Quivers, algebras and representations

Aim: Construct algebras associated to oriented graphs and study their representations.

§1 Quivers and path algebras

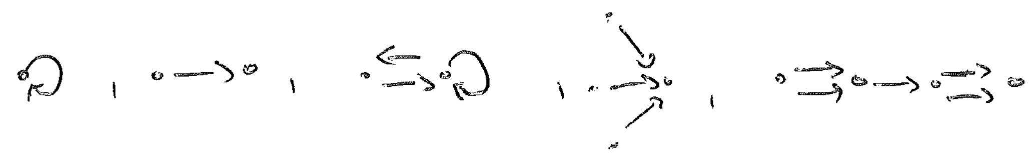
Def: A quiver $Q = (Q_0, Q_1, s, t)$ is given by two finite sets

$Q_0 \longleftrightarrow$ vertices

$Q_1 \longleftrightarrow$ arrows

and two maps $s, t : Q_1 \to Q_0$ which associate to $\alpha \in Q_1$ its source $s(\alpha) \in Q_0$ and its target $t(\alpha) \in Q_0$.

Write: $s(\alpha) \xrightarrow{\alpha} t(\alpha)$

Ex: 

A path of length $l \geq 1$ from a to b with $a, b \in Q_0$ is a sequence $a = a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} a_2 \xrightarrow{\alpha_3} \dots \xrightarrow{\alpha_l} a_l = b$ where $a_0, \dots, a_l \in Q_0$, $\alpha_1, \dots, \alpha_l \in Q_1$ and $s(\alpha_i) = a_{i-1}$, $t(\alpha_i) = a_i$ $\forall i \in \{1, \dots, l\}$.

Write: $\alpha_l \alpha_{l-1} \alpha_{l-2} \cdots \alpha_1$

To every $a \in Q_0$ we associate a path e_a of length 0.

Def: Q quiver. The path algebra $\mathbb{K}Q$ of Q over a field $\mathbb{K}$ is the $\mathbb{K}$-algebra whose underlying vectorspace has as basis the set of all paths of length $l \geq 0$ in Q and multiplication defined by

$$(\alpha_l \alpha_{l-1} \dots \alpha_1) \cdot (\beta_r \beta_{r-1} \dots \beta_1) := \begin{cases} \alpha_l \dots \alpha_1 \beta_r \dots \beta_1, & \text{if } t(\beta_r) = s(\alpha_1) \\ 0, & \text{else} \end{cases}$$

This defines a multiplication on the whole of $\mathbb{K}Q$.

Rmk: Denote by Q_ℓ the set of all paths in Q of length ℓ.

Then $\mathbb{K}Q = \mathbb{K}Q_0 \oplus \mathbb{K}Q_1 \oplus \dots \oplus \mathbb{K}Q_\ell \oplus \dots$

"decomposition of vectorspaces"

with $(\mathbb{K}Q_n) \cdot (\mathbb{K}Q_m) \subseteq \mathbb{K}Q_{n+m} \quad \forall n,m \geq 0$.

$\mathbb{K}Q$ becomes graded $\mathbb{K}$-algebra.

Ex: 1.) $Q = 1 \circlearrowleft \alpha$

Basis of $\mathbb{K}Q$ given by $\{e_1, \alpha, \alpha^2, \dots, \alpha^\ell, \dots\}$.

Multiplication: $e_1 \cdot \alpha^\ell = \alpha^\ell \cdot e_1 = \alpha^\ell \quad \forall \ell \geq 0$

$\alpha^\ell \cdot \alpha^k = \alpha^{\ell+k} \quad \forall \ell, k \geq 0$.

Get an algebra isomorphism $\mathbb{K}Q \longrightarrow \mathbb{K}[x]$ by mapping

$e_1 \longmapsto 1$

$\alpha \longmapsto x$

2.) $Q = \alpha \circlearrowright 1 \circlearrowleft \beta$

Similarly, get isomorphism of algebras $\mathbb{K}Q \longrightarrow \mathbb{K}\langle x, y\rangle$ by

$e_1 \longmapsto 1$

$\alpha \longmapsto x$

$\beta \longmapsto y$.

3.) $Q = 1 \xrightarrow{\alpha} 2$.

Basis of $\mathbb{K}Q$ given by $\{e_1, e_2, \alpha\}$.

Multiplication:

	e_1	e_2	α
e_1	e_1	0	0
e_2	0	e_2	α
α	α	0	0

Get an isomorphism of algebras

$$\mathbb{K}Q \longrightarrow \begin{pmatrix} \mathbb{K} & 0 \\ \mathbb{K} & \mathbb{K} \end{pmatrix} = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \,\middle|\, a,b,c \in \mathbb{K} \right\} \quad \text{by}$$

$$e_1 \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$e_2 \longmapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\alpha \longmapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

4.) $Q =$
$$\begin{tikzcd} & 1 \arrow[dl, "\alpha"'] \arrow[d, "\beta"] \arrow[dr, "\gamma"] & \\ 2 & 3 & 4 \end{tikzcd}$$
, basis of $\mathbb{K}Q$ is $\{e_1, e_2, e_3, \alpha, \beta, \gamma\}$.

Then $\mathbb{K}Q \cong \begin{pmatrix} \mathbb{K} & 0 & 0 & 0 \\ \mathbb{K} & \mathbb{K} & 0 & 0 \\ \mathbb{K} & 0 & \mathbb{K} & 0 \\ \mathbb{K} & 0 & 0 & \mathbb{K} \end{pmatrix}$.

<u>Lemma</u> Q quiver. Then

(1) $\mathbb{K}Q$ is an associative $\mathbb{K}$-algebra with $1_{\mathbb{K}Q} = \sum_{i \in Q} e_i$.

(2) $\mathbb{K}Q$ finite dimensional $\Leftrightarrow$ Q acyclic
(i.e. $\nexists$ path γ of length ≥ 1 with $s(\gamma) = t(\gamma)$)

(3) $\{e_i \mid i \in Q_0\}$ is a complete set of primitive orthogonal idempotents for $\mathbb{K}Q$.

<u>Pf</u> (1) * composition of paths is associative.
* $\forall i \in Q_0$, e_i an idempotent and $e_i \cdot e_j = 0$ for $j \neq i$

$\Rightarrow$ $1_{\mathbb{K}Q} = \sum_{i \in Q_0} e_i$.

(2) If γ a cycle (i.e. γ path of length ≥ 1 with $s(\gamma) = t(\gamma)$), then γ^t a basis vector of $\mathbb{K}Q$ $\forall t \geq 0$ $\Rightarrow$ $\mathbb{K}Q$ infinite dimensional.

Conversely, Q acyclic $\Rightarrow$ Q has only finitely many paths

$\Rightarrow$ $\mathbb{K}Q$ finite dimensional.

(3) enough to check: e_i primitive $\forall i \in Q_0$.

(i.e., if $e_i = f_1 + f_2$ for orthogonal idempotents $f_1, f_2 \in \mathbb{K}Q$, then $f_1 = 0$ or $f_2 = 0$)

Fix some $i \in Q_0$.

Observe: e_i primitive $\Leftrightarrow$ $e_i \mathbb{K}Q e_i$ has only $0, e_i$ as idempotents.

'$\Leftarrow$' ✓

'$\Rightarrow$' Let $f \neq 0, e_i$ be an idempotent in $e_i \mathbb{K}Q e_i$. Then

$$e_i = \underset{\neq 0}{f} + \underbrace{(e_i - f)}_{\neq 0} \quad \text{with } f \text{ and } (e_i - f) \text{ orthogonal}$$

idempotents in $\mathbb{K}Q$.

Now take an idempotent g in $e_i \mathbb{K}Q e_i$ and write $g = \lambda e_i + \omega$ for $\lambda \in \mathbb{K}$ and ω a linear combination of cycles through i of length ≥ 1. Then

$$0 = g^2 - g = \lambda^2 e_i + 2\lambda\omega + \omega^2 - \lambda e_i - \omega = (\lambda^2 - \lambda) e_i + (2\lambda - 1)\omega + \omega^2$$

$\Rightarrow$ $\omega = 0$ and $\lambda^2 = \lambda$, thus, $\lambda = 1$ or $\lambda = 0$ saying $g = e_i$ or $g = 0$.

$\square$

Rmk: $\mathbb{K}Q$ finite dimensional. Then

$${}_{\mathbb{K}Q}(\mathbb{K}Q) = \mathbb{K}Qe_1 \oplus \dots \oplus \mathbb{K}Qe_n \quad \text{for a complete set of}$$

primitive orthogonal idempotents $\{e_1, \dots, e_n\}$. The $\mathbb{K}Qe_i$ are indecomposable projective left $\mathbb{K}Q$-modules.

Exercises: (1) Let $Q = 1 \rightrightarrows 2$ be the Kronecker quiver.

Show: $\mathbb{K}Q \cong \begin{pmatrix} \mathbb{K} & 0 \\ \mathbb{K}^2 & \mathbb{K} \end{pmatrix}$ where $\mathbb{K}^2$ is viewed as a $\mathbb{K}$-$\mathbb{K}$-bimodule in the obvious way.

(2) Q quiver. Show that, in general, the set $\{e_i \mid i \in Q_0\}$ is not the unique complete set of primitive orthogonal idempotents for $\mathbb{K}Q$.

(3) A $\mathbb{K}$-algebra A is called connected if $0, 1$ are the only idempotents that lie in the center of A.

Show: Q a connected quiver $\Leftrightarrow$ $\mathbb{K}Q$ connected $\mathbb{K}$-algebra.

§2 Admissible ideals and quotients of path algebras

Q connected quiver.

The two-sided arrow ideal of $\mathbb{K}Q$ is given by

$$R_Q := \langle \alpha \mid \alpha \in Q_1 \rangle.$$

As vectorspaces $R_Q = \mathbb{K}Q_1 \oplus \dots \oplus \mathbb{K}Q_l \oplus \dots$, $\mathbb{K}Q_l$ gen. by paths of length l.

Q acyclic $\Rightarrow$ $R_Q = \operatorname{rad} \mathbb{K}Q$ (see below)

Def: A two-sided ideal $I \trianglelefteq \mathbb{K}Q$ is called admissible if $\exists\, m \geq 2$ s.th.

$$R_Q^m \subseteq I \subseteq R_Q^2 .$$

(Q, I) called <u>bound quiver</u> and $\mathbb{K}Q/_I$ <u>bound path algebra</u>.

<u>Rmk:</u> Q acyclic $\Rightarrow$ every ideal I in R_Q^2 is admissible.

<u>Ex:</u> (1): $Q =$
$$\begin{tikzcd} & 2 \arrow[dr, "\gamma"] & \\ 1 \arrow[ur, "\alpha"] \arrow[rr, "\varepsilon"] \arrow[dr, "\beta"'] & & 4 \\ & 3 \arrow[ur, "\delta"'] & \end{tikzcd}$$
.

$I_1 = \langle \gamma\alpha - \delta\beta \rangle$ is admissible

$I_2 = \langle \gamma\alpha - \varepsilon \rangle$ not admissible, in fact, $\gamma\alpha - \varepsilon \notin R_Q^2$.

(2): $Q = 1 \xrightarrow{\alpha} 2 \overset{\beta}{\underset{\gamma}{\rightrightarrows}} 3$.

Both $I_1 = \langle \beta\alpha \rangle$ and $I_2 = \langle \beta\alpha - \gamma\alpha \rangle$ are admissible. There is an isomorphism $\mathbb{K}Q/_{I_1} \longrightarrow \mathbb{K}Q/_{I_2}$ by mapping

$$\begin{aligned} e_i &\longmapsto e_i \qquad i \in \{1,2,3\} \\ \alpha &\longmapsto \alpha \\ \beta &\longmapsto \beta - \gamma \\ \gamma &\longmapsto \gamma \end{aligned}$$

(3): $Q = 1 \circlearrowleft \alpha$.

$\forall m \geq 2$: $I_m = R_Q^m = \langle \alpha^m \rangle \trianglelefteq \mathbb{K}Q$ is admissible and $\mathbb{K}Q/_{I_m} \cong \mathbb{K}[x]/_{\langle x^m \rangle}$.

In part., $\mathbb{K}Q/_{I_m}$ is local with unique maximal ideal $\langle \alpha \rangle$.

<u>Proposition:</u> Q connected quiver, $I \trianglelefteq \mathbb{K}Q$ admissible. Then

(1) $\mathbb{K}Q/I$ is a finite dimensional connected $\mathbb{K}$-algebra.

(2) The set $\{ e_i + I \mid i \in Q_0 \}$ is a complete set of primitive orthogonal idempotents for $\mathbb{K}Q/_I$.

(3) The ideal I is finitely generated.

(4) $\operatorname{rad}(\mathbb{K}Q/_I) = R_Q/_I$.

Pf (1) I admissible $\Rightarrow \exists\, m \geq 2$ such that $R_Q^m \subseteq I$.

Get surjective algebra homomorphism $\mathbb{K}Q/_{R_Q^m} \twoheadrightarrow \mathbb{K}Q/_I$.

Thus, enough to check that $\mathbb{K}Q/_{R_Q^m}$ finite dimensional.

But a basis of $\mathbb{K}Q/_{R_Q^m}$ is given by $\{\bar{\gamma} \mid \gamma$ path in Q of length $< m\}$ and this set is finite.

For connectedness see Exercise (3).

(2) Similar to statement for $\mathbb{K}Q$.

(3) Again, take $m \geq 2$ such that $R_Q^m \subseteq I$. Get short exact sequence of $\mathbb{K}Q$-modules $0 \to R_Q^m \to I \to I/_{R_Q^m} \to 0$.

Have to check: R_Q^m, $I/_{R_Q^m}$ are finitely generated $\mathbb{K}Q$-modules.

R_Q^m generated by the paths of length m ✓

Moreover, $I/_{R_Q^m}$ is an ideal of $\mathbb{K}Q/_{R_Q^m}$ and, hence, finite dimensional. In part., $I/_{R_Q^m}$ a fin. generated $\mathbb{K}Q$-module.

(4) Take $m \geq 2$ such that $R_Q^m \subseteq I$.

Thus, $(R_Q/I)^m = 0$ and R_Q/I nilpotent in $\mathbb{K}Q/I$.

$\Rightarrow R_Q/_I \subseteq \mathrm{rad}(\mathbb{K}Q/_I)$.

Conversely, we have $(\mathbb{K}Q/_I)/(R_Q/_I) \cong \mathbb{K} \times \dots \times \mathbb{K}$.

$\Rightarrow \mathrm{rad}(\mathbb{K}Q/_I) \subseteq R_Q/_I$. $\square$

Exercise (4) Consider $Q = \alpha \circlearrowleft 1 \circlearrowright \beta$.

Decide if the following ideals are admissible:

i.) $I_1 = \langle \beta^2, \alpha^3, \beta\alpha\beta \rangle$

ii.) $I_2 = \langle \alpha\beta - \beta\alpha, \beta^2, \alpha^2 \rangle$.

(5) Consider $Q = 1 \circlearrowright \alpha$. $\mathbb{C}$ = complex numbers.

Show that $0 = \operatorname{rad} \mathbb{C}Q \neq R_Q$.

(6) Give an example of a fin. dim $\mathbb{K}$-algebra A such that $(\operatorname{rad} A)^{1000} \neq 0$ and $(\operatorname{rad} A)^{1001} = 0$.

§3 The quiver of a finite dimensional algebra

$\mathbb{K}$ algebraically closed field.

A finite dimensional $\mathbb{K}$-algebra

Def A is called <u>basic</u> if $A/\operatorname{rad} A$ isomorphic to a product $\mathbb{K} \times \dots \times \mathbb{K}$ of copies of $\mathbb{K}$.

Ex (1) $A = \mathbb{K}Q/I$ a bound path algebra.

Then $\operatorname{rad} A = R_Q/I$ and therefore $A/\operatorname{rad} A \cong \mathbb{K}Q/R_Q \cong \mathbb{K} \times \dots \times \mathbb{K}$.

(2) Let $n \geq 2$. Then $M_n(\mathbb{K})$ is not basic.

We have $\operatorname{rad}(M_n(\mathbb{K})) = \{0\}$.

Rmk

(1) A a fin. dim $\mathbb{K}$-algebra. Then there is a fin. dim. basic $\mathbb{K}$-algebra A' such that $\operatorname{mod}(A) \cong \operatorname{mod}(A')$.

E.g. $\operatorname{mod}(\mathbb{K}) \cong \operatorname{mod}(M_n(\mathbb{K})) \quad \forall n \geq 1$.

(2) Simple modules over a basic $\mathbb{K}$-algebra are one-dimensional.

Theorem: A a basic and connected fin. dim. $\mathbb{K}$-algebra. Then there is a quiver Q_A and an admissible ideal $I \trianglelefteq \mathbb{K}Q_A$ such that $A \cong \mathbb{K}Q_A/I$.

Sketch of proof:

(1) Find the quiver Q_A:

Take a complete set $\{e_1, \dots, e_n\}$ of primitive orthogonal idempotents for A. Set $(Q_A)_0 := \{1, \dots, n\}$. Given $a, b \in (Q_A)_0$, the arrows $a \xrightarrow{\alpha} b$ correspond to basis vectors in $\underbrace{e_b(\operatorname{rad} A/\operatorname{rad}^2 A)e_a}_{\text{finite dimensional } \mathbb{K}\text{-vectorspace}}$.

Thus, we obtain a quiver Q_A.

One checks that Q_A is connected and does not depend on the chosen set $\{e_1, \dots, e_n\}$ of idempotents.

(2) Construct an algebra homom $\mathbb{K}Q_A \xrightarrow{\varphi} A$:

Set $\varphi_0 : (Q_A)_0 \to A$, $a \mapsto e_a$

$\varphi_1 : (Q_A)_1 \to A$, $(a \xrightarrow{\alpha} b) \longmapsto x_\alpha \in \operatorname{rad} A$ such that

$\{x_\alpha + \operatorname{rad}^2 A \mid a \xrightarrow{\alpha} b\}$ forms a basis of $e_b(\operatorname{rad} A/\operatorname{rad}^2 A)e_a$.

φ_0, φ_1 extend to an algebra homom $\varphi : \mathbb{K}Q_A \to A$ by defining

$$\varphi(\alpha_\ell \cdots \alpha_1) := \varphi_1(\alpha_\ell) \cdot \ldots \cdot \varphi_1(\alpha_1).$$

(3) Show: φ surjective.

Have a vectorspace decomposition $A = \operatorname{rad} A \oplus V$ with $V \cong A/\operatorname{rad} A$.

V generated by the e_a for $a \in (Q_A)_0$ and $\operatorname{rad} A$ generated by the x_α for $\alpha \in (Q_A)_1$.

(4) Remains to check $\ker \varepsilon =: I$ is admissible.

By construction, $\varepsilon(R_{Q_A}) \subseteq \operatorname{rad} A \Rightarrow \varepsilon(R_{Q_A}^{\ell}) \subseteq \operatorname{rad}^{\ell} A \quad \forall \ell \geq 1$.

$\operatorname{rad} A$ nilpotent $\Rightarrow \exists\, m \geq 1$ such that $R_{Q_A}^{m} \subseteq I$.

Left to show $I \subseteq R_{Q_A}^{2}$ (Exercise (7)).

$\square$

Exercise:

(8) Write the following two $\mathbb{K}$-algebras as bound path algebras:

i.) $A_1 = \begin{pmatrix} \mathbb{K} & 0 & 0 & 0 \\ \mathbb{K} & \mathbb{K} & 0 & 0 \\ 0 & 0 & \mathbb{K} & 0 \\ \mathbb{K}^3 & \mathbb{K}^3 & \mathbb{K} & \mathbb{K} \end{pmatrix}$ (view $\mathbb{K}^3$ as $\mathbb{K}$-$\mathbb{K}$-bimodule).

ii.) $A_2 = B/J$ where $B = \left\{ \begin{pmatrix} a & 0 & 0 \\ c & b & 0 \\ e & d & a \end{pmatrix} \,\middle|\, a,b,c,d,e \in \mathbb{K} \right\}$

and $J = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e & 0 & 0 \end{pmatrix} \,\middle|\, e \in \mathbb{K} \right\}$.

§4 Representations of quivers

Q connected quiver, $\mathbb{K}$ an algebraically closed field.

Def: A representation M of Q associates to each vertex $a \in Q_0$ a $\mathbb{K}$-vectorspace M_a and to each arrow $a \xrightarrow{\alpha} b$ in Q_1 a $\mathbb{K}$-linear map $\varphi_\alpha : M_a \to M_b$.

Write: $M = (M_a, \varphi_\alpha)_{a \in Q_0, \alpha \in Q_1}$ or simply $M = (M_a, \varphi_\alpha)$.

M is called finite dimensional if $\forall a \in Q_0$: M_a finite dimensional.

A morphism of representations $f : M = (M_a, \varphi_\alpha) \longrightarrow M' = (M'_a, \varphi'_\alpha)$ is a family $f = (f_a)_{a \in Q_0}$ of $\mathbb{K}$-linear maps $(f_a : M_a \to M'_a)_{a \in Q_0}$ such that $\forall (a \xrightarrow{\alpha} b) \in Q_1$ we have $\varphi_\alpha' f_a = f_b \varphi_\alpha$, i.e.

$$\text{the diagram}\quad \begin{array}{ccc} M_a & \xrightarrow{\varphi_\alpha} & M_b \\ {\scriptstyle f_a}\downarrow & \circlearrowleft & \downarrow{\scriptstyle f_b} \\ M'_a & \xrightarrow{\varphi'_\alpha} & M'_b \end{array} \quad \text{commutes.}$$

f is an isomorphism if so are all the f_a.

If $f: M \to M'$ and $g: M' \to M''$ are morphisms of representations of Q, where $f = (f_a)_{a \in Q_0}$ and $g = (g_a)_{a \in Q_0}$, then their composition $gf : M \to M''$ is defined to be $gf = (g_a f_a)_{a \in Q_0}$.

Thus, we have defined a category $\operatorname{Rep}(Q)$ of $\mathbb{K}$-linear representations with a full subcategory $\operatorname{rep}(Q)$ of finite dimensional representations.

Def: Let $M = (M_a, \varphi_\alpha)$ and $M' = (M_a', \varphi_\alpha')$ be representations of Q. The direct sum of M and M' is given by the representation

$$M \oplus M' = \left(M_a \oplus M_a', \begin{pmatrix} \varphi_\alpha & 0 \\ 0 & \varphi_\alpha' \end{pmatrix} \right).$$

A representation N of Q is called <u>decomposable</u> if $N = 0$ or $\exists$ representations M_1, M_2 of Q such that $N \cong M_1 \oplus M_2$.

(with $N, M_1, M_2 \neq 0$)

N is called <u>indecomposable</u> if N is not decomposable.

<u>General aim:</u> Describe indecomposable representations up to isomorphism!

<u>Ex:</u> (1) $Q = \bullet$

Finite dimensional representations are given by $\mathbb{K}^n$ for some $n \geq 0$. $\mathbb{K}^n \cong \mathbb{K}^m \iff n = m$ and up to isomorphism $\mathbb{K}$ is the only indecomposable representation of Q.

(2) $Q = 1 \xrightarrow{\alpha} 2$

Recall that $\mathbb{K}Q \cong \begin{pmatrix} \mathbb{K} & 0 \\ \mathbb{K} & \mathbb{K} \end{pmatrix}$.

The representations $M_1 = 0 \to \mathbb{K}$, $M_2 = \mathbb{K} \to 0$ and $M_3 = \mathbb{K} \xrightarrow{1} \mathbb{K}$ are indecomposable.

<u>Quest:</u> Are these the only ones up to isomorphism?

Consider the representation $\mathbb{K}^n \xrightarrow{\varphi} \mathbb{K}^m$ of Q for $m, n \geq 0$.

$\exists$ $\mathbb{K}$-linear isomorphisms $f_1 : \mathbb{K}^n \to \mathbb{K}^n$ and $f_2 : \mathbb{K}^m \to \mathbb{K}^m$ such that

$$\begin{array}{ccc} \mathbb{K}^n & \xrightarrow{\varphi} & \mathbb{K}^m \\ {\scriptstyle f_1}\downarrow & \circlearrowright & \downarrow{\scriptstyle f_2} \\ \mathbb{K}^n & \xrightarrow[\begin{pmatrix} 1_r & 0 \\ 0 & 0 \end{pmatrix}]{} & \mathbb{K}^m \end{array}$$

commutes. <u>Answer:</u> Yes!

<u>Quest:</u> What are the morphisms between indecomposables?

* $$\begin{array}{ccc} M_1 : & 0 \to \mathbb{K} \\ \downarrow & 0\downarrow \circlearrowright \downarrow 0 \\ M_2 : & \mathbb{K} \to 0 \end{array} \Rightarrow \mathrm{Hom}_{\mathrm{rep}(Q)}(M_1, M_2) = 0$$

(also $\mathrm{Hom}_{\mathrm{rep}(Q)}(M_2, M_1) = 0$)

* $$\begin{array}{ccc} M_1 : & 0 \to \mathbb{K} \\ \downarrow & 0\downarrow \circlearrowright \downarrow \lambda \in \mathbb{K} \\ M_3 : & \mathbb{K} \to \mathbb{K} \end{array} \Rightarrow \mathrm{Hom}_{\mathrm{rep}(Q)}(M_1, M_3) \cong \mathbb{K}.$$

* $$\begin{array}{ccc} M_3 : & \mathbb{K} \to \mathbb{K} \\ \downarrow & 0\downarrow \circlearrowright \downarrow 0! \\ M_1 : & 0 \to \mathbb{K} \end{array} \Rightarrow \mathrm{Hom}_{\mathrm{rep}(Q)}(M_3, M_1) = 0.$$

Similarly, one checks that $\mathrm{Hom}_{\mathrm{rep}(Q)}(M_3, M_2) \cong \mathbb{K}$ and $\mathrm{Hom}_{\mathrm{rep}(Q)}(M_2, M_3) = 0$.

<u>Conclusion:</u> We have good understanding of $\mathrm{rep}(1 \to 2)$!

(3) $Q = \circlearrowleft$. Recall that $\mathbb{K}Q \cong \mathbb{K}[x]$.

Finite dimensional representations are given by $\mathbb{K}^n \circlearrowleft e$ ($\mathbb{K}^n \xrightarrow{e} \mathbb{K}^n$) with $e \in \mathrm{End}_{\mathbb{K}}(\mathbb{K}^n)$.

$\forall \lambda \in \mathbb{K}$: $M_\lambda := \mathbb{K} \circlearrowleft \lambda$ is indecomposable. Take $\mu \in \mathbb{K}$ and assume that $M_\lambda \cong M_\mu$. Thus, $\exists\, v \in \mathbb{K} \setminus \{0\}$ and an isomorphism of representations $\mathbb{K} \circlearrowleft \lambda \xrightarrow{\cdot v} \mathbb{K} \circlearrowleft \mu$, i.e., $v\lambda = \mu v \Leftrightarrow \lambda = \mu$.
There are infinitely many 1-dimensional non-isomorphic representations of Q.
More generally, a representation $\mathbb{K}^n \circlearrowleft e$ of Q is isomorphic to $\mathbb{K}^n \circlearrowleft e'$ $\Leftrightarrow$ $\exists\, \varphi \in \mathrm{Aut}_{\mathbb{K}}(\mathbb{K}^n)$ such that $\varphi e \varphi^{-1} = e'$, i.e.,

e and e' admit the same Jordan normal form. $\mathbb{K}^n \circlearrowleft e$ is indecomposable if the Jordan form of ℓ consists of a single block.

(4) $Q = 1 \rightrightarrows 2$.

Recall that $\mathbb{K}Q \cong \begin{pmatrix} \mathbb{K} & 0 \\ \mathbb{K}^2 & \mathbb{K} \end{pmatrix}$.

Finite dimensional representations are given by $\mathbb{K}^n \overset{\ell_1}{\underset{\ell_2}{\rightrightarrows}} \mathbb{K}^m$ for $m, n \geq 0$ and $\ell_1, \ell_2 \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^m)$.

The representations $\mathbb{K} \rightrightarrows 0$, $0 \rightrightarrows \mathbb{K}$, $\mathbb{K} \overset{\lambda}{\underset{\mu}{\rightrightarrows}} \mathbb{K}$ (with $\lambda \neq 0$ or $\mu \neq 0$) are indecomposable. Which of the latter ones are isomorphic?

Case 1: $\lambda = 0$.

Consider isomorphism
$$\begin{tikzcd} \mathbb{K} \arrow[r, shift left, "0"] \arrow[r, shift right, "\mu"'] \arrow[d, "\alpha_1"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{K} \arrow[d, "\alpha_2"] \\ \mathbb{K} \arrow[r, shift left, "\lambda'"] \arrow[r, shift right, "\mu'"'] & \mathbb{K} \end{tikzcd}$$
Commutativity implies $\lambda' = 0$ and
$$\mu' \alpha_1 = \alpha_2 \mu \overset{\mu \neq 0}{\iff} \alpha_2 = \frac{\mu'}{\mu} \alpha_1.$$

Thus, up to isomorphism, only one such representation $\mathbb{K} \overset{0}{\underset{1}{\rightrightarrows}} \mathbb{K}$.

Case 2: $\lambda \neq 0$.

Consider isomorphism
$$\begin{tikzcd} \mathbb{K} \arrow[r, shift left, "\lambda"] \arrow[r, shift right, "\mu"'] \arrow[d, "\alpha_1"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{K} \arrow[d, "\alpha_2"] \\ \mathbb{K} \arrow[r, shift left, "\lambda'"] \arrow[r, shift right, "\mu'"'] & \mathbb{K} \end{tikzcd}$$
Thus, $\lambda' \alpha_1 = \alpha_2 \lambda \overset{\lambda \neq 0}{\iff} \alpha_2 = \frac{\lambda'}{\lambda} \alpha_1$ and $\mu' \alpha_1 = \alpha_2 \mu$.
$$\overset{\lambda' \neq 0}{\Rightarrow} \frac{\mu'}{\lambda'} = \frac{\mu}{\lambda}.$$

Thus, up to isomorphism, all these representations of the form $\mathbb{K} \overset{1}{\underset{\mu}{\rightrightarrows}} \mathbb{K}$, $\mu \in \mathbb{K}$.

There are many more indecomposable representations of Q and we will need more tools to classify them. Finding isomorphisms between them translates to finding normal forms for pairs of matrices.

Exercise:

(8) Let $Q = 1 \to 2 \to 3$.

Find all (finite dimensional) indecomposable representations of Q (up to isomorphism) and describe all possible morphisms between them.

(10) Let $Q =$

$$\begin{tikzcd}
1 \arrow[dr] & & 2 \arrow[dl] \\
& 5 & \\
3 \arrow[ur] & & 4 \arrow[ul]
\end{tikzcd}.$$

Show that there are infinitely many pairwise non-isomorphic indecomposable representations of Q.

<u>Hint:</u> Consider representations of the form:

$$M_{\lambda,\mu} = \begin{tikzcd}
\mathbb{K} \arrow[dr, "\binom{1}{0}"'] & & \mathbb{K} \arrow[dl, "\binom{1}{\mu}"] \\
& \mathbb{K}^2 & \\
\mathbb{K} \arrow[ur, "\binom{0}{1}"'] & & \mathbb{K} \arrow[ul, "\binom{1}{1}"]
\end{tikzcd} \quad \text{with } \mu \neq 0.$$

§ 5 Representations of bound quivers

Q connected quiver, $\mathbb{K}$ an algebraically closed field and $I \trianglelefteq \mathbb{K}Q$ admissible ideal (i.e. $\exists\, m \geq 2$ such that $R_Q^m \subseteq I \subseteq R_Q^2$).

I is a finitely generated $\mathbb{K}Q$-module $\Rightarrow$ $I = \langle \rho_1, \dots, \rho_m \rangle$ with

$$\rho_i = \sum_{j=1}^{\ell} \lambda_j^i \omega_j^i$$

for $\lambda_j^i \in \mathbb{K}$ and ω_j^i a path in Q of length ≥ 2 such that for $k \neq j$ the source and the target of ω_k^i and ω_j^i coincide.
ρ_i is called a <u>relation</u> in Q.

Let $M = (M_a, \varphi_\alpha)$ be a representation of Q. For any nontrivial path $\omega = \alpha_\ell \cdots \alpha_1$ we consider the $\mathbb{K}$-linear map $\varphi_\omega = \varphi_{\alpha_\ell} \circ \dots \circ \varphi_{\alpha_1}$.
For a relation $\rho = \sum_{j=1}^{\ell} \lambda_j \omega_j$ we define $\varphi_\rho = \sum_{j=1}^{\ell} \lambda_j \varphi_{\omega_j}$.

<u>Def1</u> A representation $M = (M_a, \varphi_\alpha)$ of Q is said to be <u>bound by I</u> if $\varphi_\rho = 0$ $\forall$ relations $\rho \in I$.

We denote by $\mathrm{Rep}(Q,I)$ (respectively $\mathrm{rep}(Q,I)$) the full subcategory of $\mathrm{Rep}(Q)$ consisting of the (finite dimensional) representations that are bound by I.

<u>Ex1</u> Let $Q = $
$$\begin{tikzcd} & 2 \arrow[dr, "\gamma"] & \\ 1 \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & 4 \\ & 3 \arrow[ur, "\delta"'] & \end{tikzcd}$$
and $I = \langle \gamma\alpha - \delta\beta \rangle$.

Then the representations
$$\begin{tikzcd} & \mathbb{K} \arrow[dr, "0"] & \\ \mathbb{K} \arrow[ur, "1"] \arrow[dr, "0"'] & & \mathbb{K} \\ & \mathbb{K} \arrow[ur, "1"'] & \end{tikzcd}$$
and

$$\begin{tikzcd}[row sep=small] & \mathbb{K}^2 \arrow[dr, "(1\,1)"] & \\ \mathbb{K} \arrow[ur, "\binom{1}{0}"] \arrow[dr, "\binom{1}{1}"'] & & \mathbb{K} \\ & \mathbb{K}^2 \arrow[ur, "(0\,1)"'] & \end{tikzcd}$$

are bound by I, but

the representation
$$\begin{tikzcd}[row sep=small] & \mathbb{K} \arrow[dr, "1"] & \\ \mathbb{K} \arrow[ur, "1"] \arrow[dr] & & \mathbb{K} \\ & 0 \arrow[ur] & \end{tikzcd}$$
is not.

<u>Theorem:</u> Let $A = \mathbb{K}Q/I$ be a bound path algebra. Then there is a $\mathbb{K}$-linear equivalence of categories

$$F: \mathrm{Mod}(A) \xrightarrow{\sim} \mathrm{Rep}(Q, I)$$

that restricts to an equivalence $F: \mathrm{mod}(A) \xrightarrow{\sim} \mathrm{rep}(Q, I)$.

[here $\mathrm{Mod}(A)$ (respectively, $\mathrm{mod}(A)$) denotes the category of all (finite dimensional) left A-modules]

<u>Proof:</u> (1) Construction of $F: \mathrm{Mod}(A) \to \mathrm{Rep}(Q, I)$. Let $M \in \mathrm{Mod}(A)$. Define $F(M) = (M_a, \varphi_\alpha)$ as follows:

$a \in Q_0 \rightsquigarrow \bar{e}_a = e_a + I$ primitive idempotent in $A = \mathbb{K}Q/I$.

Set $M_a := \bar{e}_a \cdot M$.

$(a \xrightarrow{\alpha} b) \in Q_1 \rightsquigarrow \bar{\alpha} = \alpha + I$. Set $\varphi_\alpha : M_a \to M_b$ with $\varphi_\alpha(x) := \bar{\alpha} \cdot x$ $(= \bar{e}_b \bar{\alpha} \bar{e}_a \cdot x)$. φ_α $\mathbb{K}$-linear, since M is an A-module.

<u>Show:</u> $F(M)$ is bound by I.

Let $\rho = \sum_j \lambda_j \omega_j$ be a relation in I. Then

$$\varphi_\rho(x) = \sum_j \lambda_j \varphi_{\omega_j}(x) = \sum_j \lambda_j \bar{\omega}_j \cdot x = \bar{\rho} \cdot x = 0.$$

Thus, F well-defined on objects.

Let $f: M \to M'$ be an A-module homomorphism. Want to define a morphism $F(f): F(M) \to F(M')$ of $\mathrm{Rep}(Q,I)$.

f restricts to a $\mathbb{K}$-linear map $f_a: M_a \to M_a'$ for all $a \in Q_0$.
$$\bar{e}_a \cdot x \longmapsto \bar{e}_a \cdot f(x)$$

Set $F(f) = (f_a)_{a \in Q_0}$. Have to check that $\ell_\alpha' f_a = f_b \ell_\alpha$ for all $(a \xrightarrow{\alpha} b) \in Q_1$. Take $x \in M_a$, then

$$f_b \ell_\alpha(x) = f_b(\underbrace{\bar{\alpha} \cdot x}_{\in M_b}) = \bar{\alpha} \cdot f(x) = \bar{\alpha} \cdot f_a(x) = \ell_\alpha' f_a(x).$$

Hence, F a $\mathbb{K}$-linear functor that restricts to $F: \mathrm{mod}(A) \to \mathrm{rep}(Q,I)$.

(2) Construct functor $G: \mathrm{Rep}(Q,I) \to \mathrm{Mod}(A)$.

Let $M = (M_a, \ell_\alpha) \in \mathrm{Rep}(Q,I)$.

Set $G(M) := \bigoplus_{a \in Q_0} M_a$. Have to define A-module structure on $G(M)$.

Take $x = (x_a)_{a \in Q_0}$ in $G(M)$.

$\forall a \in Q_0$ set $e_a \cdot x := x_a$ and if ω is a path in Q from a to b $(\omega \cdot x)_b := \ell_\omega(x_a) \in M_b$ and $(\omega \cdot x)_{\substack{c \neq b \\ c \in Q_0}} := 0$.

This turns $G(M)$ into a $\mathbb{K}Q$-module. Moreover, for a relation $\rho \in I$ we have $\rho \cdot x = 0$. Hence, $G(M)$ is an A-module by defining $\bar{y} \cdot x := y \cdot x$ for $y \in \mathbb{K}Q$.

Now let $(f_a)_{a \in Q_0}: M = (M_a, \ell_\alpha) \to M' = (M_a', \ell_\alpha')$ be a morphism in $\mathrm{Rep}(Q,I)$. Then $f := \bigoplus_{a \in Q_0} f_a : G(M) \to G(M')$ defines an

A-module homomorphism. In fact, we have to check that

$\forall x \in G(M)\ \forall \bar{y} \in A:\ f(\bar{y}x) = \bar{y} f(x)$.

Can assume that $x = x_a \in \mathcal{M}_a$ and y is a path from a to b in Q.

Then

$$f(\bar{y}x) = f(\bar{y}x_a) = f_b\, \varphi_y(x_a) = \varphi_y'\, f_a(x_a) = \bar{y} f(x).$$

It follows that G a $\mathbb{K}$-linear functor restricting to

$$G: \operatorname{rep}(Q,I) \longrightarrow \operatorname{mod}(A).$$

(3) By construction, $FG \cong \mathbb{1}_{\operatorname{Rep}(Q,I)}$ and $GF \cong \mathbb{1}_{\operatorname{Mod}(A)}$ $\square$

<u>Application I:</u> "Simple modules"

Let $A = \mathbb{K}Q/I$ be a bound path algebra.

A simple A-module S is a module over $A/\operatorname{rad} A$ $\left(\bar{a}\cdot s := a\cdot s \text{ for } a \in A,\ s \in S\right)$.

Have already checked that $A/\operatorname{rad} A \cong \mathbb{K} \times \dots \times \mathbb{K}$.

Thus, S is 1-dimensional and corresponds to a representation

$$F(S) = S(a) := \begin{tikzcd} & 0 \arrow[d] & 0 \\ 0 \arrow[r] & \mathbb{K} \arrow[ur] & \\ & & 0 \arrow[ul] \end{tikzcd} \qquad \text{for } a \in Q_0.$$

The set $\{ S(a) \mid a \in Q_0 \}$ describes a complete set of non-isomorphic simple A-modules.

<u>Application VI</u> "projective modules"

Let $A = \mathbb{K}Q/_{I}$.

$\{\bar{e}_i \mid i \in Q_0\}$ is a complete set of primitive orthogonal idempotents for A. We obtain a decomposition ${}_A A = A\bar{e}_1 \oplus \dots \oplus A\bar{e}_n$.

A basic $\Rightarrow$ $\{A\bar{e}_i \mid i \in Q_0\}$ is a complete set of non-isomorphic indecomposable projective A-modules.

Assume that Q acyclic and locally at $i \in Q_0$ of the form

$$\begin{tikzcd} i+3 \arrow[r, "\gamma"] & i \arrow[r, shift left, "\alpha_1"] \arrow[r, shift right, "\alpha_2"'] \arrow[dr, "\beta"'] & i+1 \\ & & i+2 \end{tikzcd}$$

Then the representation $F(A\bar{e}_i)$ is locally given by

$$\begin{tikzcd} \bar{e}_{i+3}A\bar{e}_i \arrow[r] & \bar{e}_iA\bar{e}_i \arrow[r, shift left, "\bar{\alpha}_1\cdot"] \arrow[r, shift right, "\bar{\alpha}_2\cdot"'] \arrow[dr, "\bar{\beta}\cdot"'] & \bar{e}_{i+1}A\bar{e}_i \cong \mathbb{K}^2 \\ 0 \arrow[u, phantom, "\shortparallel"] & \mathbb{K} \arrow[u, phantom, "\cong"] & \bar{e}_{i+2}A\bar{e}_i \cong \mathbb{K} \end{tikzcd}$$

<u>Ex:</u> $A = \mathbb{K}Q$ for $Q = 1 \to 2 \to 3$.

Then the indecomposable projective A-modules are given by

$F(Ae_1) = \mathbb{K} \xrightarrow{1} \mathbb{K} \xrightarrow{1} \mathbb{K}$

$F(Ae_2) = 0 \to \mathbb{K} \xrightarrow{1} \mathbb{K}$

$F(Ae_3) = 0 \to 0 \to \mathbb{K}$.