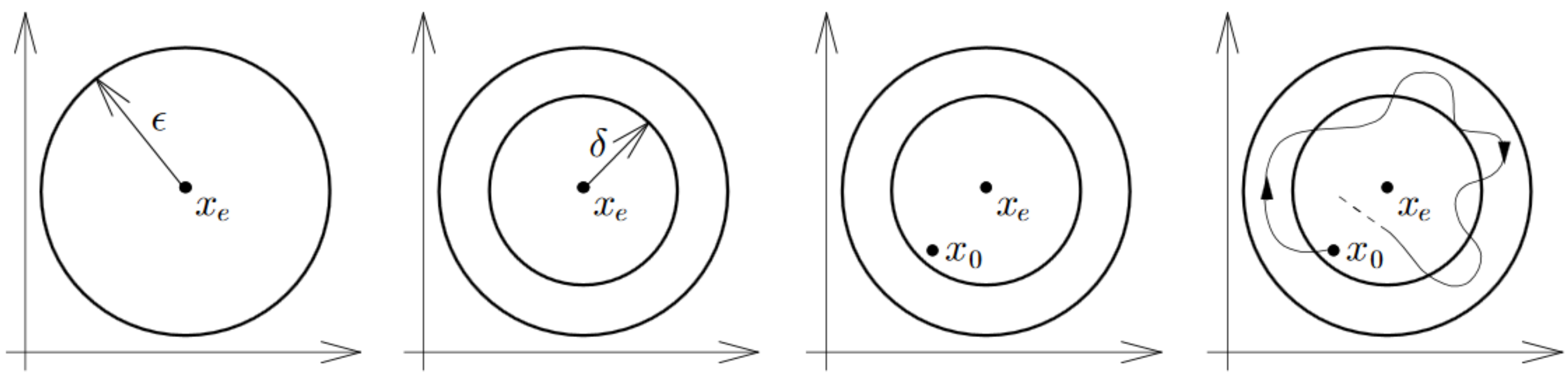
Linear vs nonlinear

Equilibria and stability

The equilibrium point x_e is **stable** if

$$\forall\epsilon, \exists\,\delta(\epsilon) : |x_0 - x_e| < \delta \Rightarrow |x(t) - x_e| < \epsilon, \forall t > 0$$

small perturbations lead to small changes in behavior



$\forall\epsilon$ | $\exists\,\delta(\epsilon)$ | $|x_0 - x_e| < \delta$ | $|x(t) - x_e| < \epsilon, \forall t > 0$

The equilibrium point x_e is **asymptotically stable** if

- it is stable
- $\exists\,\delta_a : |x_0 - x_e| < \delta_a \Rightarrow \lim_{t\to\infty} |x(t) - x_e| = 0$

small perturbations lead to small changes in behavior and are re-absorbed, in the long run

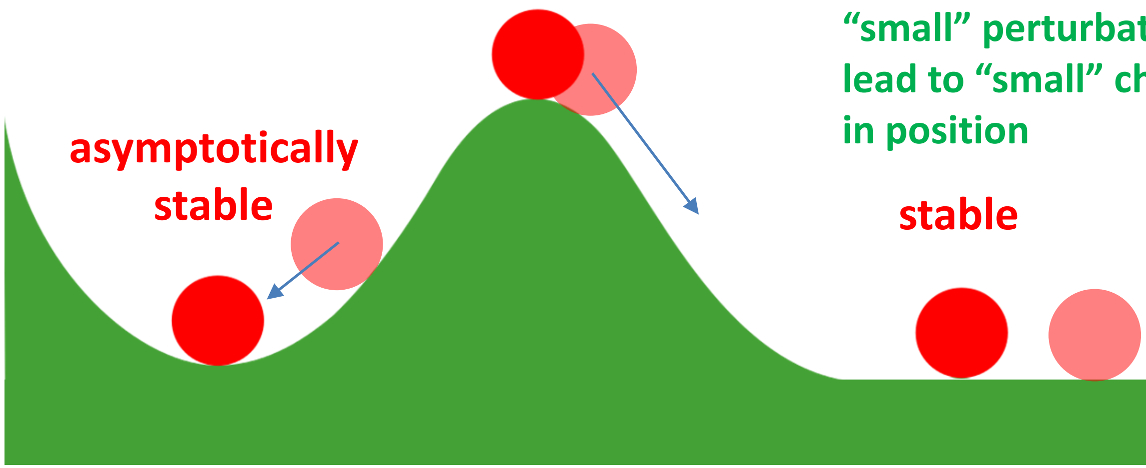
"small" perturbations
lead to "big" changes
in position

unstable

"small" perturbations
lead to "small" changes
in position

asymptotically
stable

stable



"small" perturbations lead to
"small" changes in position and
are re-absorbed, in the long run

One of the major discoveries of the twentieth century was that deterministic systems could be inherently unpredictable.

The Butterfly effect

Predictability: does the flap of a butterfly's wings in Brazil set off a tornado in Texas ?

Sensitivity to initial conditions

Systems that are sensitive to initial conditions and bounded are said to be *chaotic*

A small change in the present state can result in large differences in a later state.

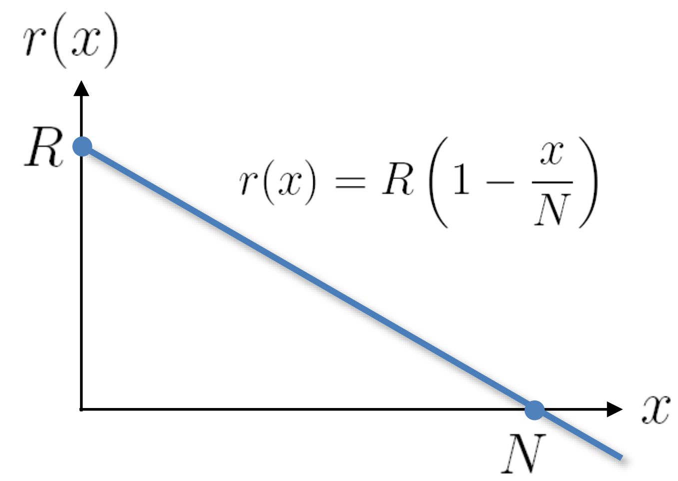
The logistic map

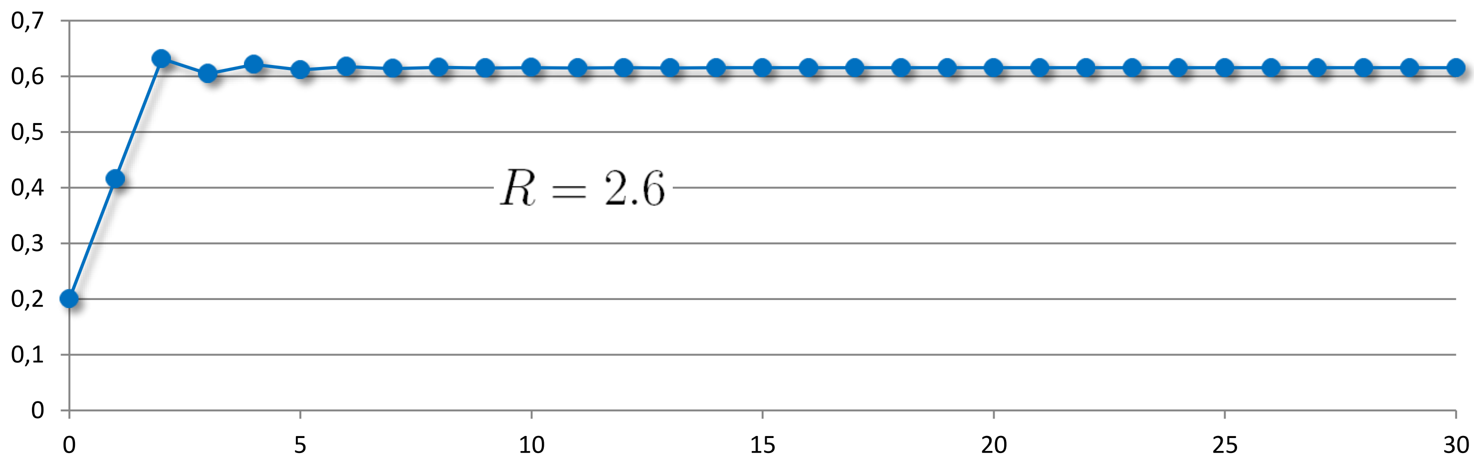
$$x(k) \to x(k)/N$$

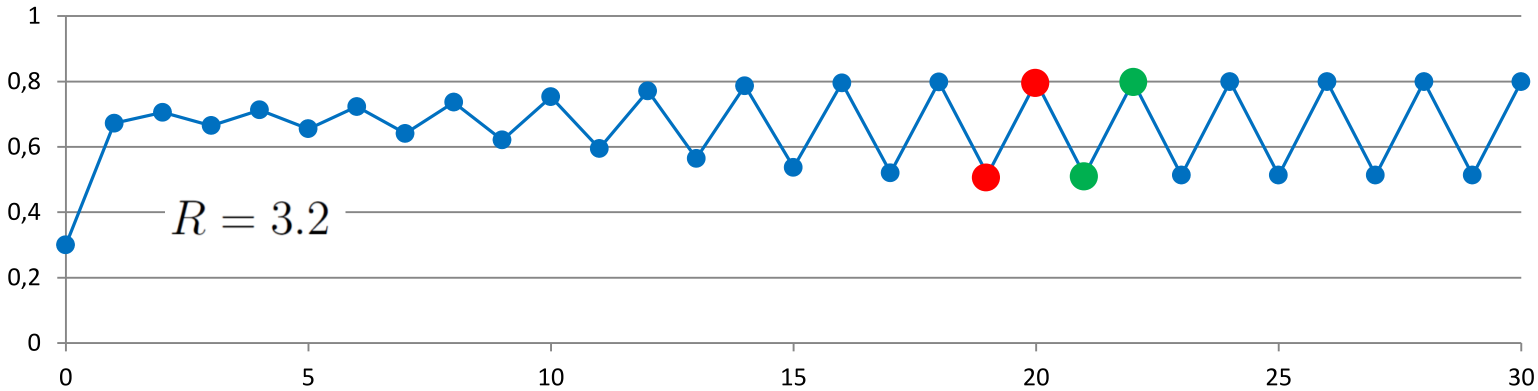
$$x(k+1) = r(x)\ x(k)$$

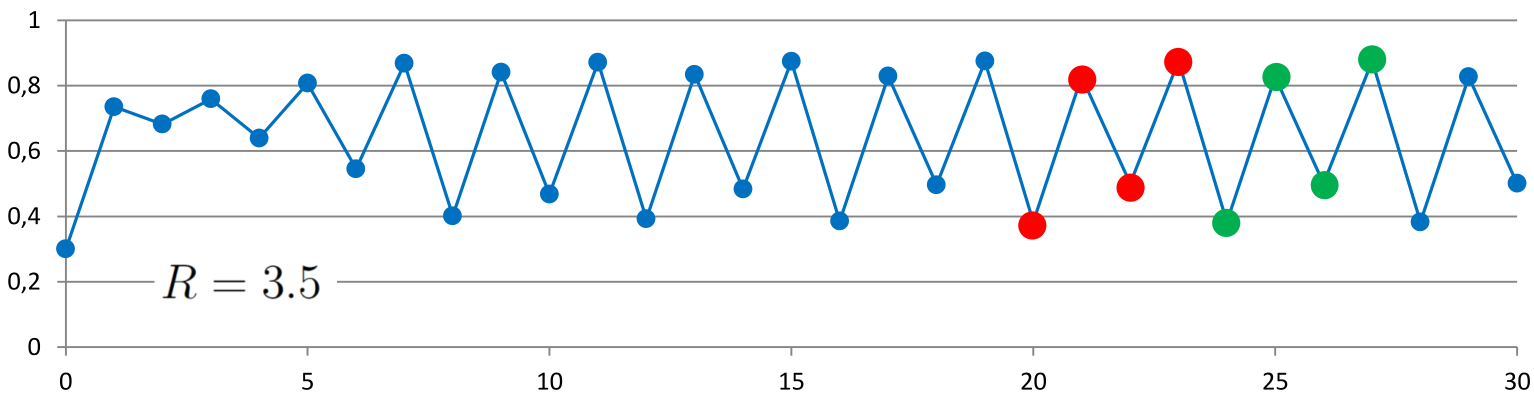
$$x(k+1) = R\ x(k)\,(1 - x(k))$$

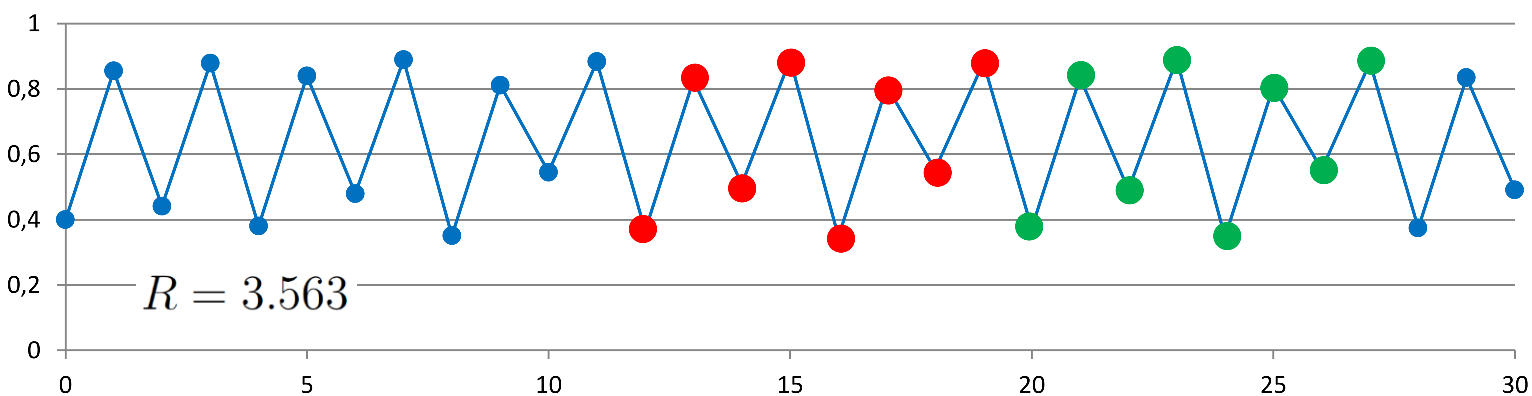
population growth rate

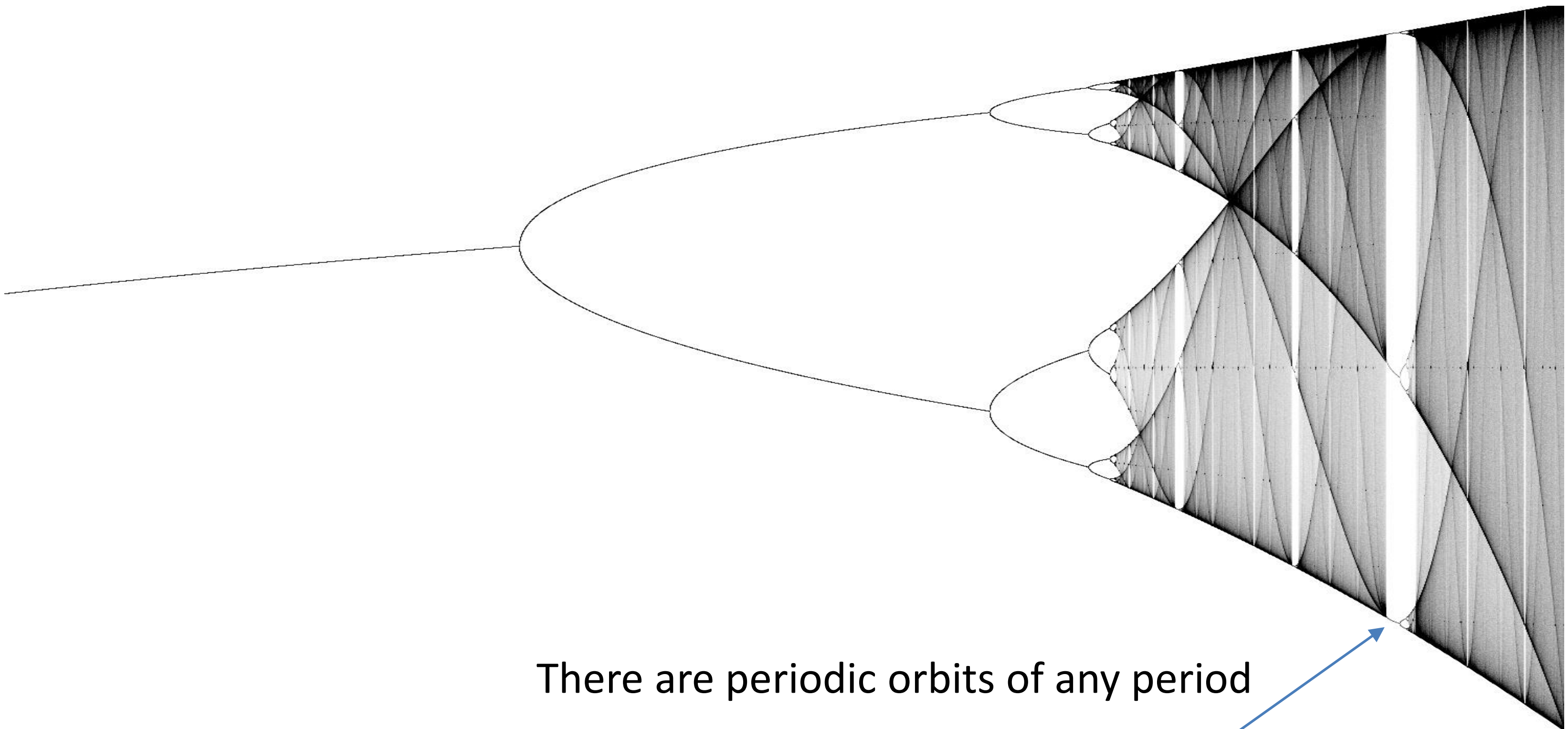




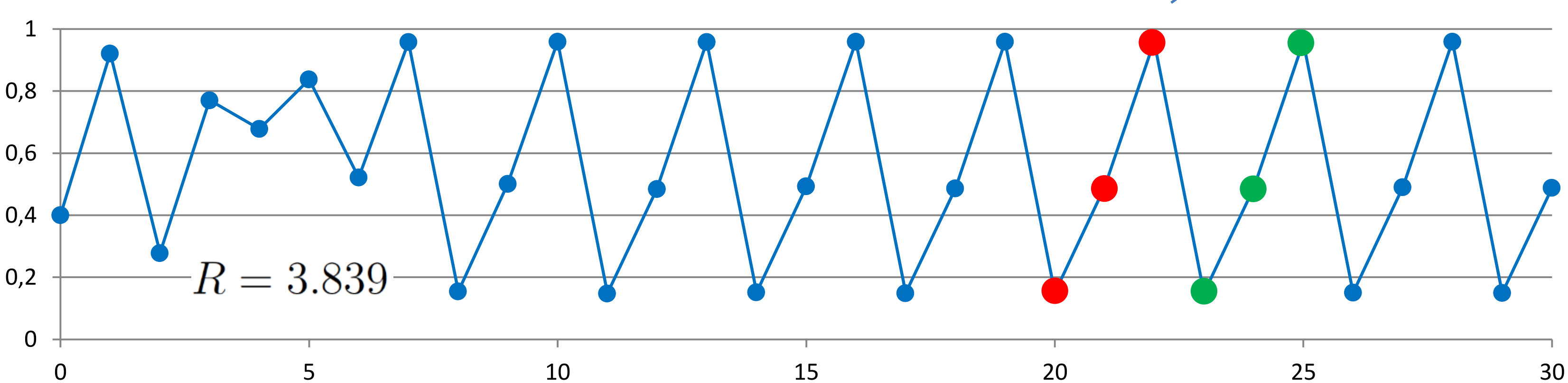

1
0,8
0,6
0,4
0,2
0
0
5
10
15
20
25
30
$R = 3.2$


1
0,8
0,6
0,4
0,2
0
0
5
10
15
20
25
30
$R = 3.5$


1
0,8
0,6
0,4
0,2
0
0
5
10
15
20
25
30
$R = 3.563$

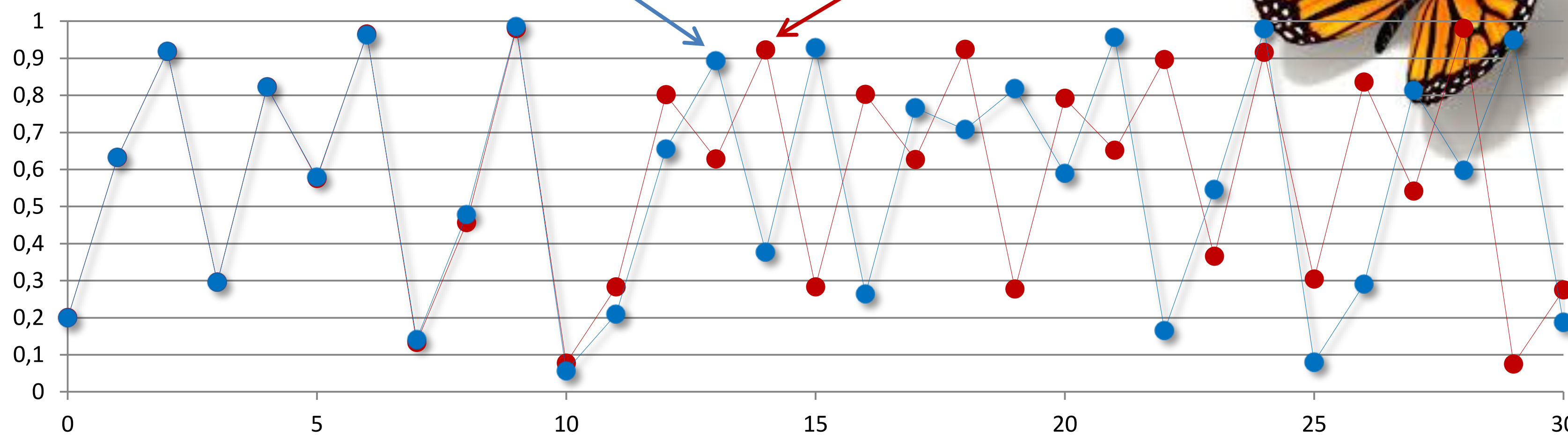


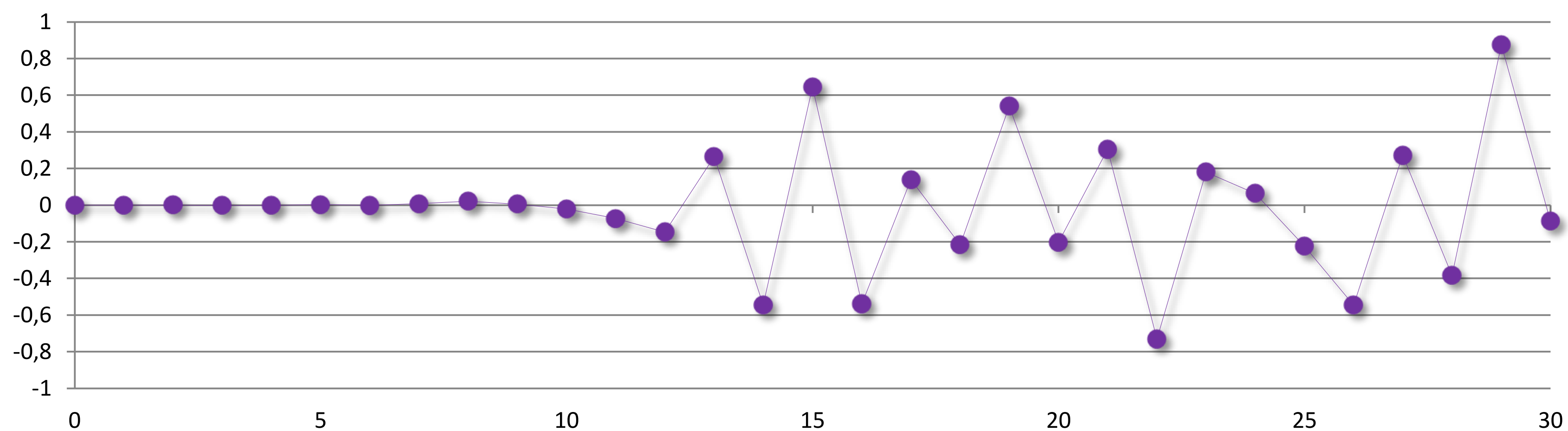
There are periodic orbits of any period



$$x(k+1) = R\, x(k)\,(1 - x(k)) \qquad R = 3.95$$

$x(0) = 0.2$ $\qquad$ $x(0) = 0.2001$





$$x(k+1) = R\,x(k)\,(1 - x(k)) \qquad R = 4$$

$x(0) = 0.4$ $x(0) = 0.4001$