INTUITIONISM

Consider the problem “Are there two irrational numbers a and b such that a^b is rational?” We apply the following smart reasoning: suppose $\sqrt{2}^{\sqrt{2}}$ is rational, then we have solved the problem. Should $\sqrt{2}^{\sqrt{2}}$ be irrational then $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$ is rational. In both cases there is a solution, so the answer to the problem is: Yes. However, should somebody ask us to produce such a pair a, b, then we have to engage in some serious number theory in order to come up with the right choice between the numbers mentioned above.

($\wedge$) a proves $\varphi \wedge \psi :=$ a is a pair $\langle b, c\rangle$ such that b proves φ and c proves ψ.

($\vee$) a proves $\varphi \vee \psi :=$ a is a pair $\langle b, c\rangle$ such that b is a natural number and if $b = 0$ then c proves φ, if $b \neq 0$ then c proves ψ.

($\rightarrow$) a proves $\varphi \rightarrow \psi :=$ a is a construction that converts any proof p of φ into a proof $a(p)$ of ψ.

($\perp$) no a proves $\perp$.

In order to deal with the quantifiers we assume that some domain D of objects is given.

($\forall$) a proves $\forall x \varphi(x) :=$ a is a construction such that for each $b \in D$ $a(b)$ proves $\varphi(\overline{b})$.

($\exists$) a proves $\exists x \varphi(x) :=$ a is a pair (b, c) such that $b \in D$ and c proves $\varphi(\overline{b})$.

1. $\varphi \wedge \psi \to \varphi$ is true, for let $\langle a, b \rangle$ be a proof of $\varphi \wedge \psi$, then the construction c with $c(a, b) = a$ converts a proof of $\varphi \wedge \psi$ into a proof of φ. So c proves $(\varphi \wedge \psi \to \varphi)$.

let c the first projection of a pair, namely c(a,b)=a (c may be defined as λ(x,y).x)

2. $(\varphi \wedge \psi \to \sigma) \to (\varphi \to (\psi \to \sigma))$. Let a prove $\varphi \wedge \psi \to \sigma$, i.e. a converts each proof $\langle b, c \rangle$ of $\varphi \wedge \psi$ into a proof $a(b, c)$ of σ. Now the required proof p of $\varphi \to (\psi \to \sigma)$ is a construction that converts each proof b of φ into a $p(b)$ of $\psi \to \sigma$. So $p(b)$ is a construction that converts a proof c of ψ into a proof $(p(b))(c)$ of σ. Recall that we had a proof $a(b, c)$ of σ, so put $(p(b)(c) = a(b, c)$; let q be given by $q(c) = a(b, c)$, then p is defined by $p(b) = q$. Clearly, the above contains the description of a construction that converts a into a proof p of $\varphi \to (\psi \to \sigma)$. (For those familiar with the λ-notation: $p = \lambda b.\lambda c.a(b, c)$, so $\lambda a.\lambda b.\lambda c.a(b, c)$ is the proof we are looking for).

$\bot \to \delta$

is intuitionistically acceptable

$\bot \to \delta$

is equivalent to the rule

$$\frac{\bot}{\varphi}$$

$\delta \vee \neg\delta$

is not intuitionistically acceptable

the principle $\delta \vee \neg\delta$

is equivalent to the rule

$$\frac{\begin{matrix}[\neg\varphi] \\ \mathcal{D} \\ \bot\end{matrix}}{\varphi}$$

A natural deduction system for intuitionism is obtained by dropping RAA and maintaining $\dfrac{\bot}{\varphi}$

Derivation with hypothesis ψ

$$\begin{matrix} \psi \\ \mathcal{D} \\ \varphi \end{matrix}$$

denotes **a** (possibly empty) set (called also parcel) of leaves labelled with the formula ψ

A derivation with hypothesis ψ cancelled

$$\dfrac{\begin{matrix} [\psi] \\ \mathcal{D} \\ \varphi \end{matrix}}{\sigma}$$

denotes **a** set of leaves labelled with the formula ψ *marked* as "*cancelled*" / "*discharged*"

Derivation with hypothesis ψ

$$\begin{array}{c} x{:}\psi \\ \mathcal{D} \\ \varphi \end{array}$$

denotes a (possibly empty) set of leaves labelled with the formula *ψ.* Each formula in the set is labelled with *x*

A derivation with hypothesis ψ cancelled

$$\begin{array}{c} [x{:}\psi] \\ \mathcal{D} \\ \dfrac{\varphi}{\sigma} \end{array}$$

denotes a set of leaves labelled with the formula *ψ* *marked* as *"cancelled"* / *"discharged".* Each discharged formula is labelled with x

(1) $\vdash \neg\varphi \leftrightarrow \neg\neg\neg\varphi$

(2) $\vdash (\varphi \wedge \neg\psi) \rightarrow \neg(\varphi \rightarrow \psi)$

(3) $\vdash (\varphi \rightarrow \psi) \rightarrow (\neg\neg\varphi \rightarrow \neg\neg\psi)$

(4) $\vdash \neg\neg(\varphi \rightarrow \psi) \leftrightarrow (\neg\neg\varphi \rightarrow \neg\neg\psi)$

(5) $\vdash \neg\neg(\varphi \wedge \psi) \leftrightarrow (\neg\neg\varphi \wedge \neg\neg\psi)$

(6) $\vdash \neg\neg\forall x\varphi(x) \rightarrow \forall x\neg\neg\varphi(x)$

$$
\cfrac{\cfrac{\cfrac{\cfrac{[\varphi]^1 \qquad \cfrac{}{\varphi \to \neg\neg\varphi}}{\neg\neg\varphi} \qquad [\neg\neg\neg\varphi]^2}{\bot}}{\neg\varphi}\,1}{\neg\neg\neg\varphi \to \neg\varphi}\,2
$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi \wedge \neg\psi]^2}{\varphi} \qquad [\varphi \to \psi]^1}{\psi} \qquad \dfrac{[\varphi \wedge \neg\psi]^2}{\neg\psi}}{\bot}}{\neg(\varphi \to \psi)}\ 1}{(\varphi \wedge \neg\psi) \to \neg(\varphi \to \psi)}\ 2$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg\neg\varphi]^3 \qquad \dfrac{\dfrac{\dfrac{[\varphi]^1 \quad [\varphi \to \psi]^4}{\psi} \qquad [\neg\psi]^2}{\bot}}{\neg\varphi}\,1}{\bot}}{\neg\neg\psi}\,2}{\neg\neg\varphi \to \neg\neg\psi}\,3}{(\varphi \to \psi) \to (\neg\neg\varphi \to \neg\neg\psi)}\,4$$

Theorem
If ϕ does not contain $\vee$ or $\exists$ and all atoms but $\bot$ in ϕ are negated, then

$$\vdash \phi \leftrightarrow \neg\neg\phi.$$

Definition 5.2.7 *The mapping* $^\circ : FORM \to FORM$ *is defined by*

(i) $\bot^\circ := \bot$ *and* $\varphi^\circ := \neg\neg\varphi$ *for atomic* φ *dinstinct from* $\bot$

(ii) $(\varphi \wedge \psi)^\circ := \varphi^\circ \wedge \psi^\circ$

(iii) $(\varphi \vee \psi)^\circ := \neg(\neg\varphi^\circ \wedge \neg\psi^\circ)$

(iv) $(\varphi \to \psi)^\circ := \varphi^\circ \to \psi^\circ$

(v) $(\forall x \varphi(x))^\circ := \forall x \varphi^\circ(x)$

(vi) $(\exists x \varphi(x))^\circ := \neg\forall x \neg\varphi^\circ(x)$

Theorem 5.2.8 $\Gamma \vdash_c \varphi \Leftrightarrow \Gamma^\circ \vdash_i \varphi^\circ$.

An **Kripke model for propositional intuitionistic logic** is a triple

$\mathcal{H} = \langle K, \leq, V, \Vdash \rangle$

s.t.

- K is a set of “worlds”
- $\leq$ is a partial order on K
- V: $K \to 2^{PROP}$ s.t. $p \in V(k)$ and $k \leq k'$ then $p \in V(k')$
- $\Vdash \subseteq K \times FORM$ is called forcing relation satisfying the following conditions:
  - $k \Vdash p \Leftrightarrow p \in V(k)$ for $p \in PROP$
  - $k \Vdash \phi \wedge \psi \Leftrightarrow k \Vdash \phi$ AND $k \Vdash \psi$
  - $k \Vdash \phi \vee \psi \Leftrightarrow k \Vdash \phi$ OR $k \Vdash \psi$
  - $k \nVdash \bot$
  - $k \Vdash \phi \to \psi \Leftrightarrow$ for each k'. $k \leq k' \Rightarrow$ if $k' \Vdash \phi$ then $k' \Vdash \psi$
  - $k \Vdash \neg\phi \Leftrightarrow$ for each k'. $k \leq k' \Rightarrow$ if $k' \nVdash \phi$

We say that $\mathcal{H} \Vdash \varphi$ iff for each k in $\mathcal{H}$ we have $k \Vdash \varphi$

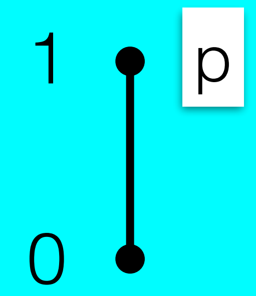
We say that $\Vdash \varphi$ iff for each $\mathcal{H}$ we have $\mathcal{H} \Vdash \varphi$

We say that $\Gamma \Vdash \varphi$ iff for each $\mathcal{H}$ and for each k in $\mathcal{H}$ we have $k \Vdash \Gamma \Rightarrow k \Vdash \varphi$

THEOREM

$\Gamma \vdash \phi$ iff $\Gamma \Vdash \phi$

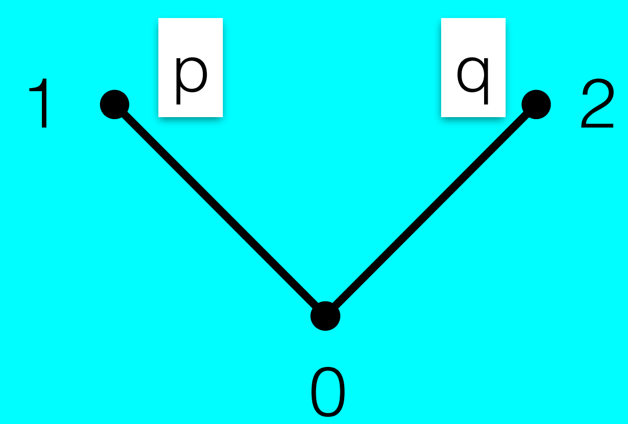
$\nvdash \phi \vee \neg \phi$



$0 \nVdash p \vee \neg p$

$\nvdash (\phi \rightarrow \sigma) \vee (\sigma \rightarrow \phi)$

$\nvdash (\phi \to \sigma) \vee (\sigma \to \phi)$



$0 \nVdash (p \to q) \vee (q \to p)$

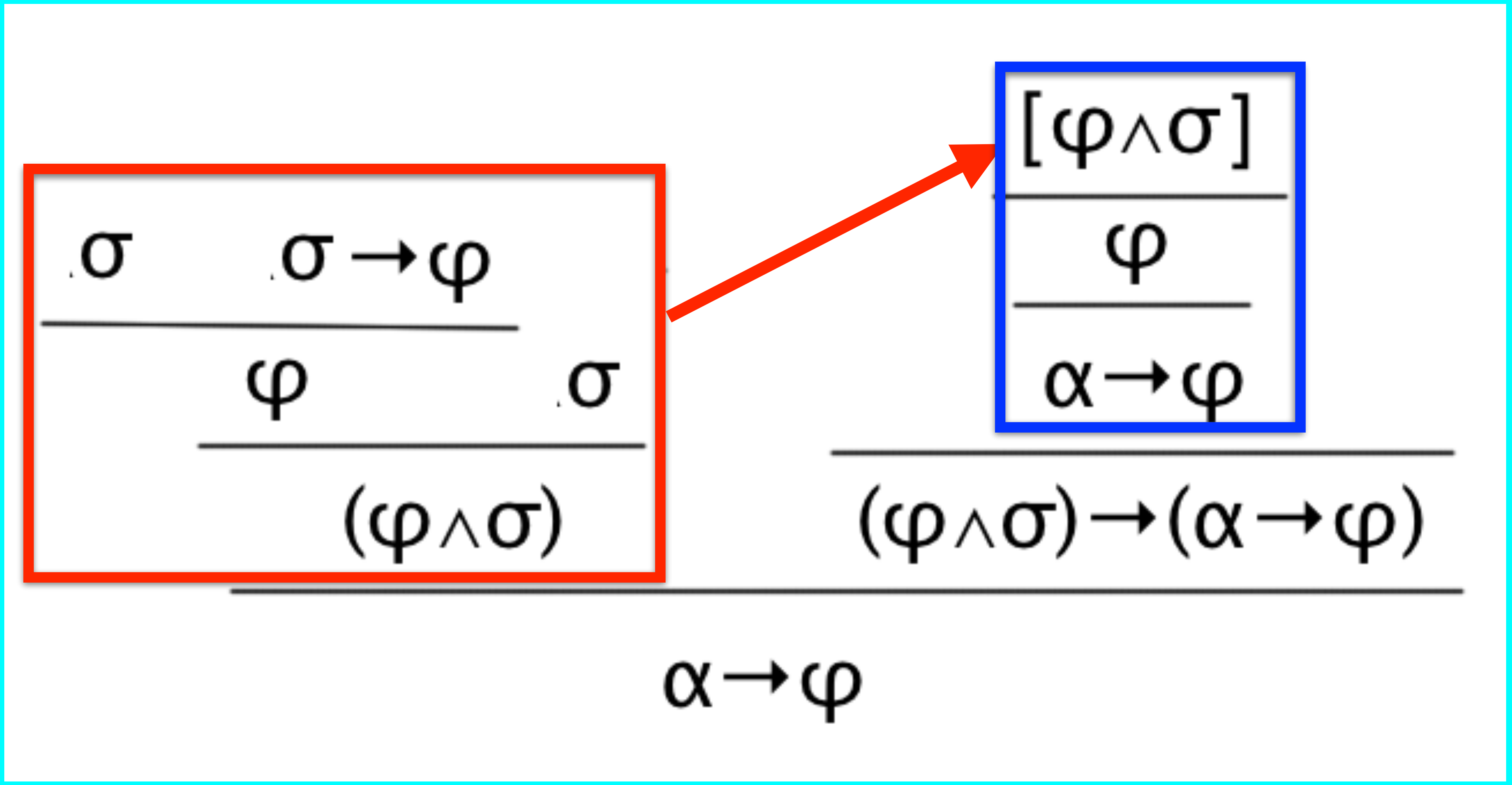
NORMALIZATION

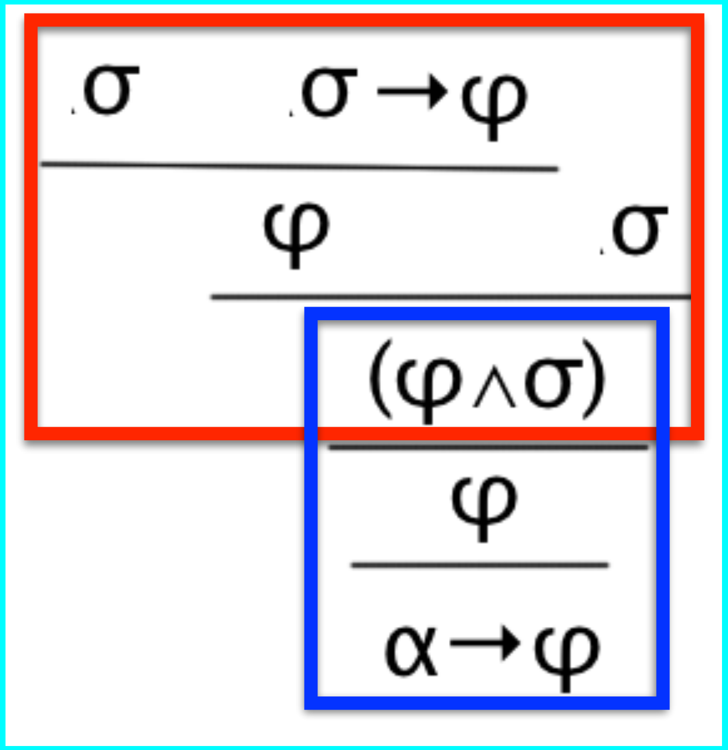
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\sigma\wedge\varphi]^2}{\varphi}\wedge E \qquad [\varphi\longrightarrow\psi]^1}{\psi}\longrightarrow E \qquad \dfrac{\dfrac{[\sigma\wedge\varphi]^2}{\sigma}\wedge E}{\psi\longrightarrow\sigma}\longrightarrow I}{\sigma}\longrightarrow E}{(\varphi\longrightarrow\psi)\longrightarrow\sigma}\longrightarrow I_1}{(\sigma\wedge\varphi)\longrightarrow((\varphi\longrightarrow\psi)\longrightarrow\sigma)}\longrightarrow I_2$$

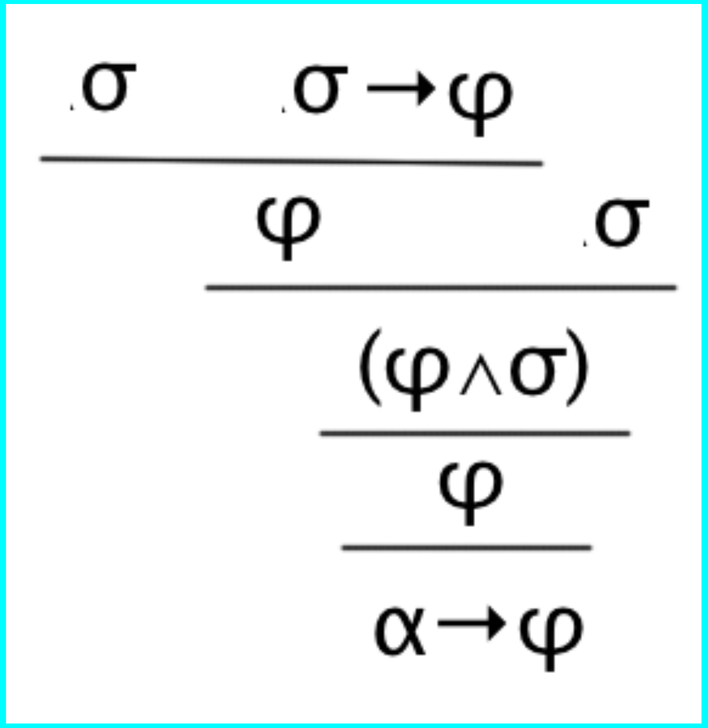
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\sigma \wedge \varphi]^2}{\varphi}\wedge E \qquad [\varphi \to \psi]^1}{\psi}\to E \qquad \dfrac{\dfrac{[\sigma \wedge \varphi]^2}{\sigma}\wedge E}{\psi \to \sigma}\to I}{\sigma}\to E}{(\varphi \to \psi) \to \sigma}\to I_1}{(\sigma \wedge \varphi) \to ((\varphi \to \psi) \to \sigma)}\to I_2$$

$$\dfrac{\dfrac{\dfrac{[\sigma \wedge \varphi]^1}{\sigma}\wedge E}{(\varphi \to \psi) \to \sigma}\to I}{(\sigma \wedge \varphi) \to ((\varphi \to \psi) \to \sigma)}\to I_1$$

$$\dfrac{\dfrac{\dfrac{\sigma \qquad \sigma\to\varphi}{\varphi} \qquad \sigma}{(\varphi\wedge\sigma)} \qquad \dfrac{\dfrac{\dfrac{[\varphi\wedge\sigma]}{\varphi}}{\alpha\to\varphi}}{(\varphi\wedge\sigma)\to(\alpha\to\varphi)}}{\alpha\to\varphi}$$


σ σ→φ
φ σ
(φ∧σ)
[φ∧σ]
φ
α→φ
(φ∧σ)→(α→φ)
α→φ

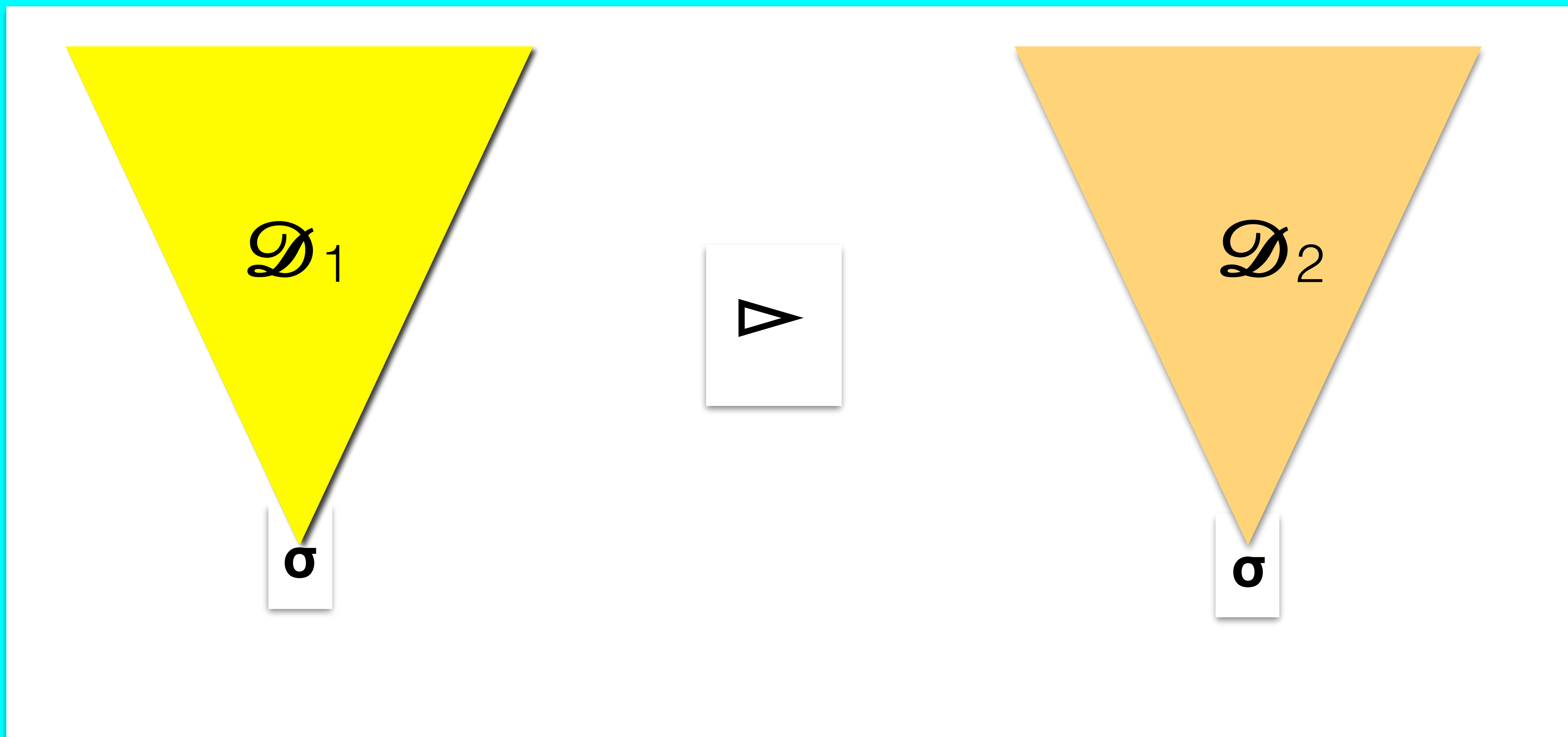

σ σ→φ
φ σ
(φ∧σ)
φ
α→φ


σ σ→φ
φ σ
(φ∧σ)
φ
α→φ

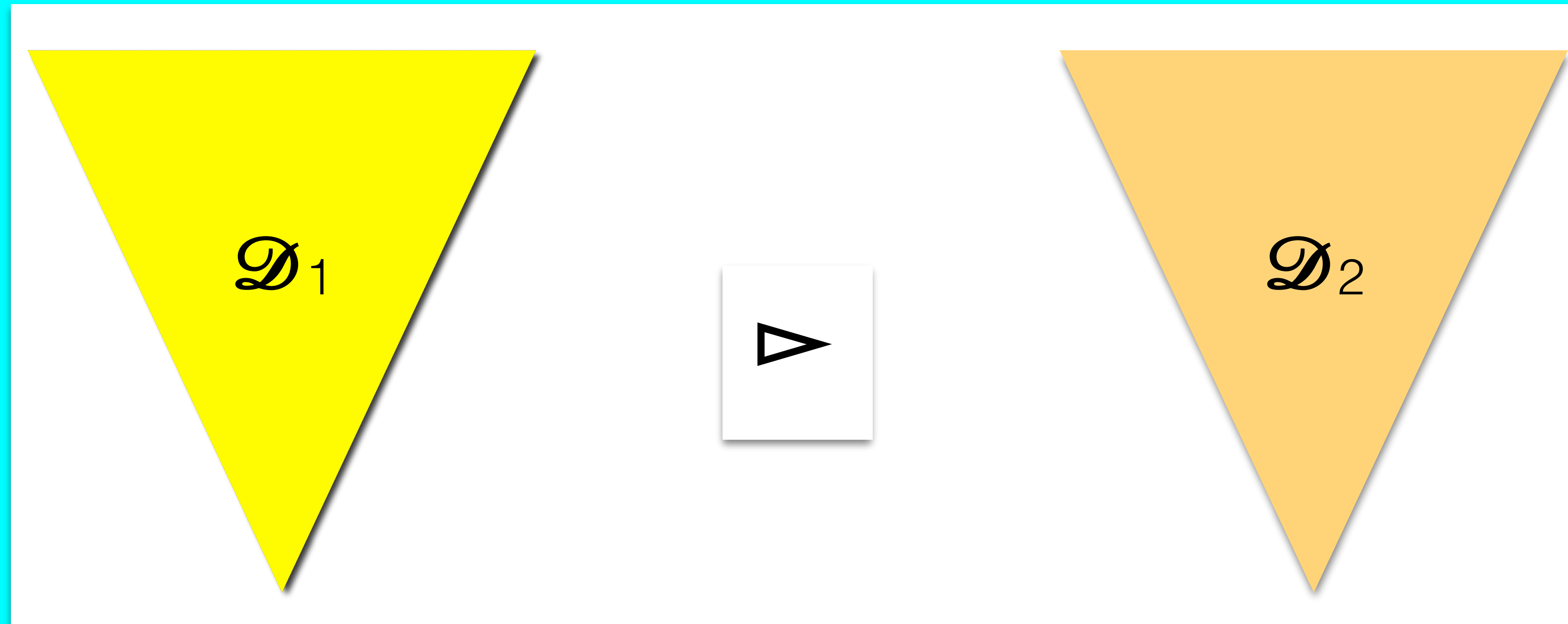
conversions

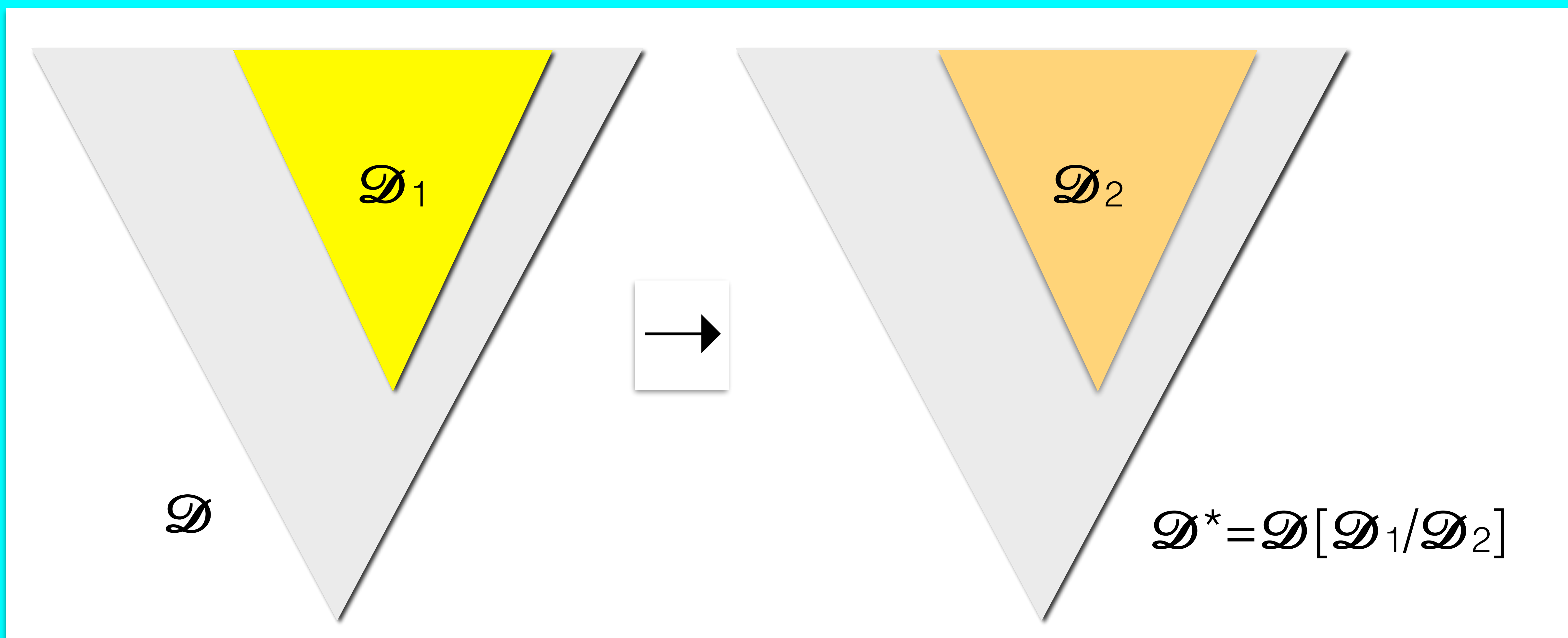
$$\dfrac{\begin{matrix}\mathcal{D}_1 \\ \psi\end{matrix} \qquad \dfrac{\begin{matrix}[\psi] \\ \mathcal{D}_2 \\ \varphi\end{matrix}}{\psi \to \varphi} \to I}{\varphi} \to E \qquad \rhd \qquad \begin{matrix}\mathcal{D}_1 \\ \psi \\ \mathcal{D}_2 \\ \varphi\end{matrix}$$

Redex/cut: sequence →I , →E



$hp\mathscr{D}_1 \subseteq hp\mathscr{D}_2$





$$\mathscr{D} \to \mathscr{D}^*$$

$\mathscr{D} \rightarrow \mathscr{D}^*$ ($\mathscr{D}$ **1-step reduces to** $\mathscr{D}^*$): $\mathscr{D}^*$ is obtained by applying a conversion to a subderivation of $\mathscr{D}$

$\mathscr{D} \twoheadrightarrow \mathscr{D}^*$ ($\mathscr{D}$ **reduces to** $\mathscr{D}^*$): $\exists\ \mathscr{D}_1 \ldots \mathscr{D}_n$ s.t. $\mathscr{D}=\mathscr{D}_1$, $\mathscr{D}^*=\mathscr{D}_n$, $\mathscr{D}_1 \rightarrow \ldots \rightarrow \mathscr{D}_n$

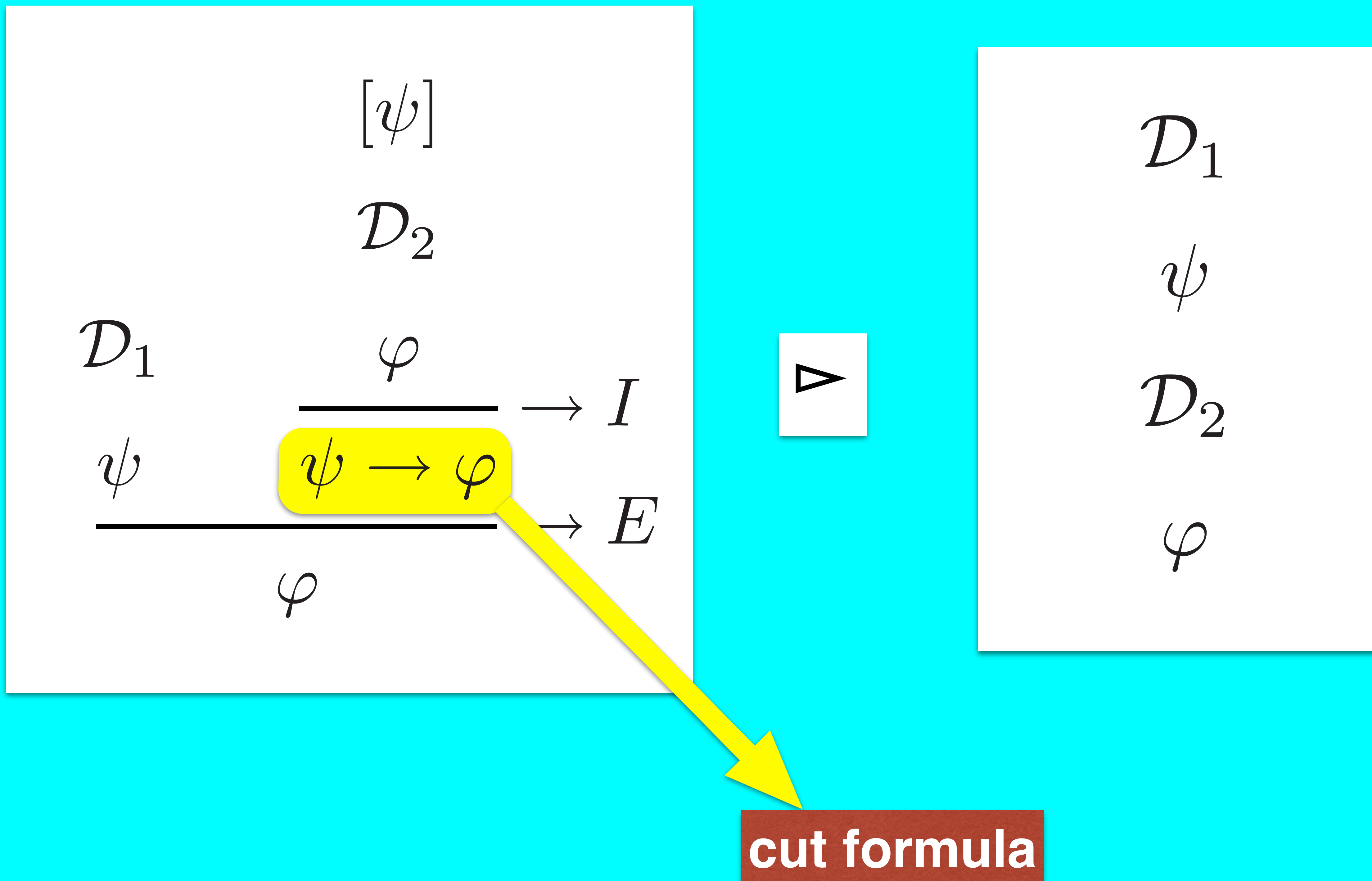
$\twoheadrightarrow$ is the reflexive and transitive closure of $\rightarrow$

$\mathscr{D}$ is in normal form (irreducible) if $\mathscr{D} \twoheadrightarrow \mathscr{D}^*$ implies that $\mathscr{D}=\mathscr{D}^*$

$\mathscr{D}$ is in normal form (irreducible) if there is no $\mathscr{D}^*$ s.t. $\mathscr{D} \rightarrow \mathscr{D}^*$

Theorem (weak normalisation)

for each $\mathscr{D}$ there is $\mathscr{D}^*$ s.t. $\mathscr{D} \twoheadrightarrow \mathscr{D}^*$ and $\mathscr{D}^*$ is in normal form



conversion with cut formula $\psi \to \varphi$

$d(\phi)$= size of ϕ

ϕ is maximal in a derivation $\mathscr{D}$ if:

1. ϕ is a cut formula
2. $d(\phi)=\max\{d(\delta): \delta \text{ is a cut formula in } \mathscr{D}\}$

Theorem (weak normalisation)

for each $\mathscr{D}$ there is $\mathscr{D}^*$ s.t. $\mathscr{D} \twoheadrightarrow \mathscr{D}^*$ and $\mathscr{D}^*$ is in normal form

d=max{d(δ): δ is a cut formula in $\mathscr{D}$}

n=#{δ : δ is an occurrence of a maximal cut}

Let call R($\mathscr{D}$) the pair (d,n) of $\mathscr{D}$.

Let us assume the lexicographic well order < for pairs of natural numbers: (d,n) < (d',n') iff d <d' or d=d' and n< n'.

The proof is by induction on R($\mathscr{D}$).

Base: if R($\mathscr{D}$)=(0,0) then $\mathscr{D}$ is in normal form;

Induction step: let us suppose that R($\mathscr{D}$)=(d,n).

Make a reduction with a maximal cut formula δ: $\mathscr{D} \rightarrow \mathscr{D}^*$, with R($\mathscr{D}^*$)=(d*,n*)

Now observe that (d*,n*)<(d,n) (if n>1 then d*=d and n*=n-1, if n=1, then d*<d)

By induction hypothesis $\mathscr{D}^* \twoheadrightarrow \mathscr{D}^{o}$

Since $\mathscr{D} \rightarrow \mathscr{D}^*$ and $\mathscr{D}^* \twoheadrightarrow \mathscr{D}^{o}$ we have the thesis.

SUBFORMULA PROPERTY

Theorem Let $\mathscr{D}$ be a normal deduction in the → fragment. Then

i) every formula in $\mathscr{D}$ is a subformula of a conclusion or a hypothesis of $\mathscr{D}$;

ii) if $\mathscr{D}$ ends in an elimination, it has a ***principal branch***,

i.e. a sequence of formulae $A_0, A_1, \ldots, A_n$ such that:

- A_0 is an (undischarged) hypothesis;
- A_n is the conclusion;
- A_i is the principal premise of an elimination of which the conclusion is A_{i+1} (for $i=0,\ldots,n-1$).

In particular A_n is a subformula of A_0

Proof 1. If $\mathscr{D}$ consists of a hypothesis, there is nothing to do.

2. If $\mathscr{D}$ ends in an introduction,

$$\cfrac{\begin{array}{c}[\psi]\\ \mathscr{D}_1\\ \varphi\end{array}}{\psi \to \varphi}$$

then it suffices to apply the induction hypothesis to $\mathscr{D}_1$.

3. If $\mathscr{D}$ ends in an elimination,

$$\frac{\psi \qquad \psi \to \varphi}{\varphi}$$

it is not possible that the proof above the principal premise ends in an introduction, so it ends in an elimination and has a principal branch, which can be extended to a principal branch of $\mathscr{D}$

lemma

if p is a propositional symbol then $\nvdash p$

Proof

Each deduction $\mathscr{D}$ with conclusion p should terminate with an elimination rule.

Therefore there is an undischarged hypothesis β s.t. p is a sub formula of p, and it is not true that $\mathscr{D}$ is a derivation without undischarged hypotheses of p.

Corollary

the natural deduction system based on → is consistent

COMPUTATIONS

finite computation

$\mathscr{D}_0 \to \mathscr{D}_1 \to \ldots \to \mathscr{D}_k$ with $\mathscr{D}_k$ in normal form

infinite computation

$\mathscr{D}_0 \to \mathscr{D}_1 \to \ldots \to \mathscr{D}_k \to \ \ldots$

A "computation"/"reduction sequence" of $\mathscr{D}$ of length $\kappa \leq \omega$ is a sequence of deductions $(\mathscr{D}_i)_{i<\kappa}$ such that $\mathscr{D}_0 = \mathscr{D}$ and $\mathscr{D}_{i-1} \to \mathscr{D}_i$ for all $i<\kappa$.

Theorem (strong normalisation)
for each $\mathscr{D}$, alle the computations of $\mathscr{D}$ are finite

Theorem (confluence)

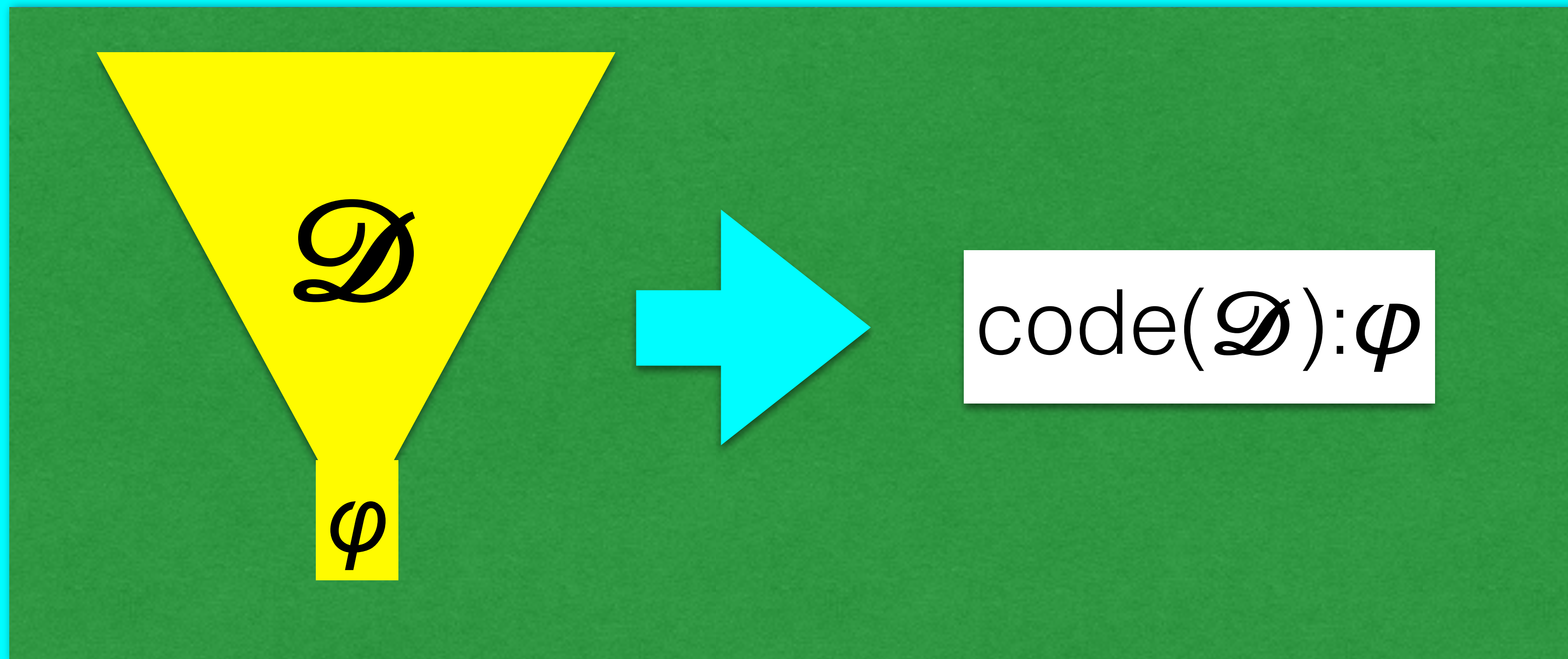
If $\mathscr{D} \twoheadrightarrow \mathscr{D}'$ and $\mathscr{D} \twoheadrightarrow \mathscr{D}''$ then there exist $\mathscr{D}^*$ s.t. $\mathscr{D}' \twoheadrightarrow \mathscr{D}^*$ and $\mathscr{D}'' \twoheadrightarrow \mathscr{D}^*$

Theorem (existence and unicity of normal form)

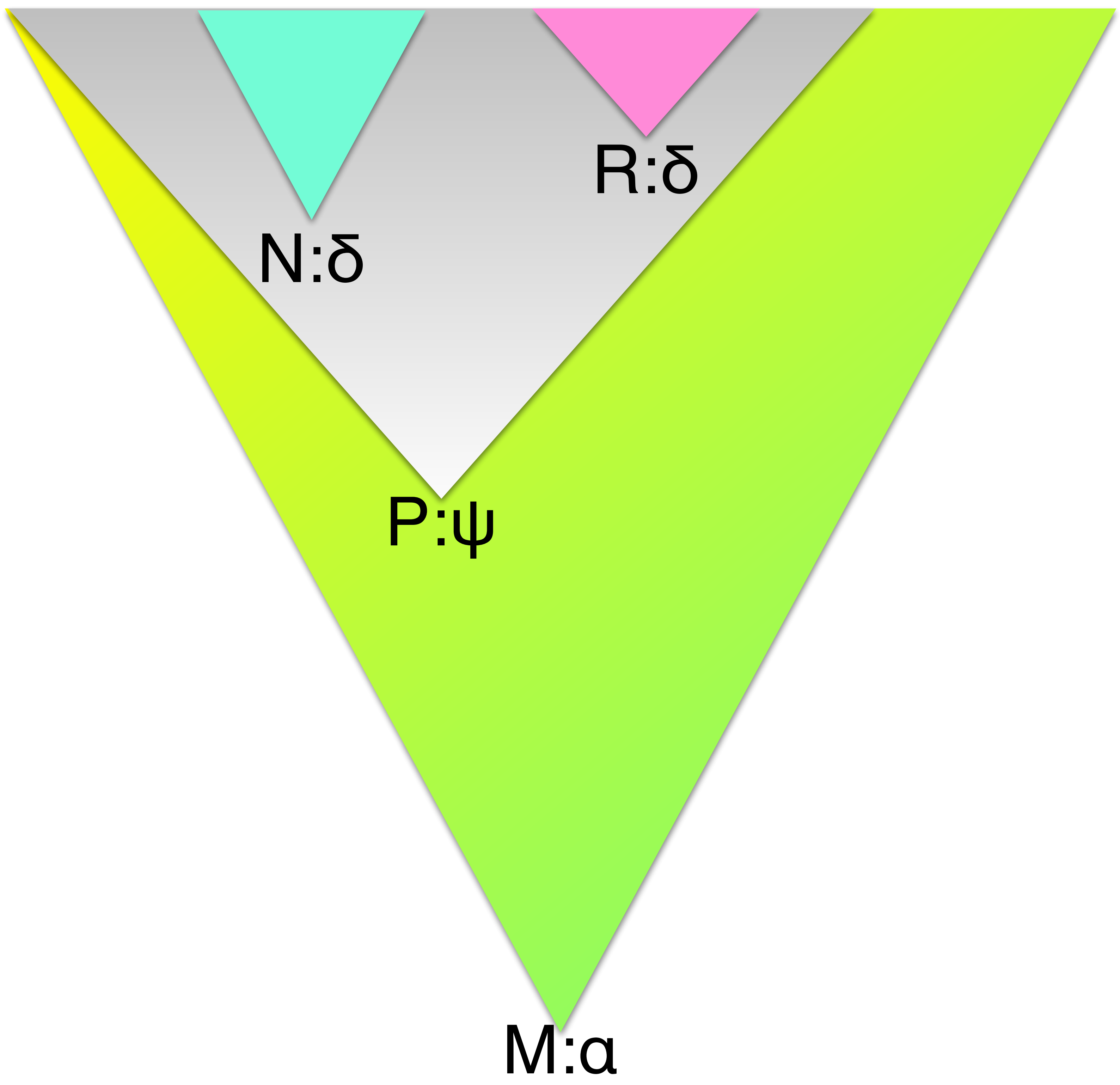
If $\mathscr{D} \twoheadrightarrow \mathscr{D}'$ and $\mathscr{D} \twoheadrightarrow \mathscr{D}''$ and $\mathscr{D}'$ and $\mathscr{D}''$ are in normal form $\mathscr{D}' = \mathscr{D}''$

For each $\mathscr{D}$ there is $\mathscr{D}'$ s.t. $\mathscr{D} \twoheadrightarrow \mathscr{D}'$ and $\mathscr{D}'$ is in normal form

coding the derivation



we inductively associate a code to each derivation, by means of a suitable decoration of formulas:
i.e. we assign to each occurrence of a formula δ in a deduction tree the code of the derivation with the occurrence δ as conclusion (root)



R:δ
N:δ
P:ψ
M:α

We assume to have a denumerable set $x_0, x_1, \ldots$ of "variables"
We decorate each assumption ψ with exactly one variable x.
Different occurrences of the same assumption ψ may be decorated with the same variable x.
Occurrences of different formulas must be decoded with different variables.

The set of **decorated derivations** is the ***smallest set Y*** such that

$x{:}\delta \in Y$

if $\begin{matrix} x{:}\delta \\ \mathscr{D} \\ M{:}\beta \end{matrix} \in Y$ then $\dfrac{\begin{matrix} [x{:}\delta] \\ \mathscr{D} \\ M{:}\beta \end{matrix}}{\lambda x^{\delta}.M{:}\delta\to\beta} \in Y$

if $\begin{matrix} \mathscr{D}_1 \\ M{:}\beta \end{matrix}$ $\begin{matrix} \mathscr{D}_2 \\ N{:}\beta\to\delta \end{matrix} \in Y$ then $\dfrac{\begin{matrix} \mathscr{D}_2 \\ N{:}\beta\to\delta \end{matrix} \quad \begin{matrix} \mathscr{D}_1 \\ M{:}\beta \end{matrix}}{NM{:}\delta} \in Y$

$$\frac{\begin{array}{c}[x:\delta]\\ \vdots \\ M:\beta\end{array}}{\lambda x^{\delta}.M:\delta\rightarrow\beta}$$

$$\frac{N:\beta\rightarrow\delta \quad M:\beta}{NM:\delta}$$

THE STRINGS USED TO CODE DERIVATIONS ARE CALLED λ-TERMS

REDUCTIONS

$$
\dfrac{\dfrac{\begin{array}{c}[x:\delta]\\ \mathcal{D}_1\\ M:\beta\end{array}}{\lambda x^{\delta}.M:\delta\to\beta} \qquad \begin{array}{c}\mathcal{D}_2\\ N:\delta\end{array}}{(\lambda x^{\delta}.M)N:\beta}
\quad \rhd \quad
\begin{array}{c}\mathcal{D}_2\\ N:\delta\\ \mathcal{D}_1[N/x]\\ M[N/x]:\beta\end{array}
$$

$$(\lambda x^{\delta}.M)N:\beta \;\rhd\; M[N/x]:\beta$$

$$\frac{(\lambda x^{\delta}.M)N:\beta \rhd M[N/x]:\beta}{(\lambda x^{\delta}.M)N:\beta \rightarrow M[N/x]:\beta}$$

$$\frac{M:\beta \rightarrow M_1:\beta}{MN:\beta \rightarrow M_1N:\beta}$$

$$\frac{M:\beta \rightarrow M_1:\beta}{NM:\beta \rightarrow NM_1:\beta}$$

$$\frac{M:\beta \rightarrow M_1:\beta}{\lambda x^{\delta}.M:\beta \rightarrow \lambda x^{\delta}.M_1:\beta}$$

$\twoheadrightarrow$ is the reflexive and transitive closure of $\rightarrow$

exercise: prove that

$\Gamma,\beta \vdash \beta$

$\Gamma,\beta \vdash \psi \;\Rightarrow\; \Gamma \vdash \beta \to \psi$

$\Gamma \vdash \beta \to \psi \;\&\; \Gamma \vdash \beta \;\Rightarrow\; \Gamma \vdash \psi$

we have the following rules for the derivation relation

$\Gamma,\beta \vdash \beta$

$$\frac{\Gamma,\beta \vdash \psi}{\Gamma \vdash \beta \to \psi}$$

$$\frac{\Gamma \vdash \beta \to \psi \qquad \Gamma \vdash \beta}{\Gamma \vdash \psi}$$

exercise: prove that

$\Gamma,\beta\vdash\beta$

$\Gamma,\beta\vdash\psi \ \Rightarrow \Gamma\vdash\beta\to\psi$

$\Gamma\vdash\beta\to\psi \ \& \ \Gamma\vdash\beta \ \Rightarrow \Gamma\vdash\psi$

we have the following rules for the derivation relation

$\Gamma,\beta\vdash\beta$

$$\frac{\Gamma,\beta\vdash\psi}{\Gamma\vdash\beta\to\psi}$$

$$\frac{\Gamma\vdash\beta\to\psi \quad \Gamma\vdash\beta}{\Gamma\vdash\psi}$$

$\Gamma,x:\beta\vdash x:\beta$

$$\frac{\Gamma,x:\beta\vdash M:\psi}{\Gamma\vdash\lambda x^{\beta}.M:\beta\to\psi}$$

$$\frac{\Gamma\vdash M:\beta\to\psi \quad \Gamma\vdash N:\beta}{\Gamma\vdash MN:\psi}$$

CURRY-HOWARD ISOMORPHISM

FORMULAS	⇔	TYPES
PROOFS	⇔	PROGRAMS (λ-TERMS)
REDUCTIONS	⇔	COMPUTATIONS

The full propositional system

we can restrict our attention to applications of the ⊥ –rule for atomic instances

$$\frac{\bot}{p} \quad p \text{ atomic}$$

The proof is done by a trivial induction (exercise)
Hint: transform e.g.

$$\begin{array}{c}\mathscr{D} \\ \dfrac{\bot}{\alpha\to\sigma}\end{array} \qquad \text{with} \qquad \begin{array}{c}\mathscr{D} \\ \dfrac{\dfrac{\bot}{\sigma}}{\alpha\to\sigma}\end{array}$$

The conversion for ∧ and ∨

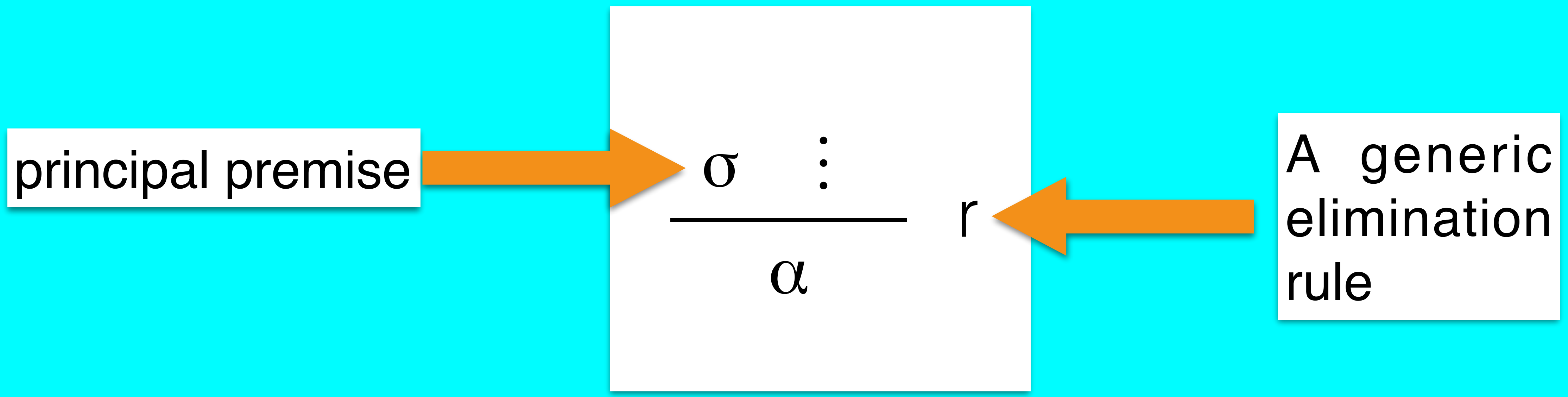
$$\dfrac{\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ \varphi_1 & \varphi_2\end{array}}{\varphi_1 \wedge \varphi_2}\wedge I}{\varphi_i}\wedge E \quad \triangleright \quad \begin{array}{c}\mathcal{D}_i\\ \varphi_i\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \varphi_i\end{array}}{\varphi_1 \vee \varphi_2}\vee I \qquad \begin{array}{c}[\varphi_1]\\ \mathcal{D}_1\\ \sigma\end{array} \qquad \begin{array}{c}[\varphi_2]\\ \mathcal{D}_2\\ \sigma\end{array}}{\sigma}\vee E \quad \triangleright \quad \begin{array}{c}\mathcal{D}_i\\ \varphi_i\\ \mathcal{D}_1\\ \sigma\end{array}$$

we need more conversions

$$\dfrac{\dfrac{\alpha\vee\alpha \qquad \dfrac{[\alpha]\quad[\alpha]}{\alpha\wedge\alpha} \qquad \dfrac{[\alpha]\quad[\alpha]}{\alpha\wedge\alpha}}{\alpha\wedge\alpha}}{\alpha}$$

failure of subformula property


principal premise
σ
⋮
α
r
A generic elimination rule

commuting conversions

$$\frac{\dfrac{\bot}{\sigma} \quad \vdots}{\alpha}\, r \quad \triangleright \quad \frac{\bot}{\alpha}$$

$$\frac{\dfrac{\alpha\vee\beta \qquad \begin{matrix}[\beta]\\ \vdots\\ \sigma\end{matrix}}{\sigma} \quad \vdots}{\gamma}\, r \quad \triangleright \quad \frac{\alpha\vee\beta \qquad \dfrac{\begin{matrix}[\alpha]\\ \vdots\\ \sigma\end{matrix} \quad \vdots}{\gamma}\,r \qquad \dfrac{\begin{matrix}[\beta]\\ \vdots\\ \sigma\end{matrix}\quad \vdots}{\gamma}\,r}{\gamma}$$

Theorem (weak normalisation)

for each $\mathscr{D}$ there is $\mathscr{D}^*$ s.t. $\mathscr{D} \twoheadrightarrow \mathscr{D}^*$ and $\mathscr{D}^*$ is in normal form

Theorem (confluence)

If $\mathscr{D} \twoheadrightarrow \mathscr{D}'$ and $\mathscr{D} \twoheadrightarrow \mathscr{D}''$ then there exist $\mathscr{D}^*$ s.t.

$\mathscr{D}' \twoheadrightarrow \mathscr{D}^*$ and $\mathscr{D}'' \twoheadrightarrow \mathscr{D}^*$

Theorem (existence and unicity of normal form)

If $\mathscr{D} \twoheadrightarrow \mathscr{D}'$ and $\mathscr{D} \twoheadrightarrow \mathscr{D}''$ and $\mathscr{D}'$ and $\mathscr{D}''$ are in normal form

$\mathscr{D}' = \mathscr{D}''$

For each $\mathscr{D}$ there is $\mathscr{D}'$ s.t. $\mathscr{D} \twoheadrightarrow \mathscr{D}'$ and $\mathscr{D}'$ is in normal form