We treat the following simple but instructive case.

Let M be a compact (smooth) manifold, equipped with a finite atlas (this can be achieved in view of compactness). We are going to construct a smooth partition of unity subordinate to it.

Let $\mathcal{A} = \{ \mathcal{U}_i, \varphi_i \}_{i=1..N}$ be the atlas in question — some N

First of all, we may alter the local charts φ_i in such a way that

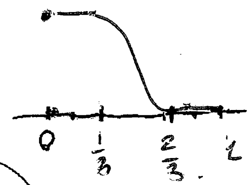
$$\varphi_i : \mathcal{U}_i \xrightarrow{\text{omeo}} B_1(0) \subset \mathbb{R}^n$$

ball of radius 1 centred at 0

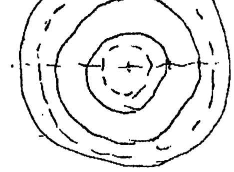
Define:

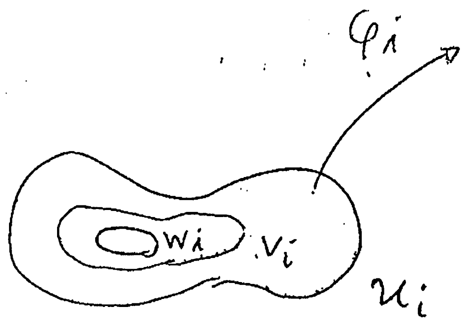
$$W_i \subset V_i \subset \mathcal{U}_i \qquad i = 1..N$$

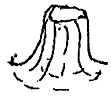
$$W_i := \varphi_i^{-1} (B_{\frac{1}{3}} (0))$$
radius



$$V_i := \varphi_i^{-1} (B_{\frac{2}{3}} (0))$$







Now let $f_i : \mathbb{R}^n \to \mathbb{R}$ a bump function adapted to

They can be equal to a single function f

see below, auxiliary constructions

Partitions of unity allow the construction of global tensors, given local ones.

Take a family of tensors t_α on $\mathcal{U}_\alpha$

Define

$$t = \sum_{\alpha \in \Omega} \rho_\alpha t_\alpha$$

(the sum is finite at each point)

This is a global tensor on M.

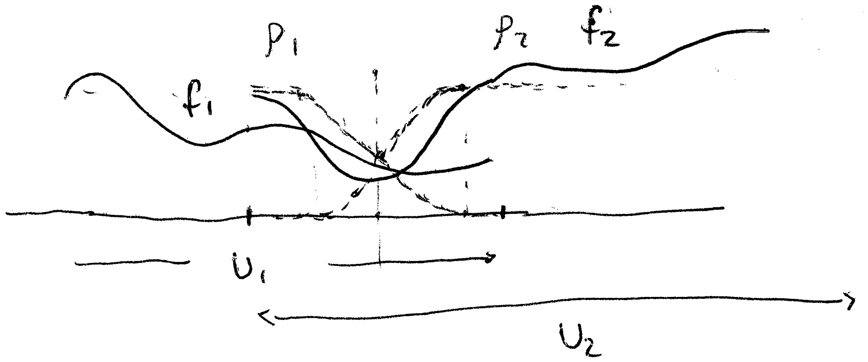
Notice that, given t, then obviously

$$t = \sum_{\alpha \in \Omega} \rho_\alpha t_\alpha$$

from $t_\alpha(x) = t_\beta(x) = t(x) \quad \forall x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$

and from $\sum_\alpha \rho_\alpha = 1$

Example



$f = \rho_1 f_1 + \rho_2 f_2$ is a global function on $\mathbb{R}$