ECOLE NATIONALE SUPERIEURE DES TRAVAUX PUBLICS

NATIONAL ADVANCED SCHOOL OF PUBLIC WORKS

MASTERS IN ENGINEERING (BAC + 5 ans)

ANNEE ACADEMIQUE/*ACADEMIC YEAR* :------------------------------

DEPARTEMENT/DEPARTMENT :------------------------------ *CLASSE/CLASS* :------------------------

COMPOSITION DE FIN DE SEMESTRE/*END OF SEMESTER EXAMINATION* : SIMULATION #2

EPREUVE/*COURSE TITLE* :------------------------------*CODE/CODE* :--------------------------

DATE/*DATE* : 24 AUGUST 2013 ------------------------------ *DUREE/DURATION* : 2 HOURS

EXAMINATEUR/*EXAMINER* :------------------------------

INSTRUCTIONS/*INSTRUCTIONS* :------------------------------

Multiple choice questions

Read <u>carefully</u> the text of each question and mark the box with the best answer.

1. The maximum floating point number x_{max} and the machine precision `eps` of the floating point system $\mathbb{F}(10, 2, -1, 1)$ are

   ☐ $x_{max} = 9.9$ $\texttt{eps} = 0.05$
   ☐ $x_{max} = 9.0$ $\texttt{eps} = 0.10$
   ☒ $x_{max} = 9.9$ $\texttt{eps} = 0.05$
   ☐ $x_{max} = 9.0$ $\texttt{eps} = 0.10$

   Answer. The floating point system has $\beta = 10$, $t = 2$, $L = -1$, $U = 1$. Thus, the maximum number and the machine precision are

$$x_{max} = \beta^U \cdot \left(1 - \beta^{-t}\right) = 10^1 \cdot \left(1 - 10^{-}2\right) = 9.9$$

$$\text{eps} = \frac{\beta^{1-t}}{2} = \frac{10^{1-2}}{2} = 0.05$$

2. Consider the fixed point iterations given by $x_{k+1} = x_k/2 + 1$. Let α be the unique fixed point. Starting at $x_0 = 1$, the absolute value of the error $e_2 = \alpha - x_2$ of the second iteration x_2 is

   ☒ 0.25   ☐ 0.50   ☐ 1.0   ☐ 1.5

   Answer. The iteration function is $\phi(x) = \frac{x}{2} + 1$; its fixed points are solutions of $x = \phi(x)$. We get $x = 2$ and so ϕ has the unique fixed point $\alpha = 2$. Starting from $x_0 = 1$, the first two iterations are

$$
\begin{aligned}
x_1 &= \phi(x_0) = \frac{x_0}{2} + 1 = \frac{1}{2} + 1 = \frac{3}{2}. \\
x_2 &= \phi(x_1) = \frac{x_1}{2} + 1 = \frac{3/2}{2} + 1 = \frac{7}{4}.
\end{aligned}
$$

So, we have

$$
|\, e_2 \,| = |\, \alpha - x_2 \,| = |\, 2 - \frac{7}{4} \,| = \frac{1}{4}.
$$

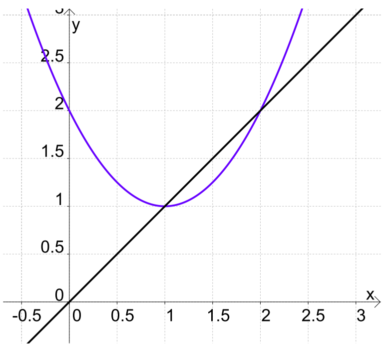
3. The order of convergence of the fixed point method $x_{k+1} = 2 - 2x_k + x_k^2$ when $x_0 = 0.5$ is

☐ 1 ☒ 2 ☐ 3 ☐ 4

Answer. The iteration function is $\phi(x) = 2 - 2x + x^2$. The corresponding fixed points are solutions of $x = \phi(x)$. We have

$$
x = \phi(x) \quad \Leftrightarrow \quad x = 2 - 2x + x^2 \quad \Leftrightarrow \quad x^2 - 3x + 2 = 0
$$

which has two solutions $x_1 = 1$ and $x_2 = 2$. Thus, the function ϕ has two fixed points: $\alpha_1 = 1$ and $\alpha_2 = 2$. So, first of all, we have to find toward which one of the two go the fixed point iterations when we start at $x_0 = 0.5$. To this aim, it is useful the geometric interpretation.



From the figure, since $x_0 = 0.5$. we see that the iterations go toward $\alpha = 1$. So, to find the order we have to look derivatives of ϕ in $\alpha_1 = 1$. We have

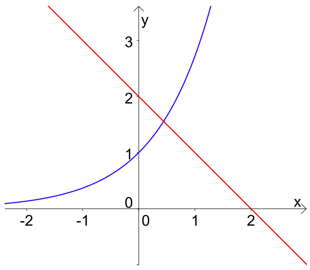
$$
\begin{array}{lll}
\phi'(x) = 2x - 2 & \Rightarrow & \phi'(1) = 0 \\
\phi''(x) = 2 & \Rightarrow & \phi''(1) \neq 0
\end{array}
$$

Thus, the order of the method is $p = 2$ since the first non zero derivative (evaluated in $\alpha_1 = 1$) of ϕ has order $p = 2$.

4. The order of convergence of the Newton method for the solution of the non linear equation $e^x = 2 - x$ is

☐ 0.5 ☐ 1 ☒ 2 ☐ more then 2

Answer. The equation has only one root since graphs $y = e^x$ and $y = 2 - x$ intersects just once. Moreover, the root is positive.



Setting $f(x) = e^x + x - 2$, we have $f'(x) = e^x + 1$ and $f''(x) = e^x$. Thus, we have

- $f'(\xi) = e^\xi + 1 > 0$. So, we have $f'(\xi) \neq 0$: this means that the root is simple (or, it has multiplicity 1) and so the order p of the method satisfies $p \geq 2$;
- $f''(\xi) > 0$. So, we have $f''(\xi) \neq 0$: as a consequence, the order ix exactly $p = 2$.

So, the Newton method for approximating the root ξ

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{e^{x_k} + 1}{e^{x_k}} = x_k - 1 - e^{-x_k}$$

has order of convergence $p = 2$ providing the starting point x_0 is sufficiently near the root ξ. We can see this behavior taking $x_0 = 1.0$; we have, for the errors

k	0	1	2	3	4
$\lvert e_k\rvert$	$5.5 \cdot 10^{-1}$	$9.5 \cdot 10^{-2}$	$2.8 \cdot 10^{-3}$	$2.3 \cdot 10^{-6}$	$1.6 \cdot 10^{-12}$

5. The order of convergence of the Newton method is always less or equal to 2

☐ True ☒ False

Answer. It's false. For example, if $f(x) = 0$ has the root ξ with $f'(\xi) \neq 0$ (and so $p \geq 2$) and $f''(\xi) = 0$ the order is $p \geq 3$. Consider $f(x) = x^3 + x$ which has the unique root $\xi = 0$. Starting from $x_0 = 1$, the behavior of the errors are

k	0	1	2	3	4	5
$\lvert e_k\rvert$	1.0	0.5	0.14	$5.5 \cdot 10^{-3}$	$3.3 \cdot 10^{-7}$	$7.7 \cdot 10^{-20}$

6. The Hilbert matrices are an example of well conditioned matrices

☐ True ☒ False

Answer. It's false. The Hilbert matrices, as well as the Vandermonde matrices, are examples of ill conditioned matrices. For example, the Hilbert matrix H_5 of order $n = 5$ has a condition number $K_2(H_5) \approx 5 \cdot 10^5$. An example of well conditioned matrix is the identity matrix which has a condition number equal to 1, the less possible value.

7. Let

$$A = \begin{pmatrix} 10 & 0 \\ 0 & 0.01 \end{pmatrix}$$

The condition number $K_2(A)$ of the matrix A is

☐ 0.01 ☐ 10 ☐ 100 ☒ 1000

Answer. The matrix A is a diagonal matrix with positive entries in the main diagonal. So, is a positive definite matrix, since all the eigenvalues are positive. For a positive definite matrix, we know that $K_2(A) = \lambda_{max}/\lambda_{min}$. In our case, we have $\lambda_{max} = 10$ and $\lambda_{min} = 0.01$. So, $K_2(A) = 10/0.01 = 1000$.

8. The LU factorization of the matrix A gives $|U| = 4$. The determinant of A^{-2} is

$$\square\ 16 \qquad \boxtimes\ \frac{1}{16} \qquad \square\ \frac{1}{4} \qquad \square\ 4$$

Answer. We have, using the Binet formula and the relation $\mid A^{-1} \mid = 1/\mid A \mid$,

$$\mid A^{-2} \mid = \mid (\ A^{-1}\ )^2 \mid = \mid A^{-1} \mid^2 = \left(\frac{1}{\mid A \mid}\right)^2 = \frac{1}{\mid A \mid^2} = \frac{1}{\mid U \mid^2} = \frac{1}{4^2} = \frac{1}{16}$$

since, from the LU factorization of $A = LU$, we have

- L is a lower triangular matrix with ones on the main diagonal. So, its determinant, which is the product of all the elements in the main diagonal, is $\mid L \mid = 1$.
- U is an upper triangular matrix. The determinant of U is again the product of the elements in the main diagonal but now this values are not known a priori (they depends on the matrix A)

So, again, from Binet, we have

$$\mid A \mid = \mid L\,U \mid = \mid L \mid \cdot \mid U \mid = 1 \cdot \mid U \mid = \mid U \mid$$

9. The L matrix of the LU-factorization of the matrix A given by

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 2 & 4 \end{pmatrix}$$

is

$$\boxtimes \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \qquad \square \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & -2 & 1 \end{pmatrix}$$

$$\square \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{pmatrix} \qquad \square \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 2 & 1 \end{pmatrix}$$

Answer. Using the Gauss algorithm we find

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 2 & 4 \end{pmatrix} \overset{\substack{(-2)\\(-3)}}{\longrightarrow} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 2 & 1 \end{pmatrix} \overset{(-2)}{\longrightarrow} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 5 \end{pmatrix}$$

So, recalling that L is a lower triangular matrix and has as entries the multiplicators (the elements above the arrows) changed in sign, and ones in the main diagonal, we get

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$$

10. Given the splitting $A = D - E - F$ where E is strictly lower triangular, F is strictly upper triangular and D is diagonal, the iteration matrix for the Jacobi method is

$$\boxtimes\ D^{-1}(E+F) \qquad \square\ D(E+F)$$

$$\square\ -D^{-1}(E+F) \qquad \square\ -D(E+F)$$

Answer. From the theory, providing that the entries of the diagonal of D are all non singular, we know that the iteration matrix of the Jacobi method is $B_J = D^{-1}(E+F)$. Indeed, just write

$$A\mathbf{x} = \mathbf{b} \quad \Leftrightarrow \quad (D - E - F)\mathbf{x} = \mathbf{b} \quad \Leftrightarrow \quad D\mathbf{x} = (E+F)\mathbf{x} + \mathbf{b} \quad \Leftrightarrow$$

Providing that D is invertible, we get

$$\mathbf{x} = D^{-1}(E+F)\mathbf{x} + D^{-1}\mathbf{b} \quad \text{and so we get} \quad \mathbf{x}_{k+1} = D^{-1}(E+F)\mathbf{x}_k + D^{-1}\mathbf{b}$$

where $B_J = D^{-1}(E+F)$ is the iteration matrix. Exactly in the same way we can find the iteration matrix of the Gauss-Seidel method: just start from $(D-E)\mathbf{x} = F\mathbf{x} + \mathbf{b}$.

11. Starting from $\mathbf{x}_0 = (0,\, 0)^T$, the norm of the residual $\mathbf{r}_1 = \mathbf{b} - A\mathbf{x}_1$ after the first Gauss-Seidel iteration for the linear system $A\mathbf{x} = \mathbf{b}$ given by

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

is

$$\square\ \frac{\sqrt{130}}{6} \qquad \boxtimes\ \frac{7}{6} \qquad 0 \qquad \left[-\frac{7}{6}\,,\, 0\right]^T$$

Answer. Setting $\mathbf{x} = [x_1\ x_2]^T$, the linear system is

$$\begin{cases} 2x_1 + x_2 = 3 \\ -x_1 + 3x_2 = 2 \end{cases} \quad \Rightarrow \quad \begin{cases} x_1 = \frac{3-x_2}{2} \\ \\ x_2 = \frac{2+x_1}{3} \end{cases}$$

Denoting with a superscript the index of the iteration, the Gauss-Seidel iterations are

$$x_1^{(k+1)} = \frac{3 - x_2^{(k)}}{2}$$

$$x_2^{(k+1)} = \frac{2 + x_1^{(k+1)}}{3}$$

So, starting from $\mathbf{x}_0 = [0\ 0]^T$ (that is, $x_1^{(0)} = 0$ and $x_2^{(0)} = 0$) we get for $\mathbf{x}_1 = [x_1^{(1)}\ x_2^{(1)}]^T$

$$x_1^{(1)} = \frac{3 - x_2^{(0)}}{2} = \frac{3-0}{2} = \frac{3}{2}$$

$$x_2^{(1)} = \frac{2 + x_1^{(1)}}{3} = \frac{2 + 3/2}{3} = \frac{7}{6}$$

The corresponding residual vector $\mathbf{r}_1$ is

$$\mathbf{r}_1 = \mathbf{b} - A\mathbf{x}_1^{(1)} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 3/2 \\ 7/6 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 25/6 \\ 2 \end{pmatrix} = \begin{pmatrix} -7/6 \\ 0 \end{pmatrix}$$

Finally, the infinity norm of the residual $\mathbf{r}_1$ is (we obtain the same result using other norms)

$$\| \mathbf{r}_1 \|_\infty = \max\left\{ \left| -\frac{7}{6} \right|,\ |\, 0\, | \right\} = \frac{7}{6}.$$

12. The Lagrange polynomials depends only on the nodes of the points (x_i, y_i), $i = 0, \ldots, n$

☒ True ☐ False

Answer. It is true: if x_i, $i = 0, \ldots, n$ are the nodes, then the Lagrange polynomials are

$$l_i(x) = \frac{\prod\limits_{k=0,k\neq i}^{n} (x - x_k)}{\prod\limits_{k=0,k\neq i}^{n} (x_i - x_k)}, \quad i = 0, 1, \ldots, n$$

13. The sum of all the Lagrange polynomials depends on the values of the function f in the interpolating points

☐ True ☒ False

Answer. It is false: from the theory, it is known that the sum of all the Lagrange polynomials is (the constant function) 1. To prove, just take $f(x) = 1$, a polynomial of degree zero. Thus,

$$f(x) = 1 = \sum_{i=0}^{n} f(x_i)l_i(x) = \sum_{i=0}^{n} 1 \cdot l_i(x) = \sum_{i=0}^{n} l_i(x)$$

14. If we want to approximate a function in an interval $[a, b]$ using equally spaced nodes, a higher degree interpolating polynomial always works better then a lower degree one

☐ True ☒ False

Answer. It is false: just remember the Runge example where the error (at the endpoints of the interval) increases with the degree of the interpolating polynomial.

15. The regression line for the set of points

x_i	-1	0	1	2
y_i	0	2	3	3

is

☒ $y = x + 1.5$ ☐ $y = 1.5x + 1$

☐ $y = x + 1$ $y = 1.5x + 1.5$

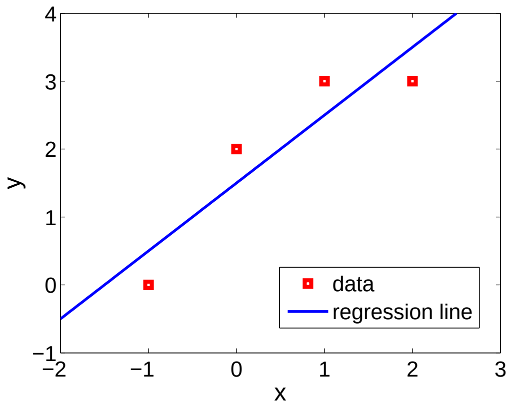
Answer. Coefficients a_0 and a_1 of the regression line $y = a_0 + a_1 x$ are solution of the linear system

$$\begin{pmatrix} m+1 & \sum_{k=0}^{m} x_k \\ \sum_{k=0}^{m} x_k & \sum_{k=0}^{m} x_k^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} \sum_{k=0}^{m} y_k \\ \sum_{k=0}^{m} x_k y_k \end{pmatrix}$$

where $m + 1$ is the number of points. In this case $m + 1 = 4$ and so, using the data in the table, we have

$$\begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 8 \\ 9 \end{pmatrix} \quad \Leftrightarrow \quad \begin{cases} 4a_0 + 2a_1 = 8 \\ 2a_0 + 6a_1 = 9 \end{cases}$$

We find $a_0 = 3/2$ and $a_1 = 1$. We can see the regression line in the next figure.



16. The quadrature formula

$$\int_0^1 \sqrt{x} f(x) \approx \sum_{i=0}^{3} \alpha_i f(x_i)$$

has the maximum degree of precision. Then the number

$$d = \left| \int_0^1 \left(\sqrt{x} + x\sqrt{x} \right) dx \; - \sum_{i=0}^{3} \alpha_i x_i \right|$$

is equal to

☐ 0   ☐ 1/3   ☒ 2/3   ☐ 1

Answer. The maximum degree of precisionof the quadrature formula is $s = 2 \cdot 3 + 1 = 7$. Since the integral in the quadrature formula is of the type

$$\int_0^1 \omega(x)\, f(x) dx$$

with $\omega(x) = \sqrt{x}$, the fist step to do is to rewrite the integral inside the expression of d in this way. We may note that

$$\sqrt{x} + x\sqrt{x} = \sqrt{x}\,(1 + x) \qquad \Rightarrow \qquad f(x) = 1 + x$$

So, f is a polynomial of degree $n = 1 < 7 = s$. Thus, the quadrature formula gives the exact result for this function f, i.e.,

$$\int_0^1 \left(\sqrt{x} + x\sqrt{x} \right) dx \; = \sum_{i=0}^{3} \alpha_i \left(1 + \; x_i \;\right)$$

Looking again to d, we have

$$\begin{aligned} d &= \left| \int_0^1 \left(\sqrt{x} + x\sqrt{x} \right) dx \; - \sum_{i=0}^{3} \alpha_i x_i \right| \\ &= \left| \int_0^1 \left(\sqrt{x} + x\sqrt{x} \right) dx \; - \sum_{i=0}^{3} \alpha_i \left(1 + x_i - 1 \right) \right| \\ &= \left| \left\{ \int_0^1 \left(\sqrt{x} + x\sqrt{x} \right) dx \; - \sum_{i=0}^{3} \alpha_i \left(1 + x_i \right) \right\} - \sum_{i=0}^{3} \alpha_i \right| = \left| \sum_{i=0}^{3} \alpha_i \right| \end{aligned}$$

Let's compute the sum of the weights. Taking in the quadrature formula $f(x) = 1$ (a polynomial of degree $n = 0 < s$ and so the formula is correct) we have

$$\frac{2}{3} = \left[\frac{x^{3/2}}{3/2} \right]_0^1 = \int_0^1 \sqrt{x}\, dx = \sum_{i=0}^{3} \alpha_i$$

So, we have $d = 2/3$.

17. Given a positive n, the sum of all Cotes numbers $C_i^{(n)}$, $i = 0, \ldots, n$ is

$\boxtimes$ 1 $\square$ n $\square$ $\sqrt{n}$ $\square$ $\frac{n}{2}$

Answer. From the theory, we know that the sum of all the Cotes numbers is 1. Recall that we have given the following definition of Cotes numbers

$$C_i^{(n)} = \frac{1}{n} \int_0^n \frac{\prod_{r=0, r\neq i}^{n} (s - r)}{\prod_{r=0, r\neq i}^{n} (i - r)} ds, \quad i = 0, \ldots, n$$

18. The error for the computation of

$$\int_0^{100} \left(x^3 + 13254x \right) dx$$

using the Cavalieri-Simpson formula is

$\square$ 10^{-3} $\square$ 10^{-2} $\square$ 10^{-1} $\boxtimes$ 0

Answer. The error E in the Cavalieri-Simpson is related to $f^{(4)}(x)$ throughout the equation

$$E = -\frac{(b - a)^5}{2880} f^{(4)}(\xi)$$

where, in our case, $a = 0$, $b = 100$, $\xi \in [0, 100]$ and $f(x) = x^3 + 13254x$. Since $f^{(4)}(x) = 0$ for all x, it is $f^{(4)}(\xi) = 0$ and so the error is zero.

19. The second derivative of f does not change much in the integration interval. Then, using the composite trapezoidal rule we expect that the ratio of the errors E_{2m}/E_m is

$\square$ 4 $\square$ 1/4 $\boxtimes$ near 1/4

Answer. From the theory, we know that

$$\frac{E_{2m}}{E_m} = \frac{-\frac{(b-a)^3}{12(2m)^3} f''(\xi_{2m})}{-\frac{(b-a)^3}{12m^3} f''(\xi_m)} = \frac{f''(\xi_{2m})}{4 \cdot f''(\xi_m)} \approx \frac{1}{4}.$$

since, if $f''(x)$ does not change much in the integration interval, it is $f''(\xi_m) \approx f''(\xi_{2m})$.

20. The Cavalieri-Simpson approximation of the integral

$$\int_0^1 \sqrt{x}\, dx$$

is

$\boxtimes$ $\frac{2\sqrt{2} + 1}{6}$ $\square$ $\frac{2}{3}$ $\square$ $\frac{1}{6}$ $\square$ 1

Answer. We have

$$x_0 = a = 0, \qquad x_1 = \frac{a+b}{2} = \frac{1}{2}, \qquad x_2 = b = 1$$

and so the Cavalieri-Simpson rule gives for the integral I

$$\begin{aligned} I_{CS} &= \frac{b-a}{2}\left[\frac{1}{3}f(x_0) + \frac{4}{3}f(x_1) + \frac{1}{3}f(x_2)\right] \\ &= \frac{1-0}{2}\left[\frac{1}{3}\sqrt{0} + \frac{4}{3}\sqrt{\frac{1}{2}} + \frac{1}{3}\sqrt{1}\right] = \frac{\sqrt{2}}{3} + \frac{1}{6} = \frac{2\sqrt{2}+1}{6} \end{aligned}$$

So, we have $I_{CS} = 0.638$ which may be comapred with the correct value $I = 2/3 = 0.667$.

21. The divided difference $f[x_0,\, x_1,\, x_2]$ of the following table

$$\begin{array}{r|l} x_0 = -1 & 2 \\ x_1 = 0 & 3 \\ x_2 = 1 & 6 \end{array}$$

is

$\boxtimes$ 1 $\quad\square$ 3 $\quad\square$ -1 $\quad\square$ 36

Answer. Completing the table, we find

$$\begin{array}{r|llll} x_0 = -1 & 2 & & \\ x_1 = 0 & 3 & f[x_0,x_1] & \\ x_2 = 1 & 6 & f[x_1,x_2] & f[x_0,x_1,x_2] \end{array} \qquad \text{or} \qquad \begin{array}{r|lll} x_0 = -1 & 2 & & \\ x_1 = 0 & 3 & 1 & \\ x_2 = 1 & 6 & 3 & 1 \end{array}$$

since we have

$$\begin{aligned} f[x_0,x_1] &= \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{3-2}{0-(-1)} = 1 \\ f[x_1,x_2] &= \frac{f(x_2)-f(x_1)}{x_2-x_1} = \frac{6-3}{1-0} = 3 \\ f[x_0,x_1,x_2] &= \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0} = \frac{3-1}{1-(-1)} = 1 \end{aligned}$$

22. Let $p(x)$ be the Newton expression of the interpolating polynomial for the points $(x_i,\, y_i)$, $i = 0, \ldots, n$. If we add a new point $(x_{n+1},\, y_{n+1})$ with $x_0 < x_{n+1} < x_1$ we have to recompute all the divided difference table

$\square$ True $\qquad\boxtimes$ False

Answer. It is false: the difference divided does not depends on the order of points. So, we can add the point $(x_{n+1},\, y_{n+1})$ at the previous table (the one we have already constructed with points $(x_i,\, y_i)$, $i = 0, \ldots, n$) and compute just the last row of the new table.

23. The composite trapezoidal formula gives the results of the following table

A_0	A_1	A_2
1	0.875	0.844

The best approximation for the integral is then

$\square$ 0.844 $\quad\boxtimes$ 0.833 $\quad\square$ 0.906 $\quad\square$ 0.875

Answer. We can apply the Romberg method to obtain

m			
1	A_0		
2	A_1	B_1	
4	A_2	B_2	C_2

where

$$
\begin{aligned}
B_1 &= \frac{4A_1 - A_0}{3} = \frac{4 \cdot 0.875 - 1}{3} = 0.833 \\
B_2 &= \frac{4A_2 - A_1}{3} = \frac{4 \cdot 0.844 - 0.875}{3} = 0.8340.833 \\
C_2 &= \frac{16B_2 - B_1}{15} = \frac{16 \cdot 0.834 - 0.833}{15} = 0.834
\end{aligned}
$$

So, the best approximation for the integral is 0.834.

24. Which is the value of `n` at the end of the Matlab code

```
toll = 1E2;
n = 5;
while( 10^n > toll & n >= 2 )
   n = n - 2;
end
```

☐ 0 ☒ 1 ☐ 2 ☐ 3

25. Consider the following Matlab code

```
S = 5;

for k=1:3
   if k>=3
      S = S*k;
   else
      S = S-k;
   end
end
```

At the end of the loop, the variable `S` is equal to

☐ 1 ☐ 4 ☒ 6 ☐ 9

26. After the execution of the following Matlab code, the variable `r` is equal to

```
A = diag( diag( [1 2; 3 4] ) );
r = eig( A );
```

☒ $[1 \;\; 4]^T$ ☐ $[1 \;\; 2]^T$ ☐ 4 ☐ 1

27. After the execution of the following Matlab code, the variable `v` is equal to

```
1.   v = [ sum( 3:4:14 )  length( 1:4 )  ];
2.   v = v.^2;
```

☒ [441 16] ☐ 7056 ☐ 4 ☐ [21 4]

28. Given the Matlab code

```
1.   v = [ 1 2 3; 4 5 6; 6 7 8 ];
2.   v = v(2,[2 3]);
```

gives

☒ [5 6] ☐ [4 5 6] ☐ $[5\ 8]^T$ ☐ $[2\ 5]^T$

29. To plot a function with the command `plot(x,y)`, the vector `x` and `y` must have the same size

☒ True
☐ False
☐ It depends on the function

30. The command `clear all` makes the command Window clear but does not clear the variables in the Workspace

☐ True ☒ False

Open questions

Write clearly all the answers in the exam's booklet.

1. Prove that the condition number $K(A)$ fulfills $K(A) \geq 1$ for each matrix A. Give an example of well conditioned matrix and one of an ill conditioned matrix.

   Answer. From the theory, we have

   $$1 = \|I_n\| = \left\|A \cdot A^{-1}\right\| \leq \|A\| \cdot \left\|A^{-1}\right\| = K(A).$$

   where I_n is the identity matrix of order n. The Hilbert matrices are ill conditioned whereas the identity matrix is well conditioned.

2. Consider the iterative method $\mathbf{x}_{k+1} = B\mathbf{x}_k + \mathbf{f}$ to solve the linear system $A\mathbf{x} = \mathbf{b}$. Prove the relationship $\mathbf{e}_k = B\mathbf{e}_{k-1}$, $k = 1, 2, \ldots$ where $\mathbf{e}_k = \mathbf{x} - \mathbf{x}_k$ is the error at the k-th step. Give necessary and sufficient conditions on the iteration matrix B in order to have a convergent sequence for each starting point $\mathbf{x}_0$. Write the iteration matrix for the Jacobi method. *Answer.*

3. Given the set of points $(\ x_i,\ y_i\ )$, $i = 0, \ldots, 3$ in the following table

x_i	-2	0	1	2
y_i	1	1	2	3

   write the Newton expression of the interpolation polynomial. Compute the minimum value of the function $S(m, q)$

   $$S(m, q) = \sum_{k=0}^{3} [\ y_i - m x_i - q\ ]^2$$

   and give the values of m and q for which this minimum is reached.

4. Show the composite trapezoidal rule using m intervals . Recalling that the error for the trapezoidal rule is

   $$E = -\frac{h^3}{12} f''(\xi)$$

   where h is the amplitude of the integration interval and ξ is a suitable point inside the integration interval, find the expression for the error in the composite trapezoidal formula.

5. Write a Matlab code for the computation of the sum of elements of the vector `x` using just a for loop.