ECOLE NATIONALE SUPERIEURE
DES TRAVAUX PUBLICS

NATIONAL ADVANCED SCHOOL
OF PUBLIC WORKS

MASTERS IN ENGINEERING (BAC + 5 ans)

ANNEE ACADEMIQUE/*ACADEMIC YEAR* : 2012 - 2013

DEPARTEMENT/DEPARTMENT : CLASSE/*CLASS* :

COMPOSITION DE FIN DE SEMESTRE/*END OF SEMESTER EXAMINATION* : FINAL EXAM

EPREUVE/*COURSE TITLE* : CODE/*CODE* :

DATE/*DATE* : 28 AUGUST 2013 DUREE/*DURATION* : 3 HOURS

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INSTRUCTIONS/*INSTRUCTIONS* :

Multiple choice questions

Read **carefully** the text of each question and mark the box with the best answer.

1. Consider the floating point system $\mathbb{F}(2, 4, -2, 2)$. The maximum floating point number x_{max} and the number of elements[1] N_{el} of $\mathbb{F}$ are

   ☐ $x_{max} = 3.75$ $N_{el} = 80$
   ☐ $x_{max} = 3.75$ $N_{el} = 81$
   ☐ $x_{max} = 4.00$ $N_{el} = 80$
   ☐ $x_{max} = 4.00$ $N_{el} = 81$

2. Noting that 1 and 2 are two consecutive numbers in $\mathbb{F}(10, 1, -1, 1)$, the representation of $x = 1.68$ in $\mathbb{F}$ using rounding is

   ☐ 1.6 ☐ 1.7 ☐ 2 ☐ overflow

3. The bisection method is used to approximate the root ξ of the equation $x - 1 = 0$ starting from the interval $[\ a_0,\ b_0\ ] = [\ 0.8,\ 1.6\ ]$. Denoting by x_0 and x_1 the first two iterates, the error $e_1 = \xi - x_1$ is

   ☐ 0 ☐ 0.1 ☐ 0.2 ☐ 0.4

4. How many fixed points has the function $f(x) = 2 - x^2$?

   ☐ 0 ☐ 1 ☐ 2 ☐ 4

5. Consider the fixed point iterations

   $$\begin{cases} x_0 &= 1.5 \\ x_{k+1} &= x_k^2 - 2x_k + 2 \end{cases}$$

   The order p of convergence of the iterates x_k, $k = 0, 1, \ldots$ is

   ☐ 1 ☐ 2 ☐ 3 ☐ does not converge

6. The number of iterations needed by the Newton method to find the approximation x_k of the root ξ of the equation $2x-3 = 0$ starting from $x_0 = 2$ and with an absolute value of the error $|\xi - x_k| < 10^{-6}$ is

   ☐ 1 ☐ 4 ☐ 9 ☐ infinity

7. The Newton method for the approximation of the root $\xi = 1$ of the equation $(x-1)^2\ln(x) = 0$ has order of convergence p equal to

   ☐ 1 ☐ 2 ☐ 3 ☐ more then 3

8. The Newton method for the approximation of the root ξ of the equation $xe^x = 1$ has order of convergence p equal to

   ☐ 1 ☐ 2 ☐ 3 ☐ more then 3

[1] That is, how many elements has the set $\mathbb{F}$. For example, $\mathbb{F} = \{-1, 0, 1\}$ has 3 elements.

9. Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0.1 \end{pmatrix}$$

The condition number $K_2(A)$ is

☐ 0.1 ☐ 1 ☐ 10 ☐ $\sqrt{101}/10$

10. Let $A = L\,U$ the LU-factorization of the non singular matrix A. If L and U are known, the smallest number of operations needed to solve all linear systems

$$A\mathbf{x} = \mathbf{b}_k, \quad k = 1, \ldots, M$$

is about

☐ n^2 ☐ $M\,n^2$ ☐ $2M\,n^2$

☐ none of above answers are correct

11. Let A be the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix}$$

Is it possible to find the Cholesky factorization $A = H\,H^T$ of A?

☐ Yes ☐ No

12. The Gauss-Seidel iteration matrix B_{GS} for the linear system $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

is

☐ $\begin{pmatrix} 0 & 1/2 \\ 0 & -1/4 \end{pmatrix}$ ☐ $\begin{pmatrix} 0 & -1/2 \\ 0 & 1/4 \end{pmatrix}$

☐ $\begin{pmatrix} 0 & 1/2 \\ 0 & 0 \end{pmatrix}$ ☐ $\begin{pmatrix} 0 & -1/2 \\ 0 & 0 \end{pmatrix}$

13. The number of arithmetic operations required for the computation of the sum S given by

$$S = \sum_{k=1}^{n} a_k \cdot b_k$$

is about

☐ n ☐ $2n-1$ ☐ n^2 ☐ $k\,n$

14. The matrix A has the LU-factorization with $|U| = -1$ (determinant of U) and L a lower triangular matrix with all ones in the main diagonal. The determinant of the matrix A^{51} is

☐ -1 ☐ 1 ☐ -51 ☐ 51

15. The Jacobi method is used to solve the linear system $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

starting from $\mathbf{x}_0 = (\,0 \;\; 0\,)^T$. The 2-norm of the residual $\mathbf{r}_1 = \mathbf{b} - A\mathbf{x}_1$ of the first iterate $\mathbf{x}_1$ is

☐ 1 ☐ $5/2$ ☐ $\pm 5/2$ ☐ $2\sqrt{2}$

16. The iterative method $\mathbf{x}_{k+1} = B\,\mathbf{x}_k + \mathbf{f}$, $k = 0, 1, \ldots$ with

$$B = \begin{pmatrix} 1/2 & 1 \\ 0 & -1/4 \end{pmatrix}, \qquad \mathbf{f} = \begin{pmatrix} 3/2 \\ -1/4 \end{pmatrix}$$

is convergent for each starting point $\mathbf{x}_0$

☐ True ☐ False

17. The Lagrange polynomials associated to the three nodes $x_0 = -1$, $x_1 = 0$, $x_2 = 1$ are

$$l_0(x) = \frac{1}{2}(x^2 - x) \qquad l_1(x) = 1 - x^2$$

$$l_2(x) = \frac{1}{2}(x^2 + x)$$

☐ True ☐ False

18. Let $f(x) = x^4 - x^2 + 5$. Consider $n = 4$ equally spaced nodes x_i, $i = 0, \ldots, n$ in the interval $[-1, 1]$. Let $p(x)$ be the corresponding interpolating polynomial, i.e. $p(x_i) = f(x_i)$, $i = 0, \ldots, n$. Then, the interpolation error in $x_0 = 0.5$ defined as $E = f(x_0) - p(x_0)$ is

☐ 0 ☐ 0.10 ☐ 0.12

☐ impossible to compute

19. Consider the points in the following table

$x_0 = 0$	1
$x_2 = 2$	3
$x_3 = 3$	-1

The divided difference $f[\,x_0,\ x_1,\ x_2\,]$ is

☐ -4 ☐ $-5/3$ ☐ $-2/3$ ☐ 1

20. If we want to approximate an arbitrary function in an interval $[a, b]$ using equally spaced nodes, a higher degree interpolating polynomial always gives a better approximation then a lower degree one

☐ True ☐ False

21. The sum of all Lagrange polynomials depends on the values of the function f in the interpolating points

☐ True ☐ False

22. The absolute value of the error in the Cavalieri-Simpson formula for the computation of $\int_a^b f(x)dx$ is

$$E = \frac{\mid b-a \mid^3}{12} \mid f''(\xi) \mid \qquad \text{where } \xi \in [a, b]$$

☐ True ☐ False

23. The composite trapezoidal formula with two equally length intervals (i.e., $m = 2$) is used to compute

$$\int_{-1}^{1} (\ 1 - x^4\ )\, dx$$

The approximation given by he trapezoidal rule is

☐ -2 ☐ -1 ☐ 1 ☐ 2

24. The maximum degree of precision of the quadrature formula

$$\int_a^b \omega(x)\, f(x)dx \approx \sum_{i=0}^{n} \alpha_i f(x_i)$$

where $\omega(x) > 0$ in $[a, b]$ is

☐ n ☐ $n+1$ ☐ $2n$ ☐ $2n+1$

25. The Cavalieri-Simpson approximation of the integral

$$\int_0^1 \sqrt{x}\, dx$$

is (Cotes numbers are $C_0^{(2)} = 1/6$, $C_1^{(2)} = 4/6$, $C_2^{(2)} = 1/6$)

☐ $\frac{2\sqrt{2}+1}{6}$ ☐ $\frac{2}{3}$ ☐ $\frac{1}{6}$ ☐ 1

26. To plot a function with the command `plot(x,y)`, the vector `x` and `y` must have the same size

☐ True
☐ False
☐ It depends on the function

27. Which of the following MatLab instructions is correct to extract the second row from the matrix A and store it in the vector v?

☐ `v = A( 2, :  )`
☐ `v = A( :, 2 )`
☐ `A( :, 2 ) = v`
☐ `v = A( 2; :  )`

28. Which is `c` at the end of the Matlab code

```
a = 1:3:8;
b = linspace( 0, 2, 3 );
c = ( a .* b ) * [ 1; 1; 1 ];
```

☐ 12 ☐ 14 ☐ 16 ☐ 18

29. Consider the following Matlab code

```
S = 5;
for k=1:4
   if k>=3
      S = S + k;
   else
      S = S - k;
   end
end
```

At the end of the loop, the variable `S` is equal to

☐ -5 ☐ 0 ☐ 5 ☐ 9

30. Which is the value of `n` at the end of the Matlab code

```
toll = 1E-6;
n = 1;
while( 10^n > toll & n <= 1 )
   n = n - 2;
end
```

☐ -3 ☐ -4 ☐ -6 ☐ -7

Open questions

Justify, in the exam's booklet, carefully all you answers, which must be short and clear.

1. Consider the Newton method for the approximation of the root ξ of the equation $f(x) = 0$.
   (a) Using the geometric interpretation, find the equation for the iterates given by
   $$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, \cdots$$
   (b) Show a graph of a function f where iterations x_k go toward the root ξ in a monotonically fashion with $x_k < x_{k+1}$, $k = 0, 1, \ldots$.
   (c) Consider equations (i) $\ln(x) = 0$ and (ii) $(x-1)\ln(x) = 0$. Sketch, for each one, a qualitative $\log_{10}(\mid e_k \mid)$ plot.
   (d) Give an example where the Newton method does not converge.

2. Consider the linear system $A\mathbf{x} = \mathbf{b}$ where
   $$A = \begin{pmatrix} 2 & 1 \\ 2 & 4 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$
   (a) Just looking to A, explain why Jacobi and Gauss-Seidel are convergent methods.
   (b) Setting $\mathbf{x}^{(k)} = [x_1^{(k)} \ x_2^{(k)}]^T$, write $x_1^{(k+1)}$ and $x_2^{(k+1)}$ for Jacobi and Gauss-Seidel methods and find, for both methods, $\mathbf{x}^{(1)}$ starting from $\mathbf{x}^{(0)} = [0 \ 0]^T$.
   (c) Let B_J be the iteration matrix of the Jacobi method. Using the relation
   $$\| \mathbf{e}_k \|_\infty \le \rho(B_J) \| \mathbf{e}_{k-1} \|_\infty$$
   estimate the number k of iterations needed to have $\| \mathbf{e}_k \|_\infty / \| \mathbf{e}_0 \|_\infty \le 10^{-3}$.

3. Consider the set of points in the following table

x_i	-1	0	1	2
y_i	-4	0	1	4

   (a) Write the Newton expression (assume $x_0 = -1$, $x_1 = 0$, $x_2 = 1$, $x_3 = 2$)
   $$\begin{aligned} P(x) &= f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ &+ f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2) \end{aligned}$$
   for the interpolating polynomial. Show that P satisfies the interpolation conditions $P(x_i) = y_i$ for the given set of points.
   (b) Find the regression line $y = a_1 x + a_0$ and draw it among with points in the same plane.

4. Consider the composite trapezoidal formula for the computation of
   $$I = \int_a^b f(x)\, dx$$
   (a) Write the formula for $m = 1$ (trapezoidal formula). Give the geometrical interpretation of the formula. Let I_T be the approximation of I given by the trapezoidal formula. Sketch three graphs of **three different functions** where $I_T < I$, $I_T = I$, $I_T > I$.
   (b) Starting from the equation of I_T, find the composite trapezoidal formula when the interval $[a, \ b]$ is divided into $m > 1$ intervals.
   (c) Given the error in the composite trapezoidal formula using m intervals as
   $$E_{CT}^{(m)} = -\frac{(b-a)^3}{12m^2} f''(\xi)$$
   where $\xi \in [a, b]$ is a suitable point, show that $E_{CT}^{(2m)} / E_{CT}^{(m)} \approx 1/4$ and use this result to prove the Richardson extrapolation formula.