Multimedia communications

to become

Multiresolution analysis: theory and applications

Comunicazione multimediale

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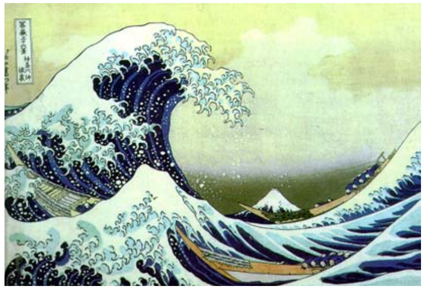


Course overview

Course structure

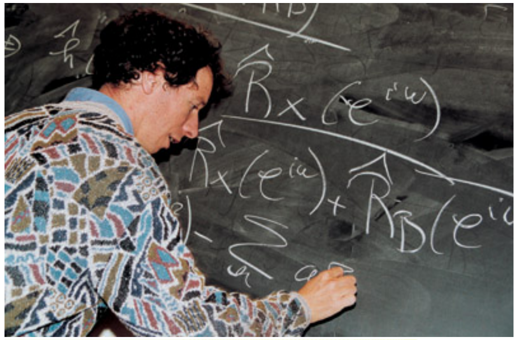
- The course is about wavelets and multiresolution
  - Theory: 4 hours per week
    - Mon. **11.30-13.00**, room I
    - Thu. 16.30-18.00, room C
  - Laboratory
    - Mon**. 14.00-15.30** (Lab. Gamma) LM32
    - Fri. 14.00-15.30 (Lab. Alpha) LM9
- Exam
  - Theory: Oral (in general)
  - Lab: Evaluation of lab. sessions and questions during the exam
  - Projects: only in case of diploma project

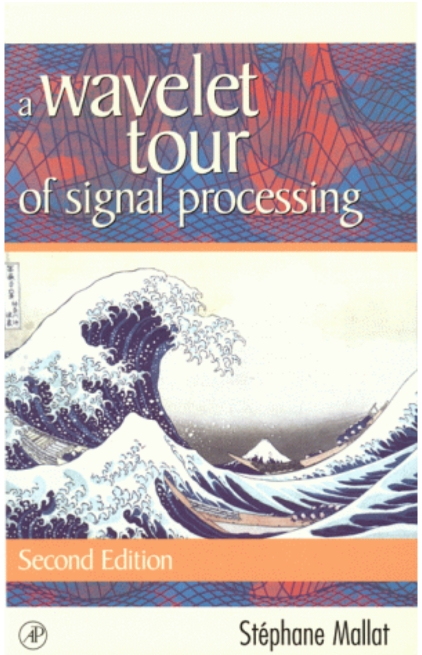
Contents

- Review of Fourier theory
- Wavelets and multiresolution
- Review of Information theoretic concepts
- Applications
  - Image coding (JPEG2000)
  - Feature extraction and signal/image analysis
- Wavelets and sparsity in neuroimaging and vision sciences
- Seminal lectures

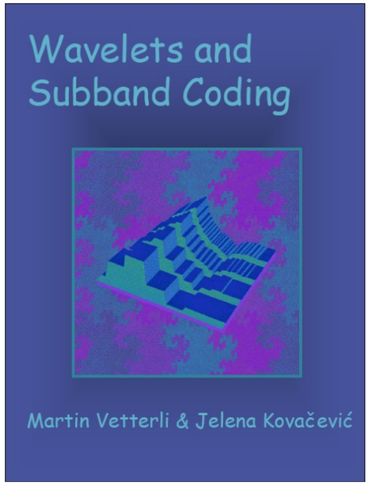
Books

Stephane Mallat
(Ecole Polytechnique)





Martin Vetterli (EPFL)



“Scale”

“Scale”

"Scale"

Telecommunications for Multimedia

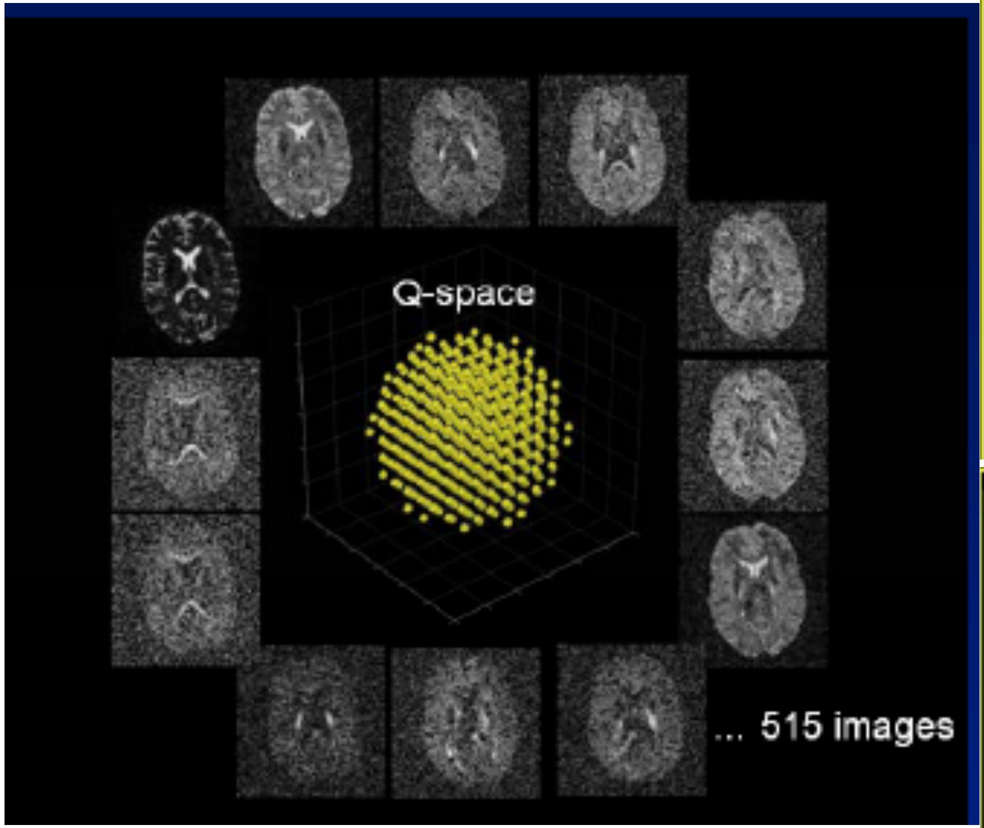
Good news

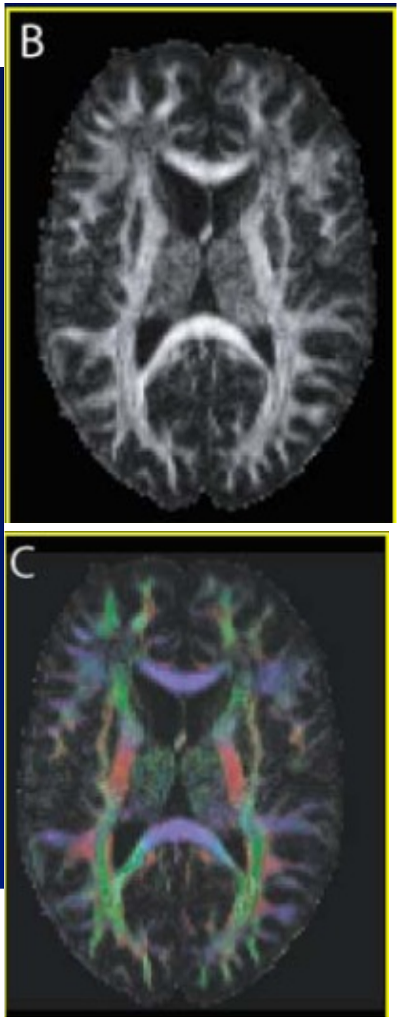
- It is fun!
- Get in touch with the state-of-the-art technology
- Convince yourself that the time spent on maths&stats was not wasted
- Learn how to map theories into applications
- Acquiring the tools for doing good research!

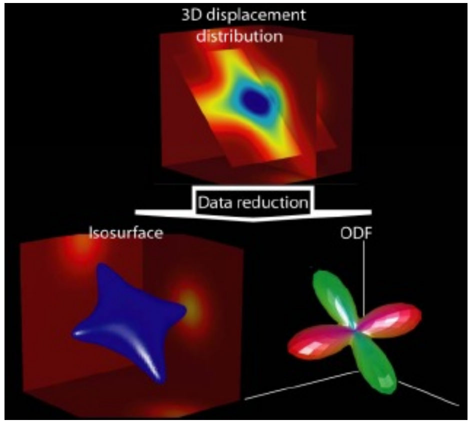
Bad news

- Some theoretical background is unavoidable
  - Mathematics
    - Fourier transform
    - Linear operators
    - Digital filters
    - Wavelet transform
  - (some) Information theory

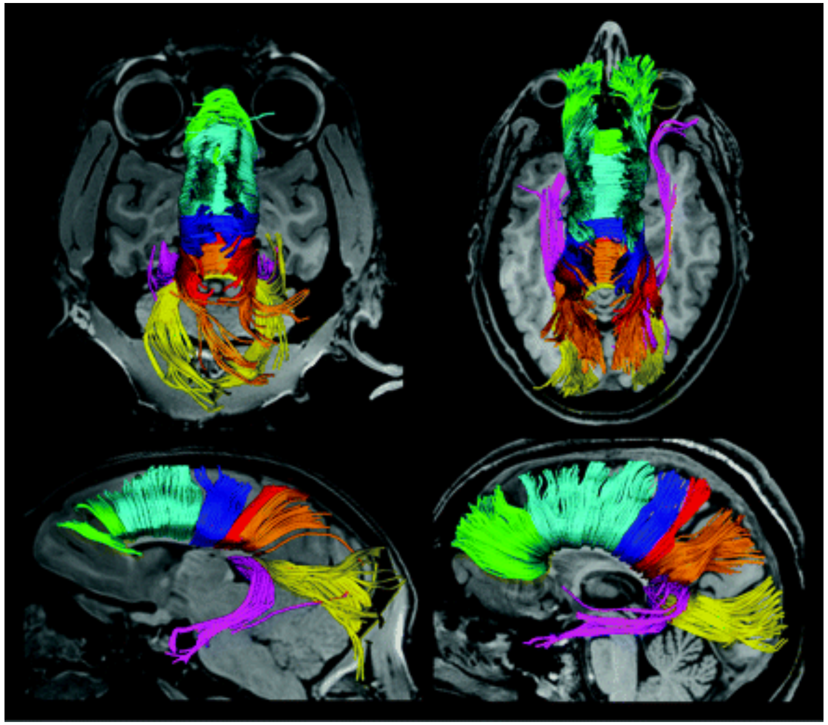
A glance on applications

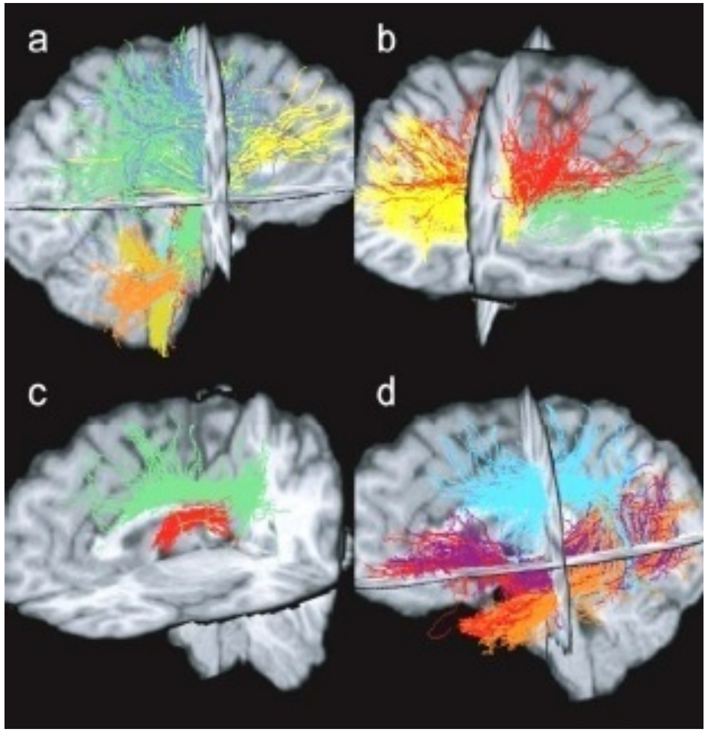




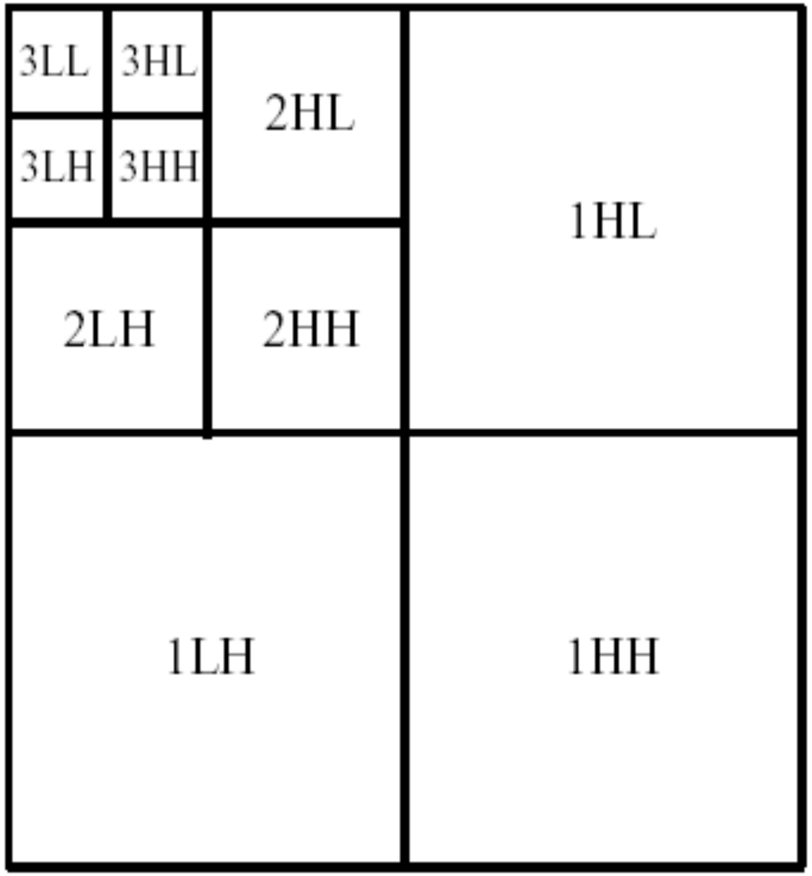


A glance on applications





JPEG2000


3LL
3HL
3LH
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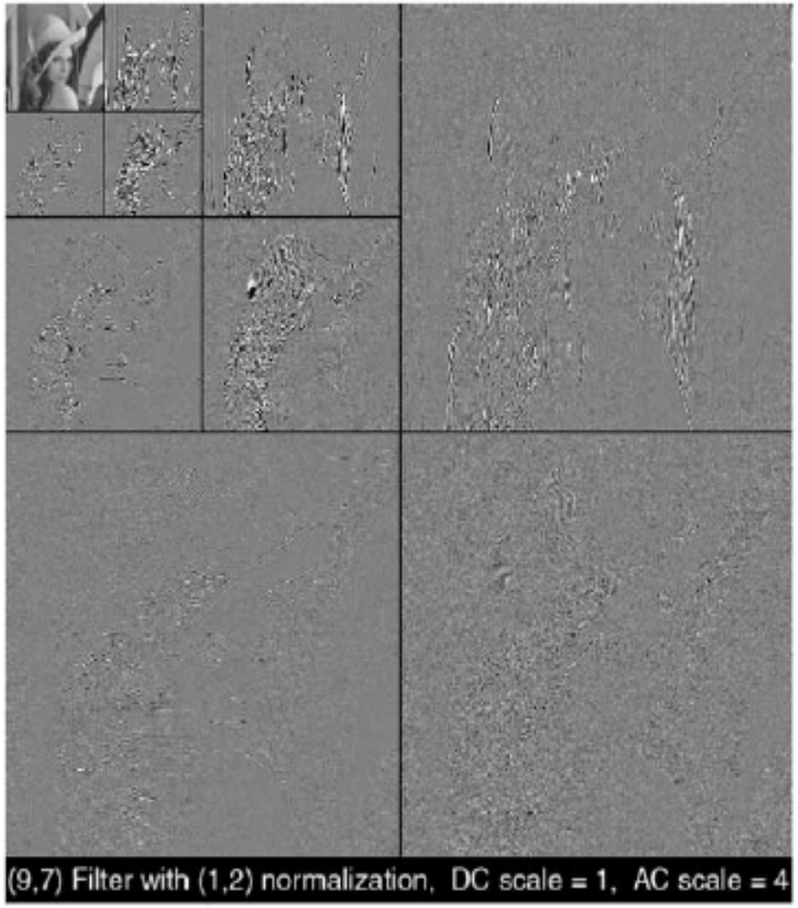
2HL
2LH
2HH
1HL
1LH
1HH


(9,7) Filter with (1,2) normalization, DC scale = 1, AC scale = 4

Mathematical tools

Introduction

- Sparse representations: few coefficients reveal the information we are looking for.
  - Such representations can be constructed by decomposing signals over elementary waveforms chosen in a family called a *dictionary*.
  - An **orthogonal** basis is a dictionary of **minimum size** that can yield a sparse representation if designed to concentrate the signal energy over a set of few vectors. This set gives a *geometric* signal description.
    - Signal compression and noise reduction
  - Dictionaries of vectors that are **larger** than bases are needed to build sparse representations of complex signals. But choosing is difficult and requires more complex algorithms.
    - Sparse representations in redundant dictionaries can improve pattern recognition, compression and noise reduction
- Basic ingredients: Fourier and Wavelet transforms
  - They decompose signals over oscillatory waveforms that reveal many signal properties and provide a path to sparse representations

Signals as functions

- CT analogue signals (real valued functions of continuous independent variables)
  - 1D: $f=f(t)$
  - 2D: $f=f(x,y)$ x,y
  - Real world signals (audio, ECG, pictures taken with an analog camera)
- DT analogue signals (real valued functions of discrete variables)
  - 1D: $f=f[k]$
  - 2D: $f=f[i,j]$
  - *Sampled* signals
- Digital signals (discrete valued functions of DT variables)
  - 1D: $y=y[k]$
  - 2D: $y=y[i,j]$
  - *Sampled and discretized* signals

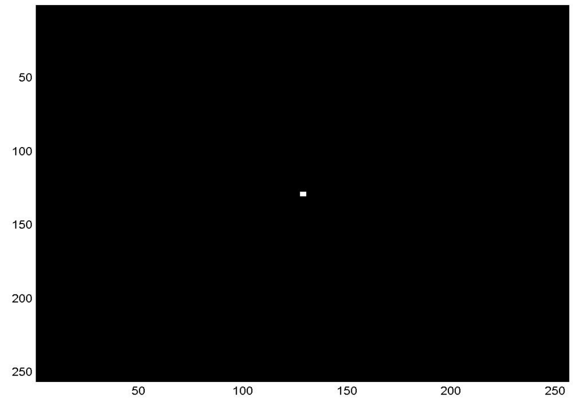
Images as functions

- Gray scale images: 2D functions
  - Domain of the functions: set of (x,y) values for which $f(x,y)$ is defined : 2D lattice $[i,j]$ defining the pixel locations
  - Set of values taken by the function : gray levels

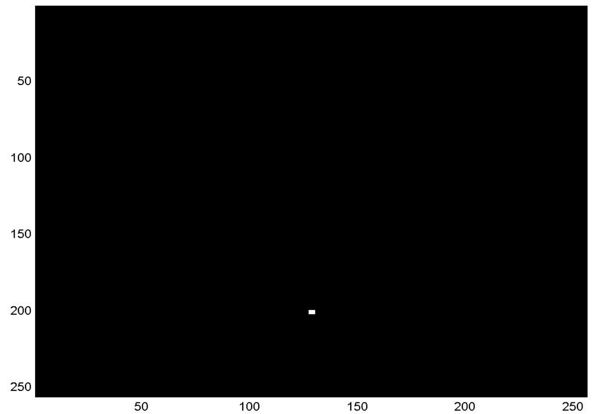
- Digital images can be seen as functions defined over a discrete domain $\{i,j: 0<i<I, 0<j<J\}$
  - I,J: number of rows (columns) of the matrix corresponding to the image
  - $f=f[i,j]$: gray level in position $[i,j]$

Example 1: δ function

$$\delta[i,j] = \begin{cases} 1 & i = j = 0 \\ 0 & i,j \neq 0; i \neq j \end{cases}$$



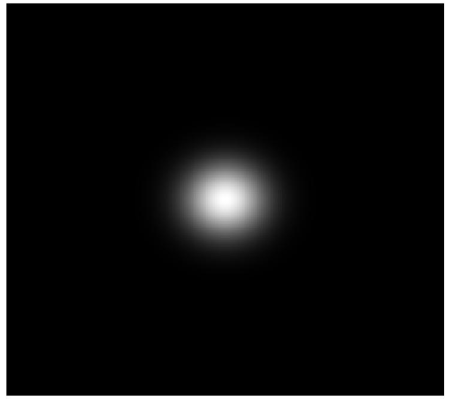
$$\delta[i,j-J] = \begin{cases} 1 & i = 0; j = J \\ 0 & otherwise \end{cases}$$



Example 2: Gaussian

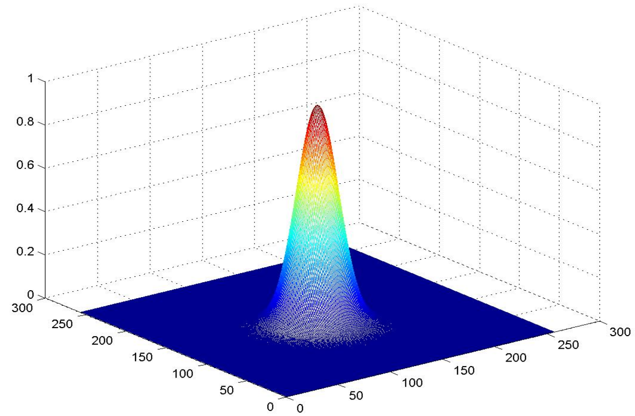
Continuous function

$$f(x,y) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{x^2+y^2}{2\sigma^2}}$$

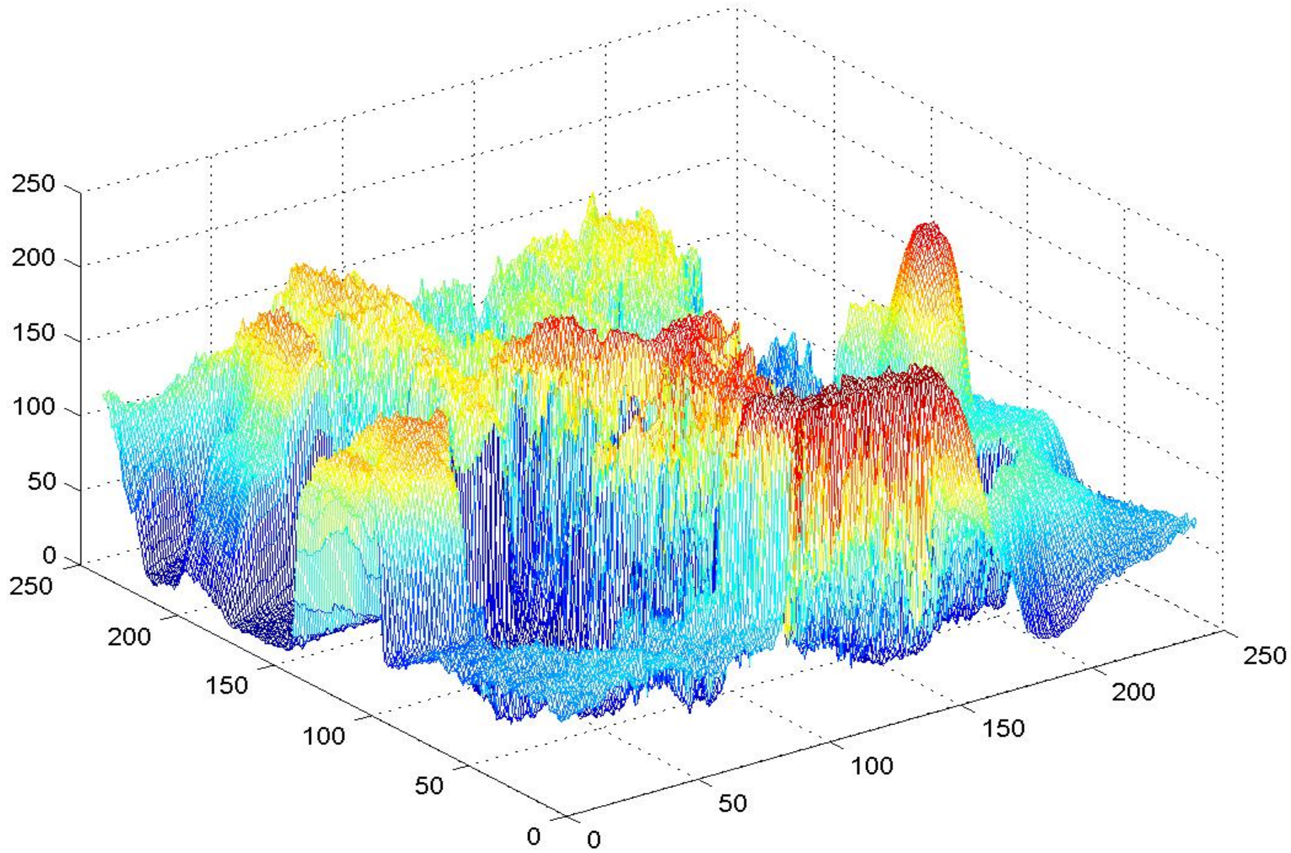


Discrete version

$$f[i,j] = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{i^2+j^2}{2\sigma^2}}$$



Example 3: Natural image



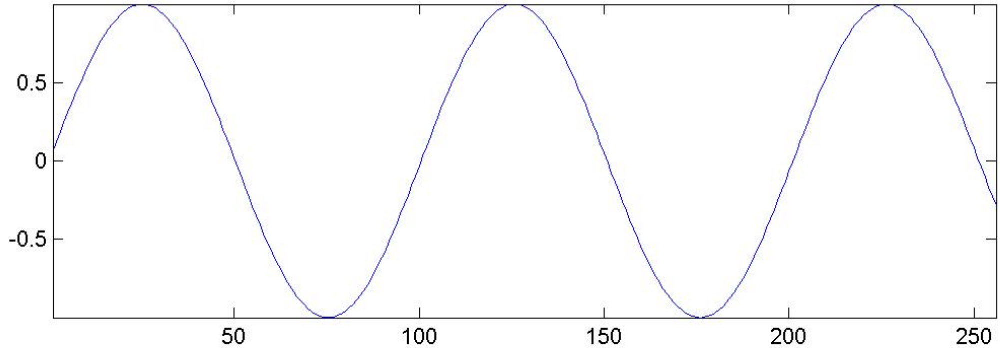
Example 3: Natural image

The Fourier kingdom

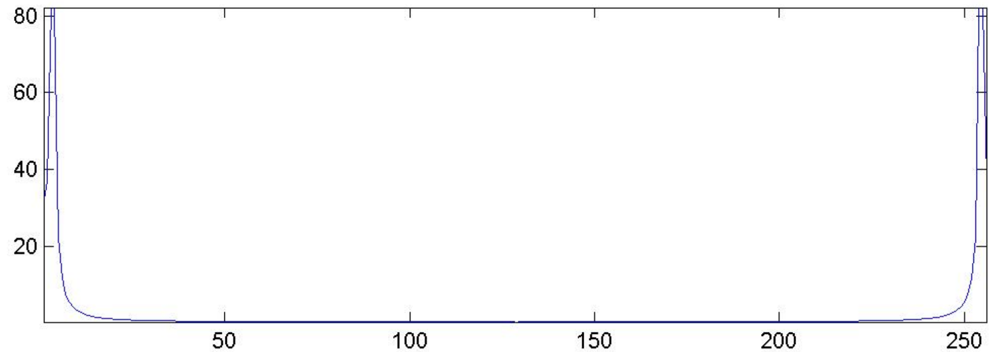
- Frequency domain characterization of signals

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt$$

$$f(t) = \int_{-\infty}^{+\infty} F(\omega) e^{j\omega t} dt$$

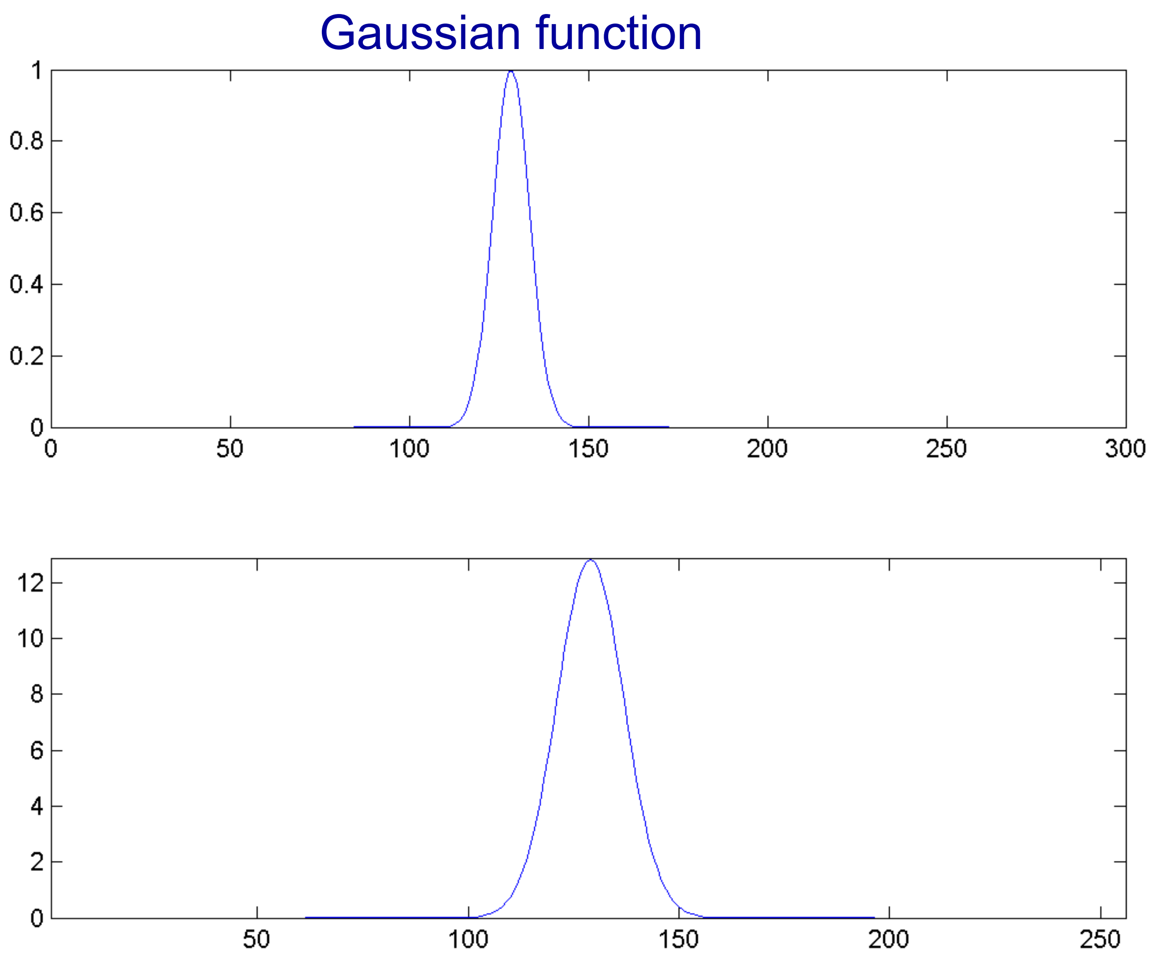


Signal domain



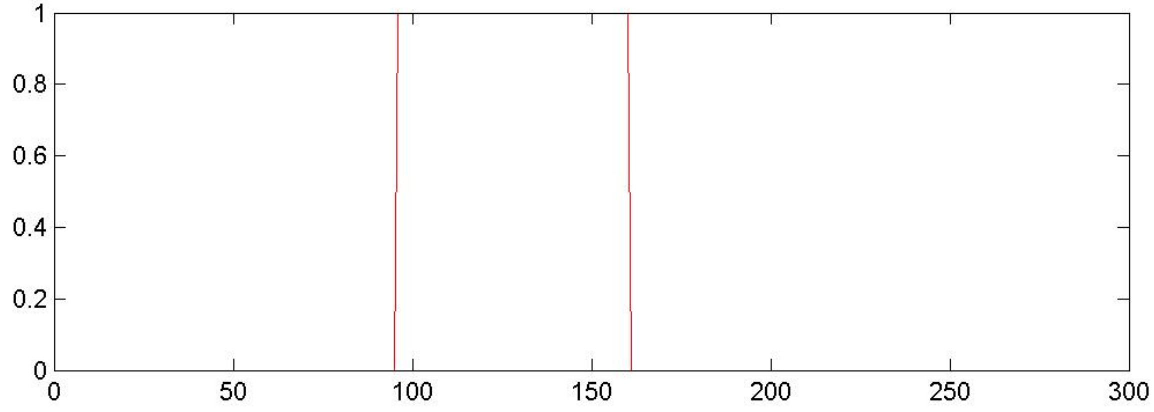
Frequency domain

The Fourier kingdom

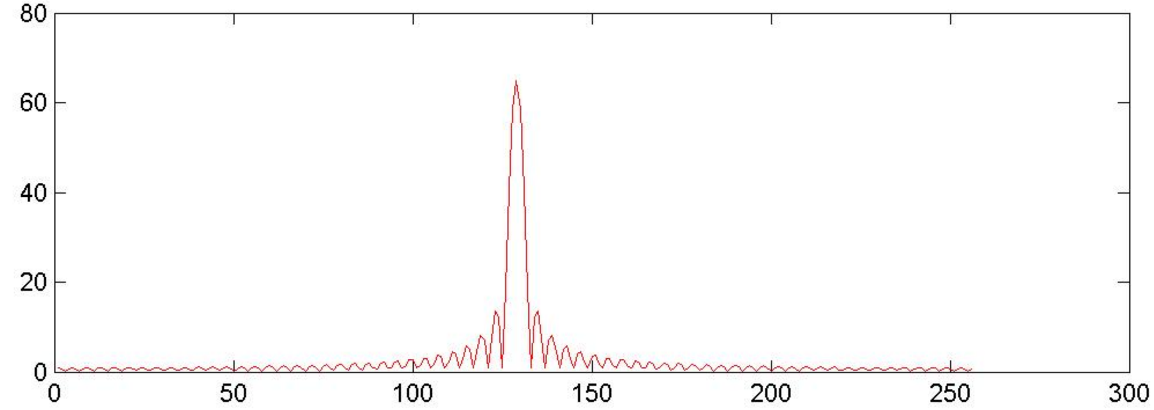


The Fourier kingdom

rect function



sinc function



2D Fourier transform

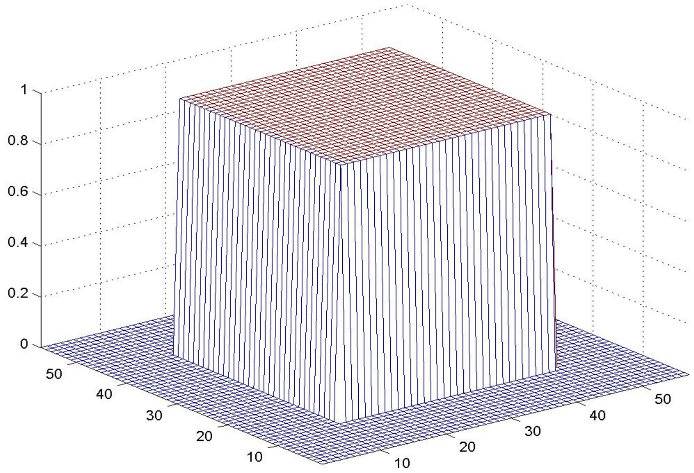
$$\hat{f}\left(\omega_x,\omega_y\right)=\int_{-\infty}^{+\infty} f\left(x,y\right)e^{-j\left(\omega_x x+\omega_y y\right)}dxdy$$

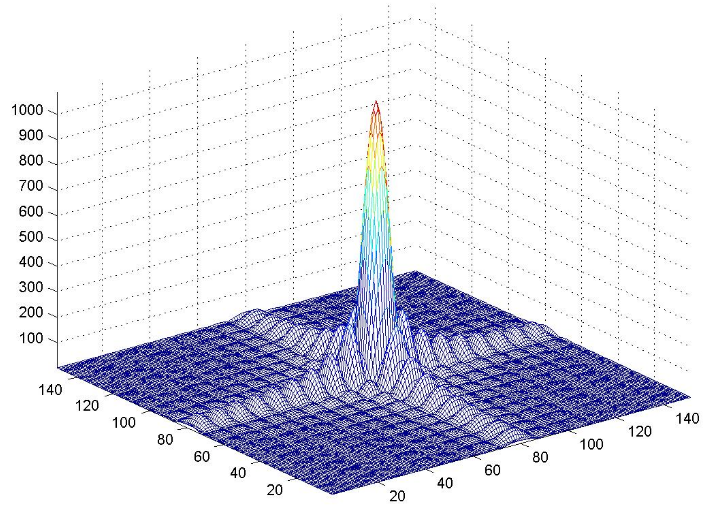
$$f\left(x,y\right)=\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\hat{f}\left(\omega_x,\omega_y\right)e^{j\left(\omega_x x+\omega_y y\right)}d\omega_x d\omega_y$$

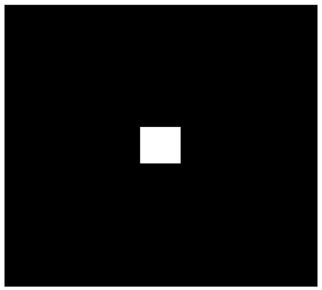
$$\iint f\left(x,y\right)g^*\left(x,y\right)dxdy=\frac{1}{4\pi^2}\iint\hat{f}\left(\omega_x,\omega_y\right)\hat{g}^*\left(\omega_x,\omega_y\right)d\omega_x d\omega_y$$ Parseval formula

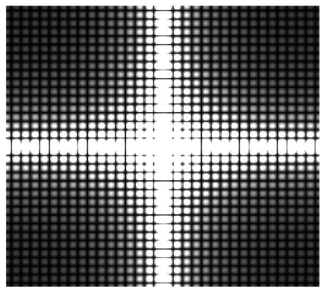
$$f=g\rightarrow\iint\left|f\left(x,y\right)\right|^2dxdy=\frac{1}{4\pi^2}\iint\left|\hat{f}\left(\omega_x,\omega_y\right)\right|^2d\omega_x d\omega_y$$ Plancherel equality

The Fourier kingdom

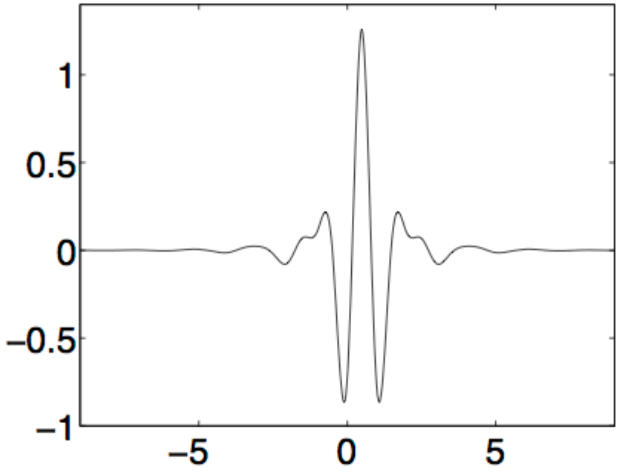




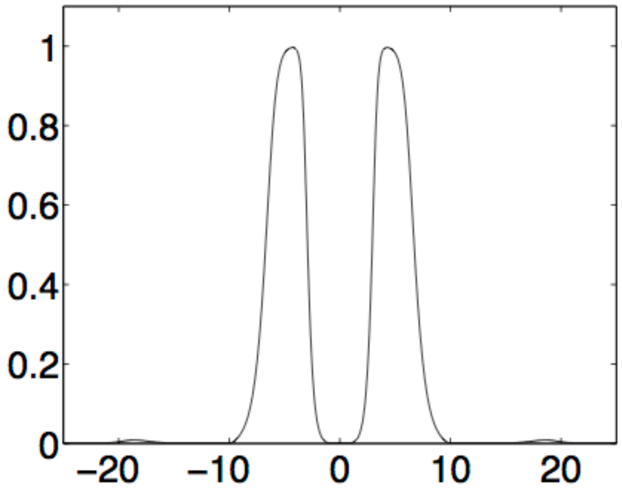




Wavelets

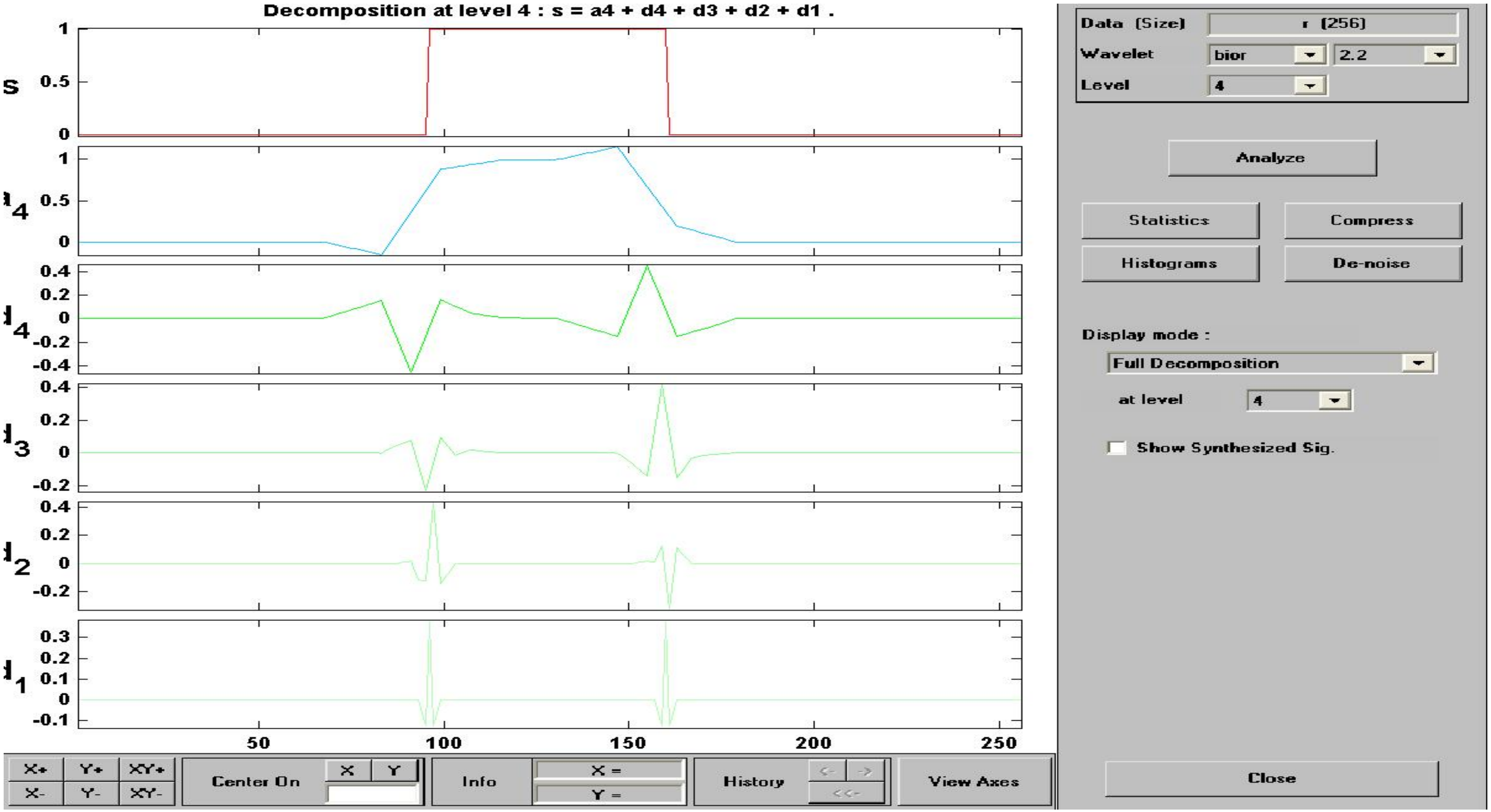


Wavalet in signal (time or space) domain



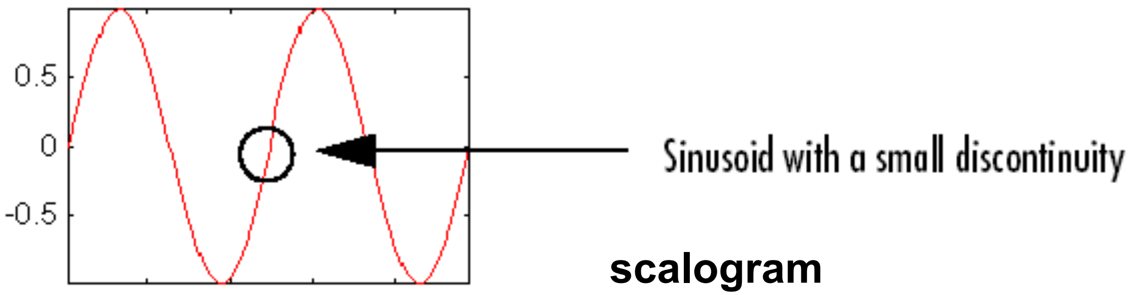
Wavalet in frequency (Fourier) domain

Wavelet representation

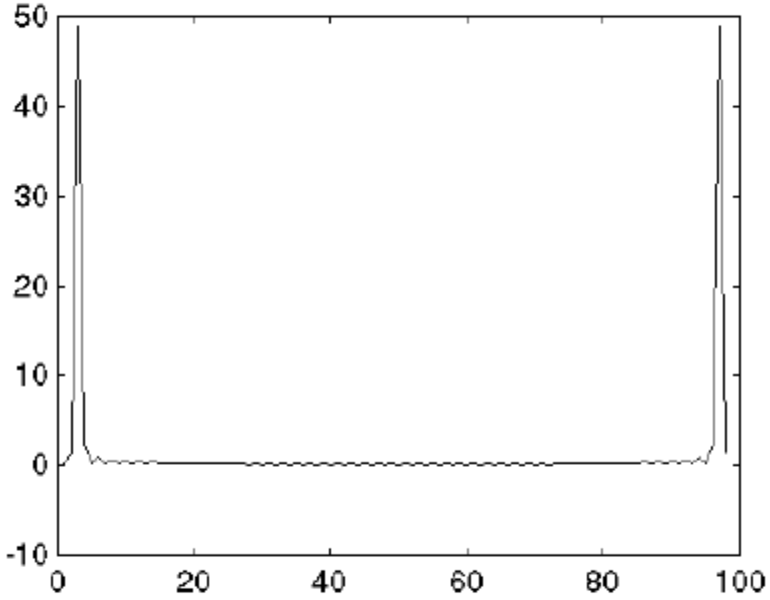


Wavelet representation

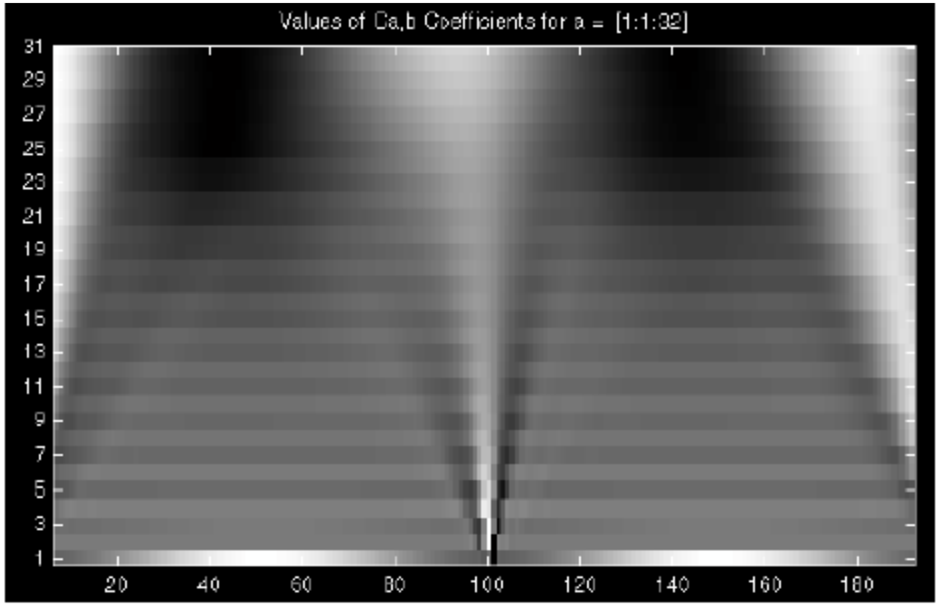
Wavelets are good for transients



scalogram

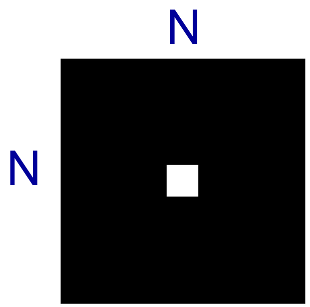


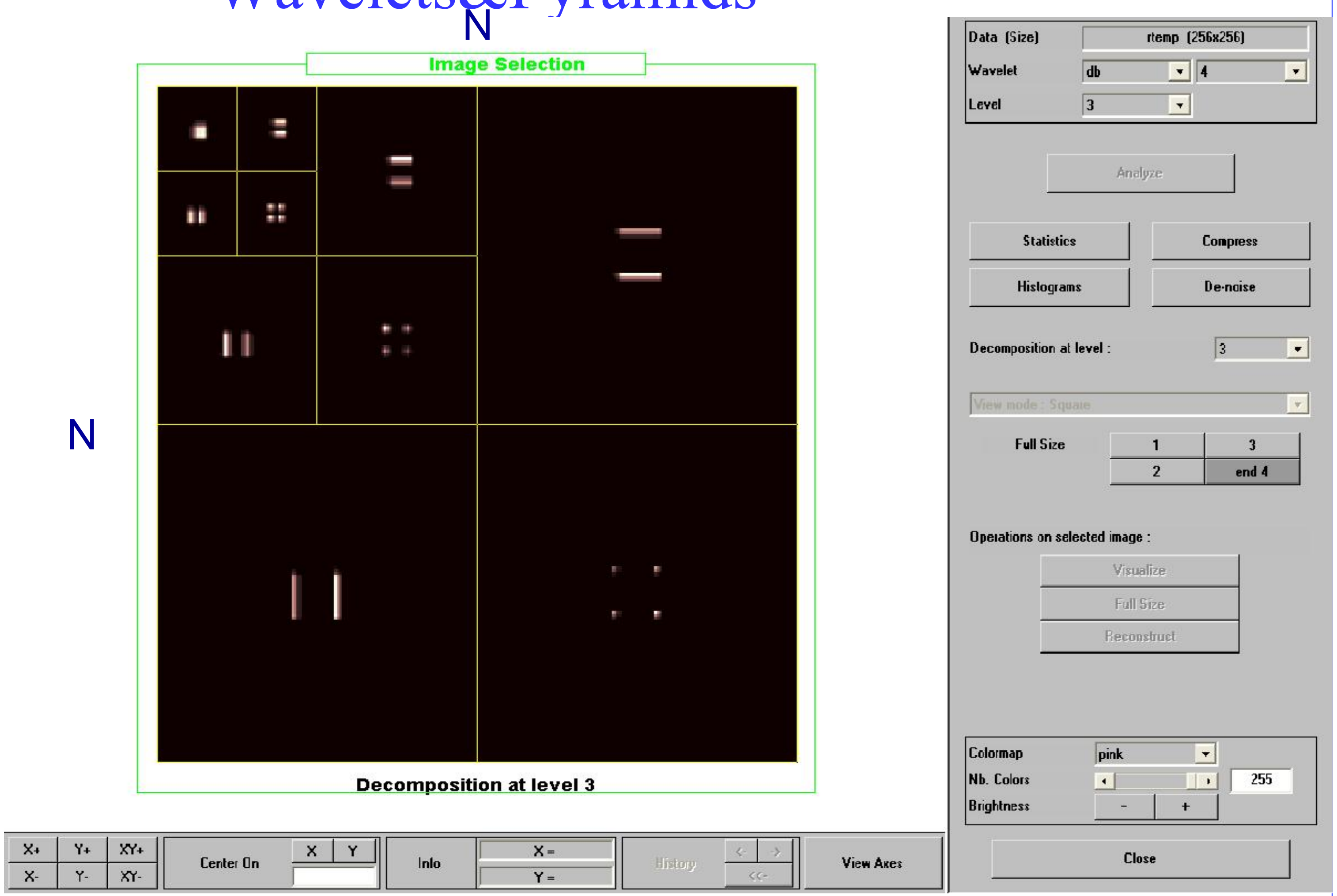
Fourier Coefficients



Wavelet Coefficients

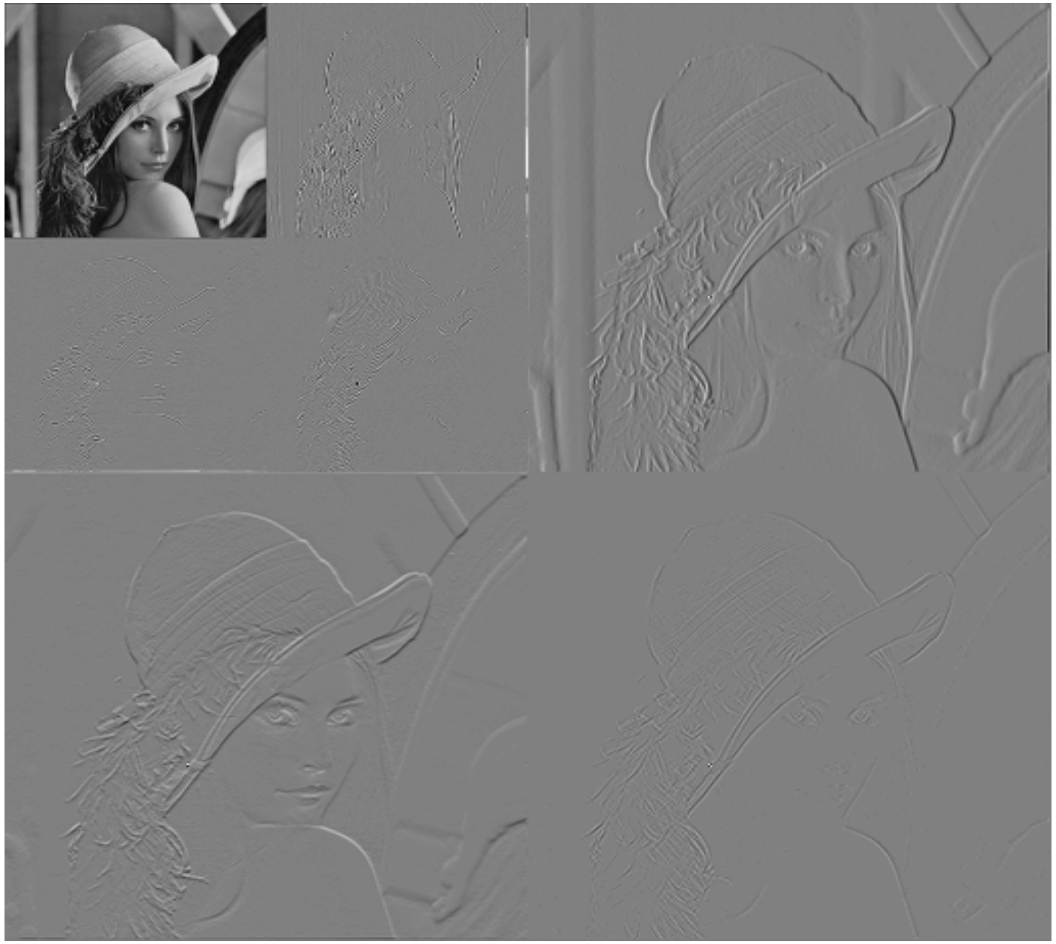
Wavelets&Pyramids



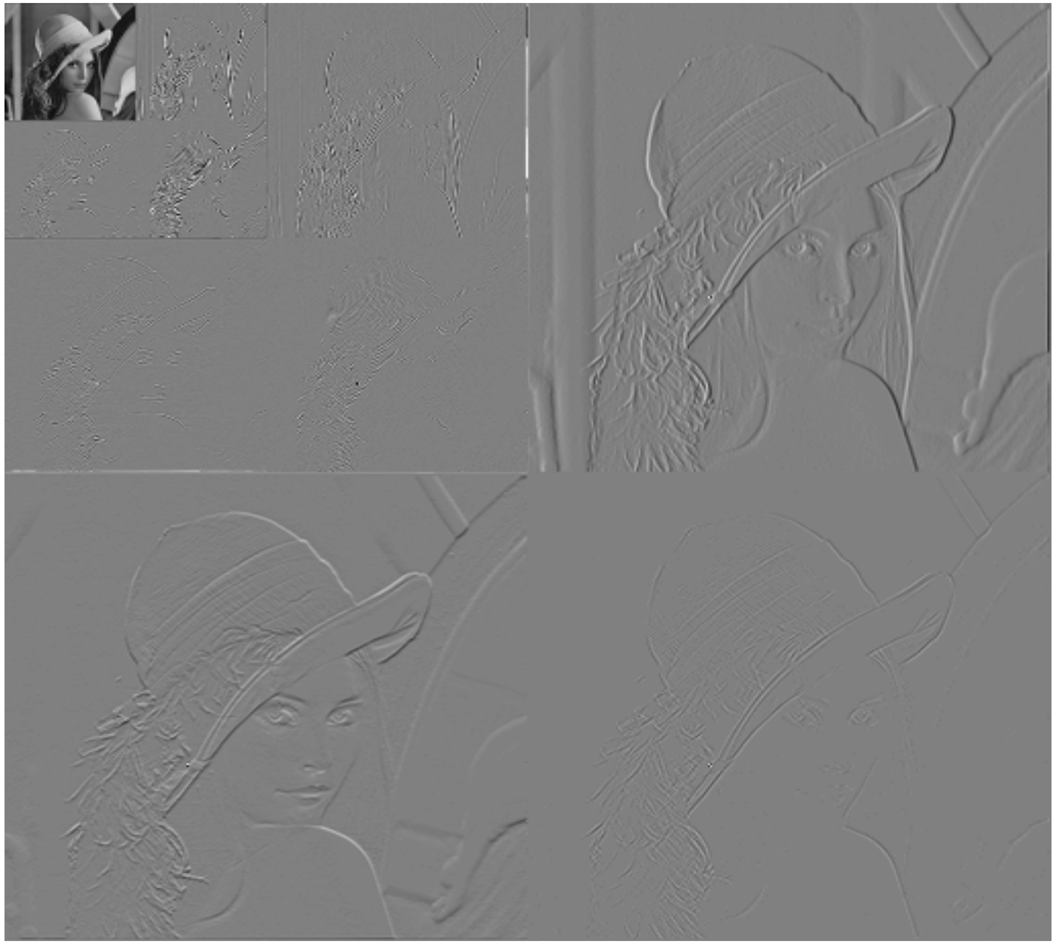


Waveletts&Pyramids

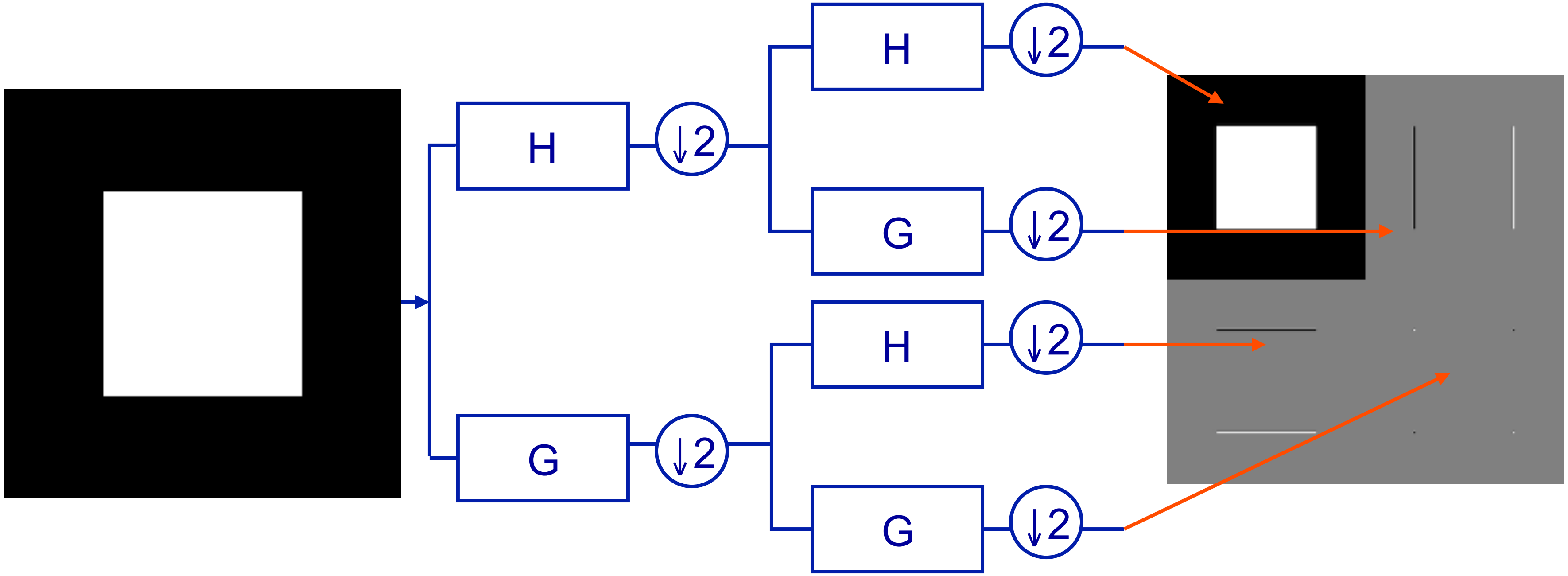
Wavelets&Pyramids



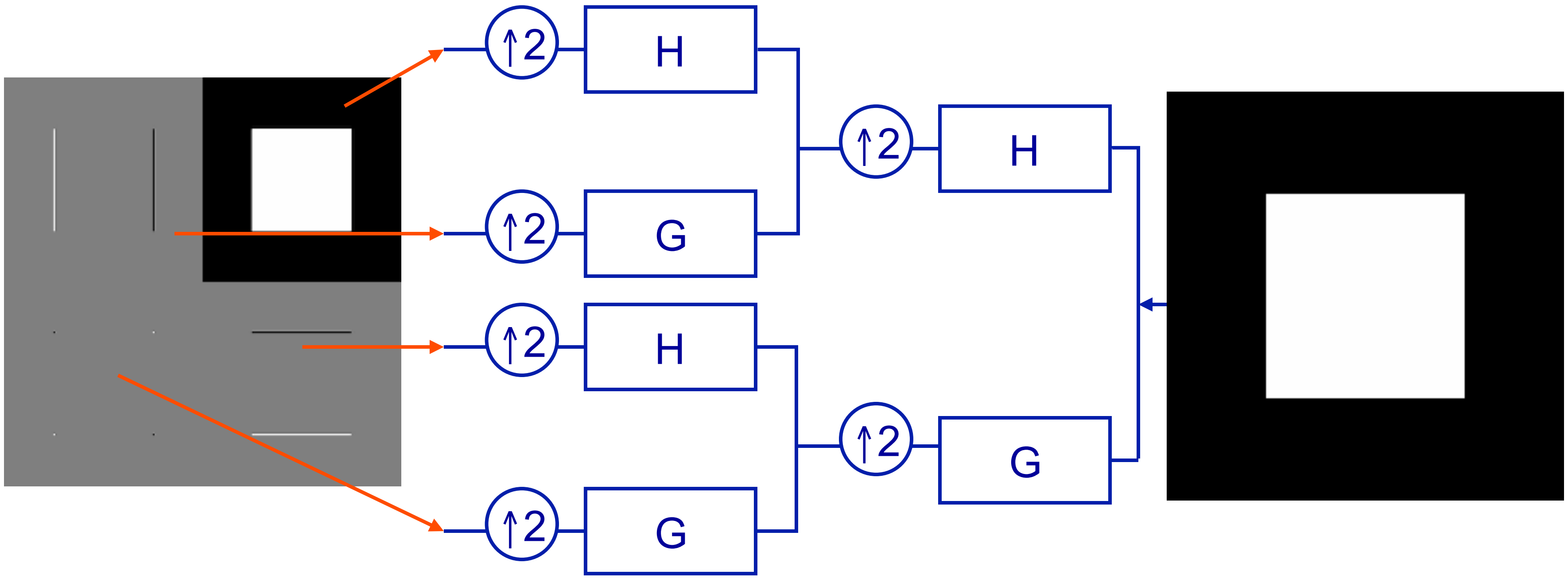
Wavelets&Pyramids



Wavelets&Filterbanks



Waveletss&Filterbanks



Very efficient implementation by recursive filtering

Fourier versus Wavelets

Fourier

- Basis functions are sinusoids
  - More in general, complex exponentials
- Switching from signal domain *t* to frequency domain *f*
  - Either spatial or temporal
- Good localization either in time or in frequency
  - Transformed domain: Information on the sharpness of the transient but not on its position
- Good for stationary signals but unsuitable for transient phenomena

Wavelets

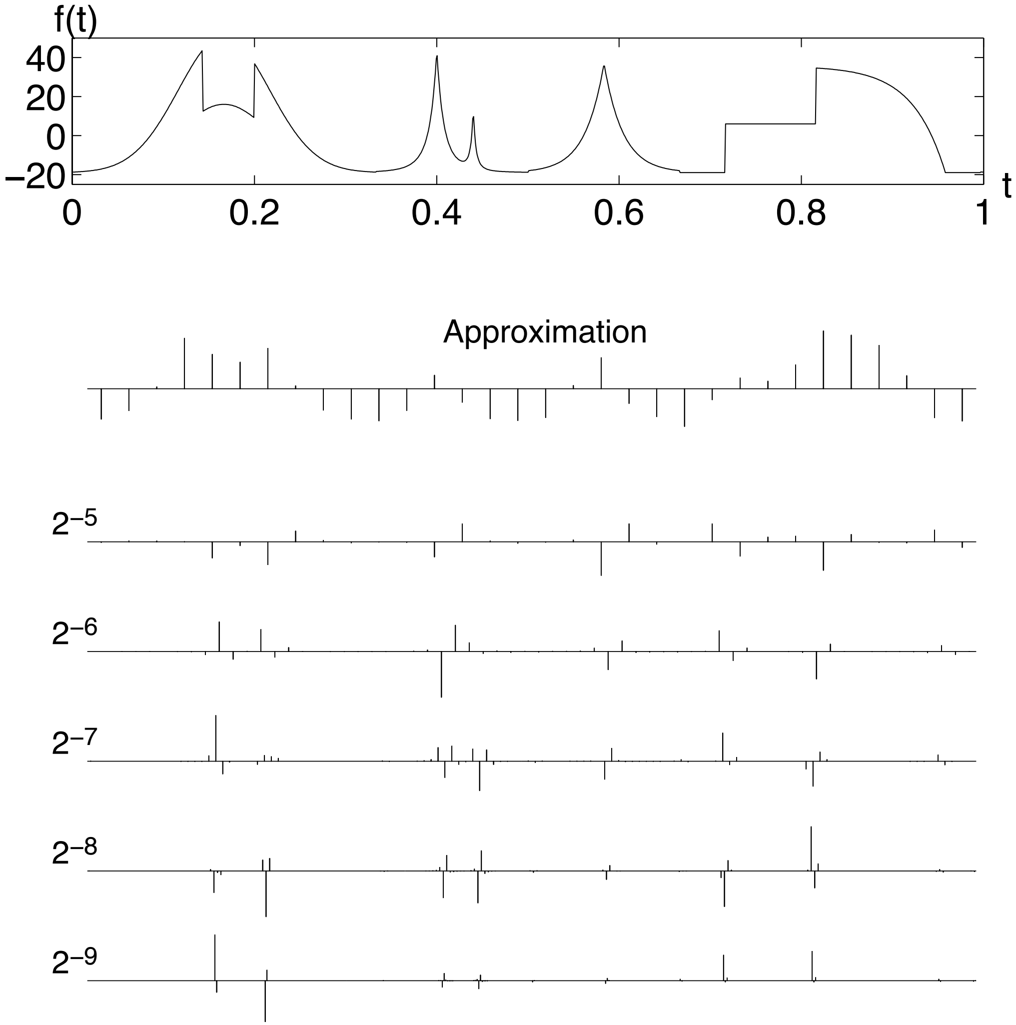
- Different families of basis functions are possible
  - Haar, Daubechies', biorthogonal
- Switching from the signal domain to a *multiresolution* representation
- *Good localization in time and frequency*
  - Information on *both* the *sharpness* of the transient and the *point* where it happens
- Good for any type of signal

Applications

- Compression and coding
  - Critically sampled representations (discrete wavelet transforms, DWT)
- Feature extraction for signal analysis
  - Overcomplete bases (continuous wavelet transform, wavelet frames)
- Image modeling
  - Modeling the human visual system: Objective metrics for visual quality assessment
  - Texture synthesis
- Image enhancement
  - Denoising by wavelet thresholding, deblurring, hole filling
- Image processing on manyfolds
  - Wavelet transform on the sphere: applications in diffusion MRI

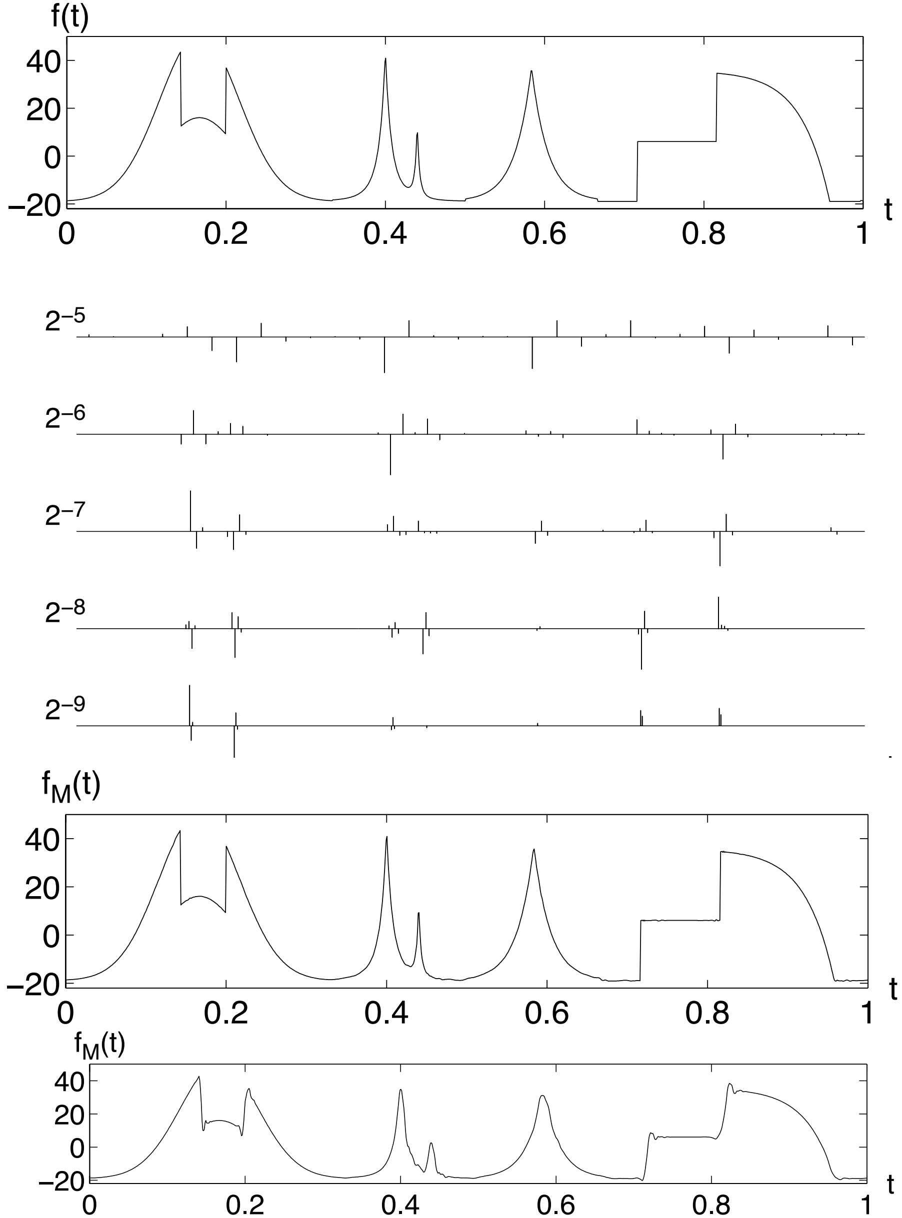
Wavelet Coefficients

•



Wavelet coefficients $\langle f, \psi_{j,n} \rangle$

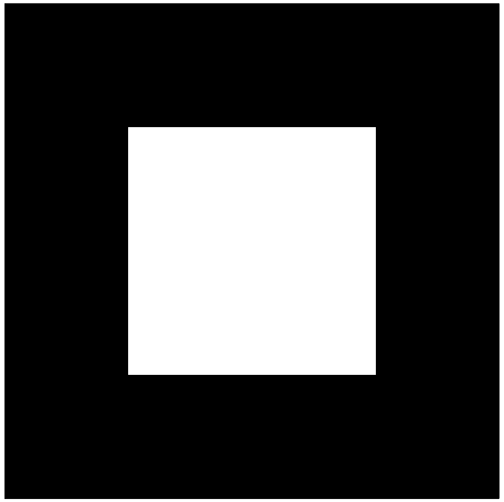
Non-Linear Wavelet Approximation

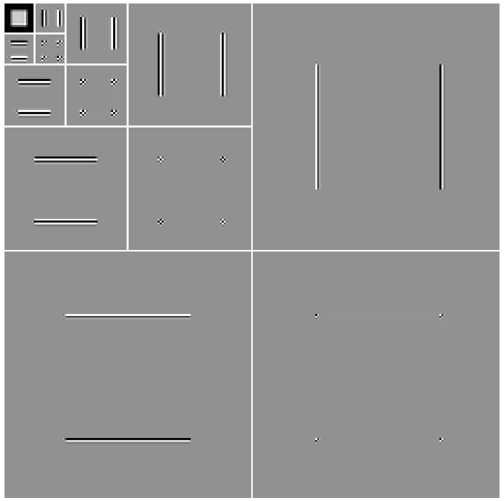


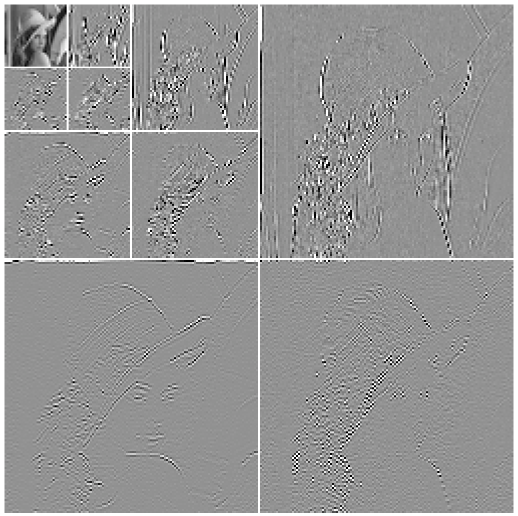
$$Non-linear:$$
$$\|f - f_M\|^2 = 5.1\,10^{-3}$$
$$Linear:$$
$$\|f - f_M\|^2 = 8.5\,10^{-2}$$

Wavelet Bases of Images

- Wavelet basis of $\mathbf{L^2}[0,1]^2$: $\left\{ \frac{1}{2^j}\psi^k\left(\frac{x-2^j n}{2^j}\right)\right\}_{\substack{1\le k\le 3,\, j<0 \\ 2^j n\in[0,1]^2}}$







Wavelet coefficients

$k = 1, 2, 3$

$j = -1, -2, -3, -4$

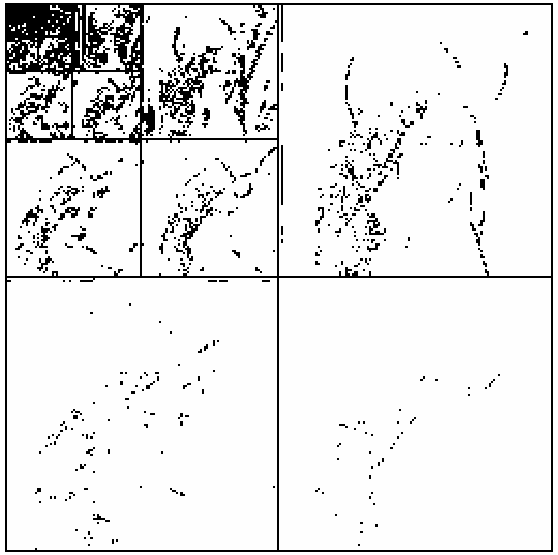
$2^j n \in [0,1]^2$

Wavelet Image Approximations

Original Image

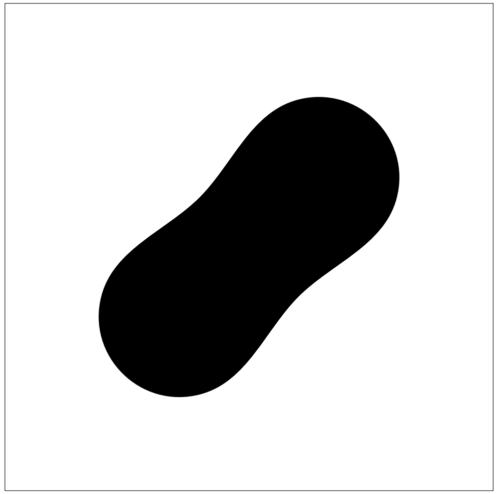
Non-linear Approximation

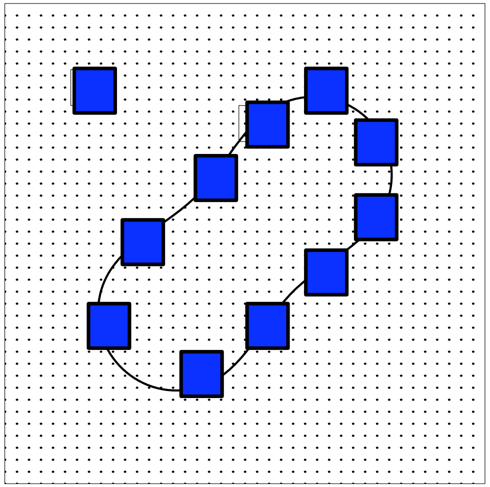
Linear Approximation



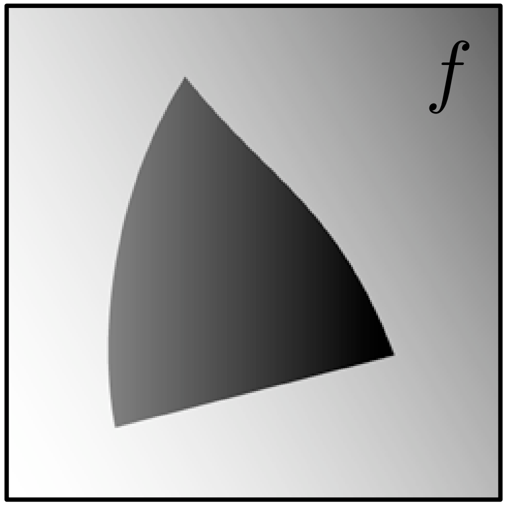
M = N/16 largest wavelet coeffs.

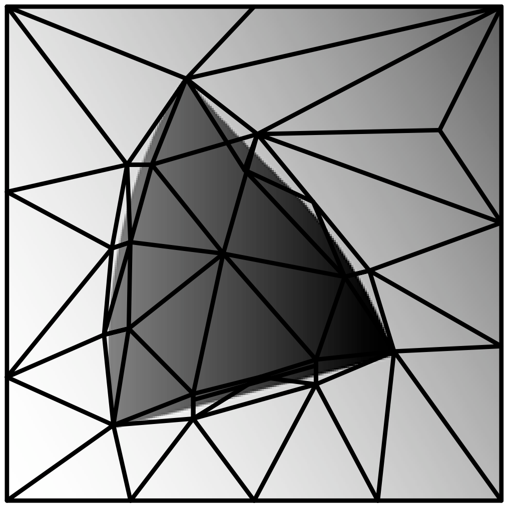
Good but Not Optimal





The number of large wavelet coefficient is proportional to the length of the contour.

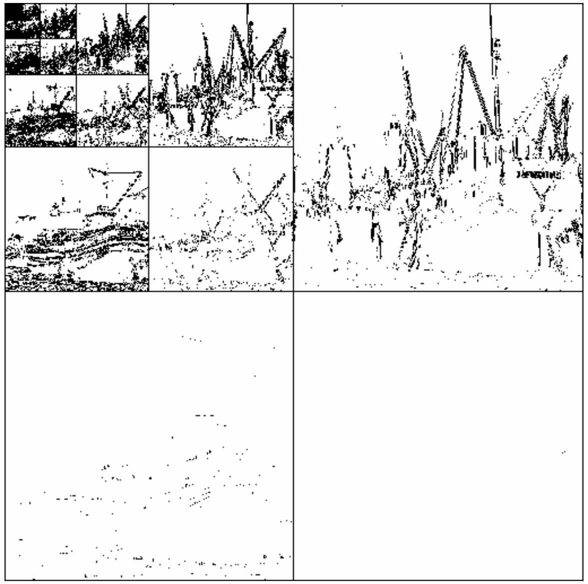




Need less adapted triangles if the contour geometry is regular.

Compression with JPEG-2000

Non-zero wavelet coefficients

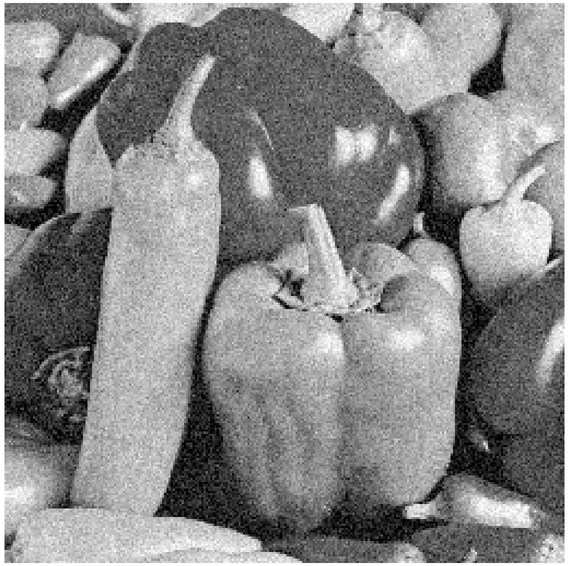


0.2 bit/pixel

0.05 bit/pixel

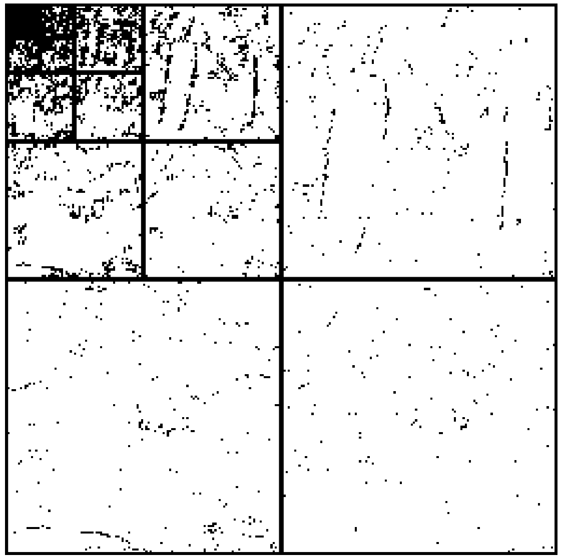
Wavelet Image Thresholding

Original image f



Noisy image X

Translat. Invariant Thesh. estim. DX



Wavelet coeff. above T