Signal representation by Fourier series

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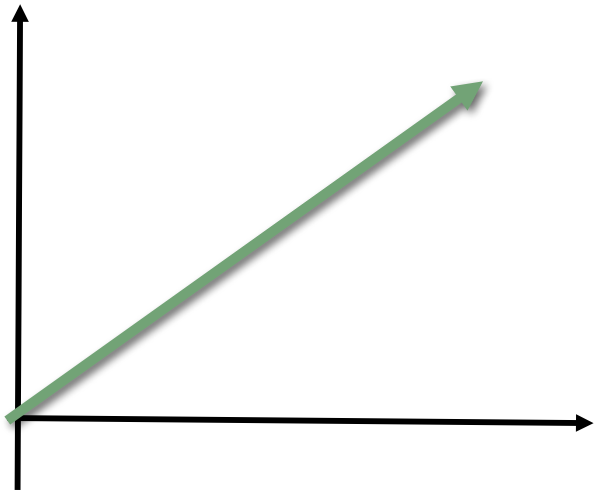
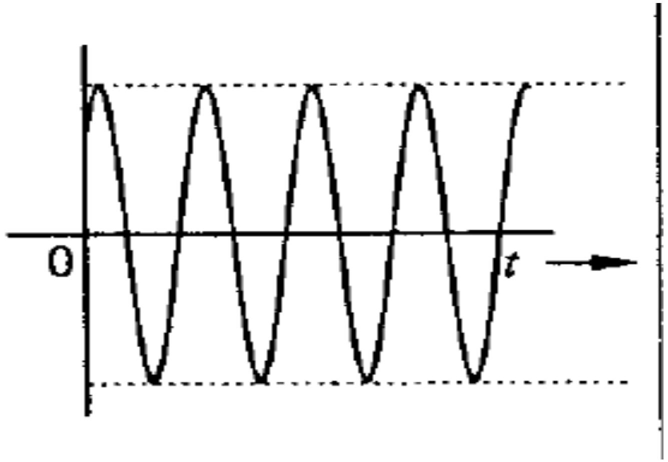
Signals as vectors





- Signals ⟷ Vectors
- The operations among signals can be easily interpreted as operations among vectors
  - Component of a vector
  - Inner product
  - Norm

Component of a vector

- Orthogonal projection: scalar product

$$\mathbf{f} \cdot \mathbf{x} = |\mathbf{f}||\mathbf{x}| \cos \theta$$

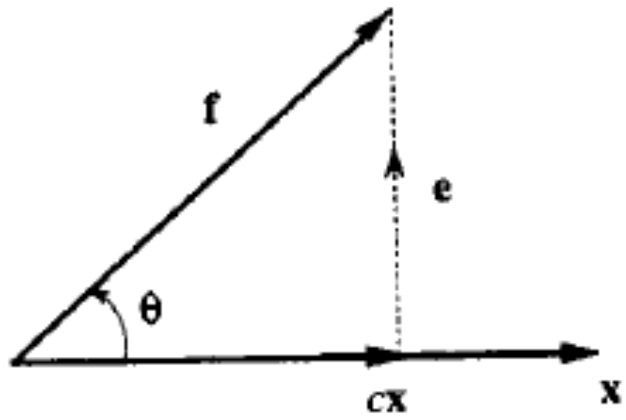


Fig. 3.1 Component (projection) of a vector along another vector.

- Norm $|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$
- Orthogonality $\mathbf{f} \cdot \mathbf{x} = 0$

Orthogonal projection

- The orthogonal projection of a vactor f over a vector x approximates f with its component along x with minimum

Fig. 3.2 Approximation of a vector in terms of another vector.

Component of a signal

The concept of a vector component and orthogonality can be extended to signals. Consider the problem of approximating a real signal $f(t)$ in terms of another real signal $x(t)$ over an interval $[t_1, t_2]$:

$$f(t) \simeq cx(t) \qquad t_1 \leq t \leq t_2 \tag{3.8}$$

The error $e(t)$ in this approximation is

$$e(t) = \begin{cases} f(t) - cx(t) & t_1 \leq t \leq t_2 \\ 0 & \text{otherwise} \end{cases} \tag{3.9}$$

"Best approximation" criterion: energy of the error signal

$$\begin{aligned} E_e &= \int_{t_1}^{t_2} e^2(t)\,dt \\ &= \int_{t_1}^{t_2} [f(t) - cx(t)]^2\,dt \end{aligned}$$

Component of a signal

- By minimizing the energy of the error signal one can show that the optimal value of the constant c is given by

$$c = \frac{1}{E_x} \int_{t_1}^{t_2} f(t)x(t)$$

$$E_x = \int_{t_1}^{t_2} x^2(t)dt$$

- Then the approximation is $f(t) \approx cx(t)$
- In vector terminology:
  - c x(t) is the projection of f(t) on x(t)
  - If c=0 then the vectors are orthogonal

Component of a signal

- Based on such analogy
  - Inner product of f(t) on x(t)

$$\int_{t_1}^{t_2} f(t)x(t)dt$$

  - Orthogonality

$$\int_{t_1}^{t_2} f(t)x(t)dt = 0$$

  - For complex signals

$$E_x = \int_{t_1}^{t_2} |x(t)|^2 \, dt \qquad c = \frac{1}{E_x}\int_{t_1}^{t_2} f(t)x^*(t)\, dt$$

  - Orthogonality

$$\int_{t_1}^{t_2} x_1(t)x_2^*(t)\, dt = 0 \quad \text{or} \quad \int_{t_1}^{t_2} x_1^*(t)x_2(t)\, dt = 0$$

Energy of the sum of orthogonal signals

- The energy of the sum is the sum of the energies for both vectors and signals

$$|\mathbf{z}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2$$

$$E_z = E_x + E_y$$

- Proof

$$\int_{t_1}^{t_2} |x(t) + y(t)|^2 dt = \int_{t_1}^{t_2} |x(t)|^2 dt + \int_{t_1}^{t_2} |y(t)|^2 dt + \int_{t_1}^{t_2} x(t) y^*(t) dt + \int_{t_1}^{t_2} x^*(t) y(t) dt$$

$$= \int_{t_1}^{t_2} |x(t)|^2 dt + \int_{t_1}^{t_2} |y(t)|^2 dt \qquad (3.22)$$

Correlation coefficient: vectors

- The larger the value of c, the higher the resemblance. Thus, a suitable measure for vector similarity could be

$$c_n = \cos\theta = \frac{\mathbf{f}\cdot\mathbf{x}}{|\mathbf{f}|\,|\mathbf{x}|}$$

This similarity measure c_n is known as the **correlation coefficient.** Observe that

$$-1 \leq c_n \leq 1 \tag{3.24}$$

Thus, the magnitude of c_n is never greater than unity. If the two vectors are aligned, the similarity is maximum ($c_n = 1$). Two vectors aligned in opposite directions have the maximum dissimilarity ($c_n = -1$). If the two vectors are orthogonal, the similarity is zero.

Correlation coefficient: signals

- Following the same line, we can define the similarity among signals as

$$c_n = \frac{1}{\sqrt{E_f E_x}} \int_{-\infty}^{\infty} f(t)x(t)\,dt$$

Observe that multiplying either $f(t)$ or $x(t)$ by any constant has no effect on this index. It is independent of the size (energies) of $f(t)$ and $x(t)$. Using the Schwarz inequality,† we can show that the magnitude of c_n is never greater than 1

$$-1 \leq c_n \leq 1 \tag{3.26}$$

Best friends, worst enemies and complete strangers

$f(t) = Kx(t) \rightarrow c = 1$ Best friends

$f(t) = -Kx(t) \rightarrow c = -1$ Worst enemies

$f(t)$ is orthogonal to $x(t) \rightarrow c = 0$ Complete strangers

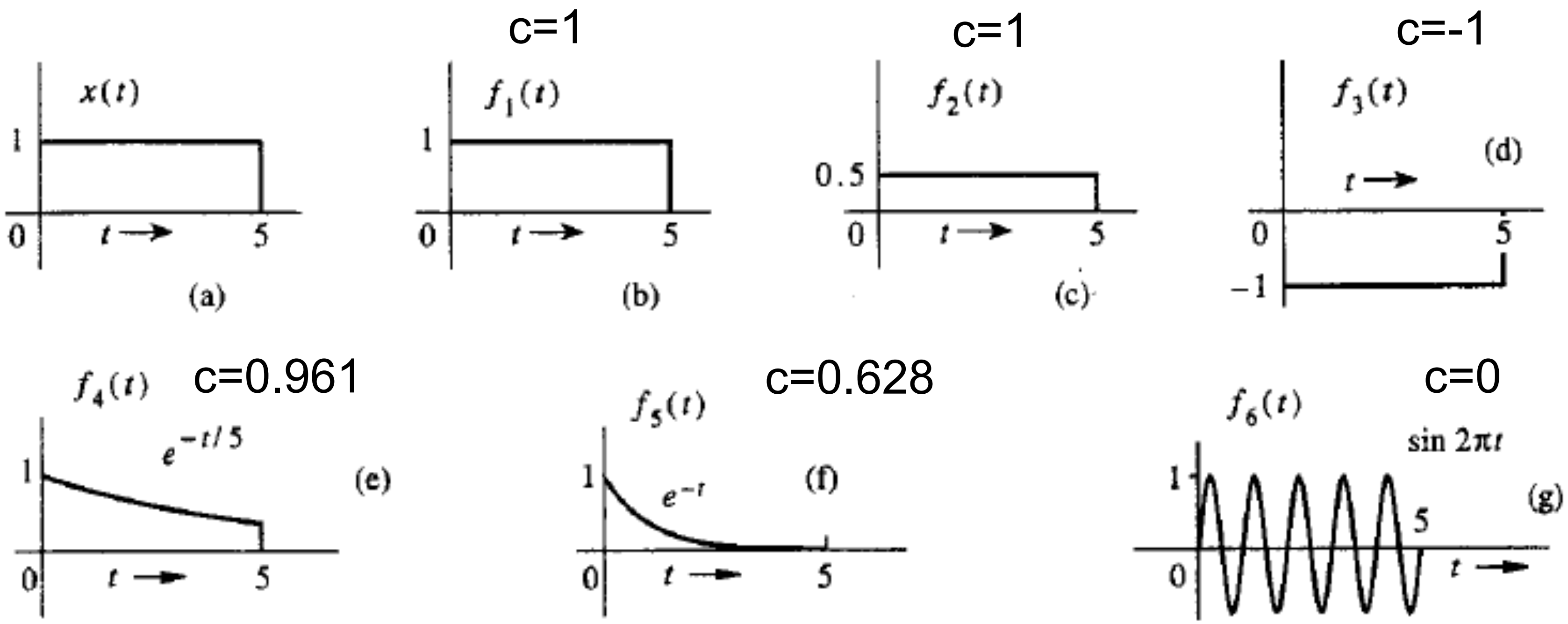


Fig. 3.4 Signals for Example 3.2.

Correlation function

- Allows to measure the similarity among signals irrespectively of their time-shift

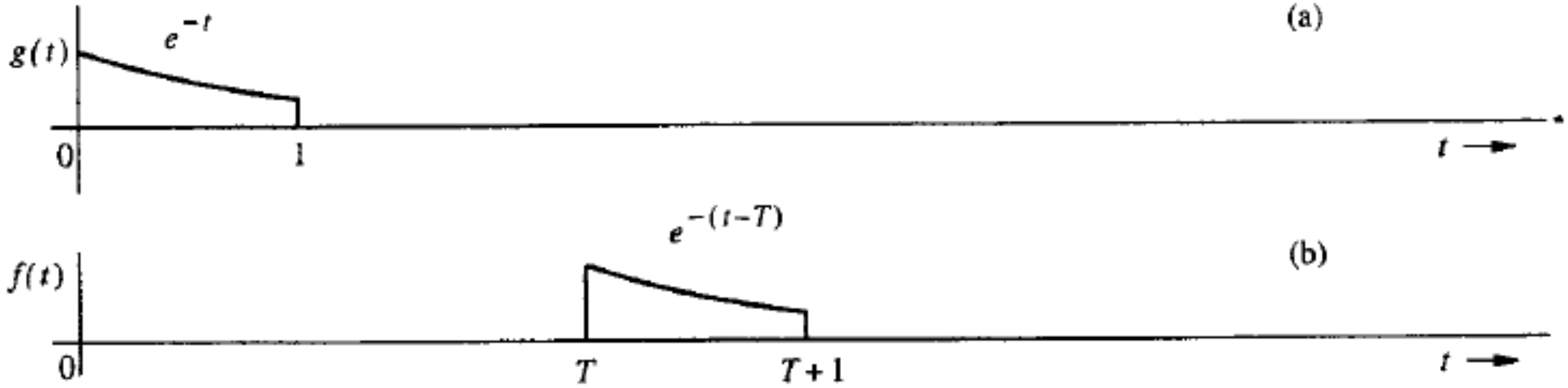


Fig. 3.5 Physical explanation of the correlation function.

Real signals

$$\psi_{fg}(t) \equiv \int_{-\infty}^{\infty} f(\tau) g(\tau - t)\, d\tau$$

Complex signals

$$\psi_{fg}(t) \equiv \int_{-\infty}^{\infty} f^*(\tau) g(\tau - t)\, d\tau$$

Convolution and correlation

Convolution (zero—state response of LTIS)

$$f(t) * x(t) = \int_{-\infty}^{+\infty} f(\tau)x(t-\tau)d\tau$$

$$\Psi(f,x)(t) = f(t) * g(-t)$$

Correlation (signal similarity)

$$f(t) \cdot x(t) = \int_{-\infty}^{+\infty} f(\tau)x(\tau - t)d\tau$$

$$f(t) * g(-t) = f(t) * w(t) = \int_{-\infty}^{\infty} f(\tau)w(t-\tau)\,d\tau = \int_{-\infty}^{\infty} f(\tau)g(\tau - t)\,d\tau = \psi_{fg}(t)$$

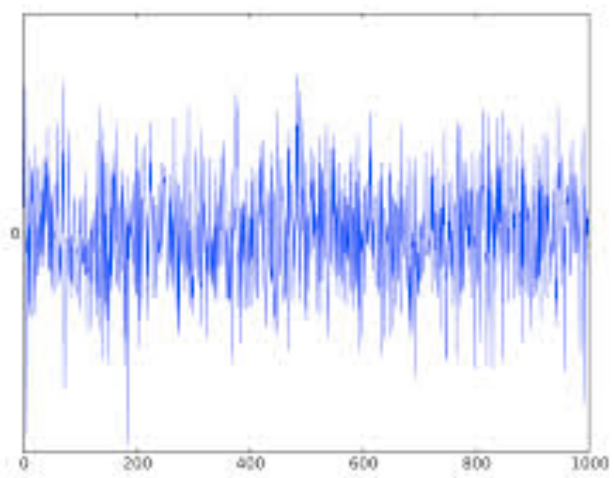
Autocorrelation function

- Correlation function of the signal with itself

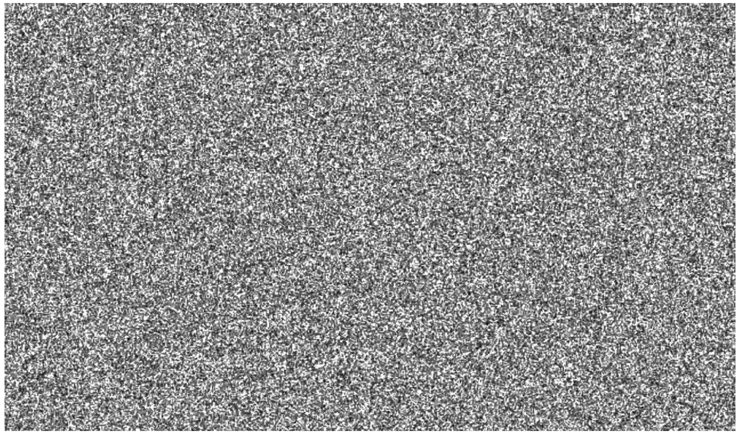
$$\psi_f(t) \equiv \int_{-\infty}^{\infty} f(\tau) f(\tau - t)\, d\tau$$

- Examples
  - Delta
  - Box
  - Sinusoid
  - White noise

1D



2D



Orthogonal vector space

- Extending to more than one dimension

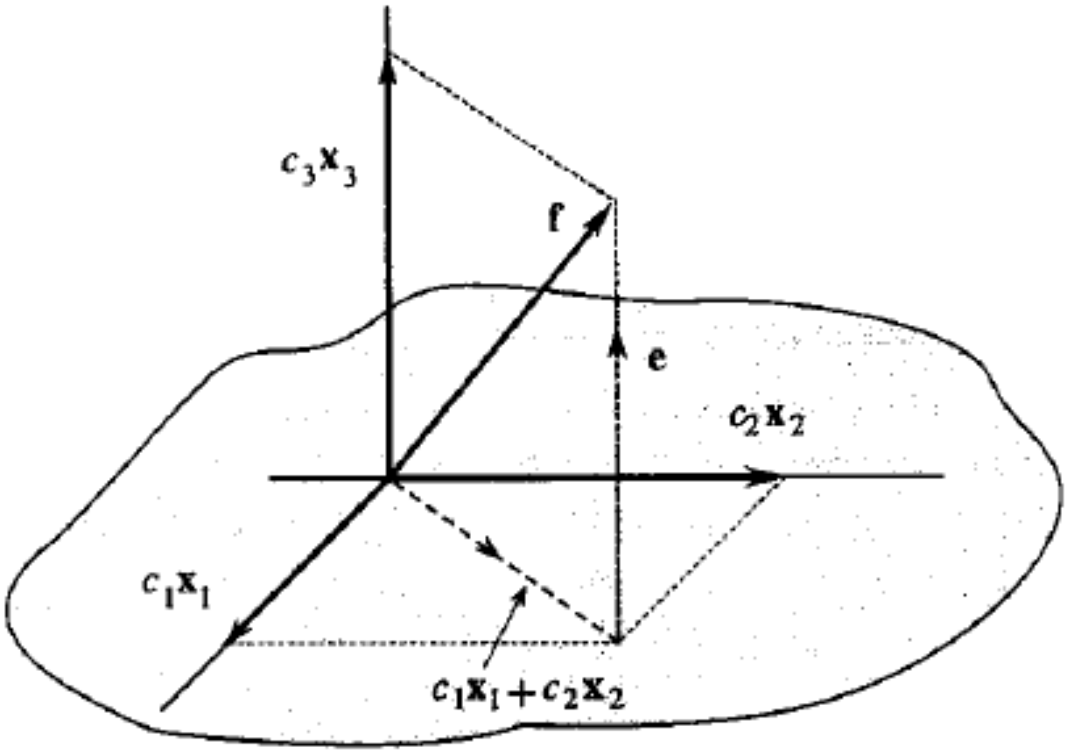


Fig. 3.6 Representation of a vector in three-dimensional space.

Orthogonal vector space

$$\mathbf{f} \simeq c_1\mathbf{x}_1 + c_2\mathbf{x}_2$$

The error $\mathbf{e}$ in this approximation is

$$\mathbf{e} = \mathbf{f} - (c_1\mathbf{x}_1 + c_2\mathbf{x}_2)$$

or

$$\mathbf{f} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \mathbf{e}$$

Now, let us determine the 'best' approximation to $\mathbf{f}$ in terms of all three mutually orthogonal vectors $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$:

$$\mathbf{f} \simeq c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 \tag{3.33}$$

Orthogonal vector space

In this case, $c_1\mathbf{x}_1, c_2\mathbf{x}_2$, and $c_3\mathbf{x}_3$ are the projections (components) of $\mathbf{f}$ on $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$, respectively; that is,

$$c_i = \frac{\mathbf{f} \cdot \mathbf{x}_i}{\mathbf{x}_i \cdot \mathbf{x}_i} \tag{3.35a}$$

$$= \frac{1}{|\mathbf{x}_i|^2} \mathbf{f} \cdot \mathbf{x}_i \qquad i = 1, 2, 3 \tag{3.35b}$$

- Following the analogy, we can define a “signal basis” such that
  - The signals $x_i(t)$ of the basis are mutually orthogonal
  - Each signal f(t) can be expressed as a linear combination of the basis signals weighted by coefficients
  - The coefficients are the correlation coefficients of the signal f(t) with each signal of the basis $x_i(t)$

Orthogonal signal spaces

- Signal basis

$$\{x_i(t)\}, i = 1, \ldots, n$$

- Orthogonality

$$\int_{t_1}^{t_2} x_i(t) x_j(t) dt = 0, \forall i \neq j$$

- Unit norm

$$\int_{-\infty}^{+\infty} x_i^2(t) dt = 1$$

- Approximation of f(t) using the considered basis

$$f(t) \approx c_1 x_1(t) + c_2 x_2(t) + \cdots + c_n x_n(t) = \sum_{i=1}^{n} c_i x_i(t)$$

Signal approximation in an orthonormal basis

- Approximation error

$$e(t) = f(t) - \sum_{n=1}^{N} c_n x_n(t)$$

- Coefficients of the summation

$$c_n = \frac{\displaystyle\int_{t_1}^{t_2} f(t) x_n(t)\, dt}{\displaystyle\int_{t_1}^{t_2} {x_n}^2(t)\, dt}$$

$$= \frac{1}{E_n} \int_{t_1}^{t_2} f(t) x_n(t)\, dt \qquad n = 1, 2, \ldots, N$$

Signal approximation in an orthonormal basis

- The error decreases as the number of basis elements increases

$$E_e = \int_{t_1}^{t_2} f^2(t)\,dt - \sum_{n=1}^{N} {c_n}^2 E_n$$

- When the error goes to zero for N going to infinity the basis is said to be complete
- When the basis is complete the signal is approximated without error and the equality holds

$$f(t) = \sum_{n=0}^{+\infty} c_n x_n(t) \qquad c_n = \int_T f(t) x_n(t) dt$$

Generalization to complex signals

The above results can be generalized to complex signals as follows: A set of functions $x_1(t)$, $x_2(t)$, ..., $x_N(t)$ is mutually orthogonal over the interval $[t_1, t_2]$ if

$$\int_{t_1}^{t_2} x_m(t)x_n^*(t)\,dt = \begin{cases} 0 & m \neq n \\ E_n & m = n \end{cases} \tag{3.43}$$

If this set is complete for a certain class of functions, then a function $f(t)$ in this class can be expressed as

$$f(t) = c_1x_1(t) + c_2x_2(t) + \cdots + c_ix_i(t) + \cdots \tag{3.44}$$

where

$$c_n = \frac{1}{E_n}\int_{t_1}^{t_2} f(t)x_n^*(t)\,dt \tag{3.45}$$

Ready for Fourier!

Trigonometric Fourier series

- Consider a signal set

$$\{1,\ \cos \omega_0 t,\ \cos 2\omega_0 t,\ \ldots,\ \cos n\omega_0 t,\ \ldots;$$
$$\sin \omega_0 t,\ \sin 2\omega_0 t,\ \ldots,\ \sin n\omega_0 t,\ \ldots\} \tag{3.46}$$

- The set is orthonormal over every interval $T_0 = 2\pi/\omega_0$

$$\int_{T_0} \cos n\omega_0 t \cos m\omega_0 t \, dt = \begin{cases} 0 & n \neq m \\ \frac{T_0}{2} & m = n \neq 0 \end{cases} \tag{3.47a}$$

$$\int_{T_0} \sin n\omega_0 t \sin m\omega_0 t \, dt = \begin{cases} 0 & n \neq m \\ \frac{T_0}{2} & n = m \neq 0 \end{cases} \tag{3.47b}$$

and

$$\int_{T_0} \sin n\omega_0 t \cos m\omega_0 t \, dt = 0 \quad \text{for all } n \text{ and } m \tag{3.47c}$$

Trigonometric Fourier series

$$f(t) = a_0 + a_1 \cos \omega_0 t + a_2 \cos 2\omega_0 t + \cdots$$
$$+ b_1 \sin \omega_0 t + b_2 \sin 2\omega_0 t + \cdots \qquad t_1 \le t \le t_1 + T_0 \tag{3.48a}$$

or

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \qquad t_1 \le t \le t_1 + T_0 \tag{3.48b}$$

where

$$\omega_0 = \frac{2\pi}{T_0} \tag{3.49}$$

Using Eq. (3.39), we can determine the Fourier coefficients a_0, a_n, and b_n. Thus

$$a_n = \frac{\int_{t_1}^{t_1+T_0} f(t) \cos n\omega_0 t \, dt}{\int_{t_1}^{t_1+T_0} \cos^2 n\omega_0 t \, dt} \tag{3.50}$$

Trigonometric Fourier series

The integral in the denominator of Eq. (3.50) as seen from Eq. (3.47a) (with $m = n$) is $T_0/2$ when $n \neq 0$. Moreover, for $n = 0$, the denominator is T_0. Hence

$$a_0 = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} f(t)\, dt \tag{3.51a}$$

and

$$a_n = \frac{2}{T_0} \int_{t_1}^{t_1+T_0} f(t) \cos n\omega_0 t\, dt \qquad n = 1, 2, 3, \ldots \tag{3.51b}$$

Arguing the same way, we obtain

$$b_n = \frac{2}{T_0} \int_{t_1}^{t_1+T_0} f(t) \sin n\omega_0 t\, dt \qquad n = 1, 2, 3, \ldots \tag{3.51c}$$

Compact Trigonometric FS

where

$$C_n = \sqrt{{a_n}^2 + {b_n}^2} \tag{3.53a}$$

$$\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right) \tag{3.53b}$$

For consistency, we denote the dc term a_0 by C_0, that is

$$C_0 = a_0 \tag{3.53c}$$

Using the identity (3.52), the trigonometric Fourier series in Eq. (3.48) can be expressed in the **compact form** of the trigonometric Fourier series as

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n) \qquad t_1 \leq t \leq t_1 + T_0 \tag{3.54}$$

where the coefficients C_n and θ_n are computed from a_n and b_n using Eqs. (3.53).

Equation 3.51a shows that a_0 (or C_0) is the average value of $f(t)$ (averaged over one period). This value can often be determined by inspection of $f(t)$.

Example qui

We choose $T_0 = \pi$

$$\omega_0 = \frac{2\pi}{T_0} = 2$$

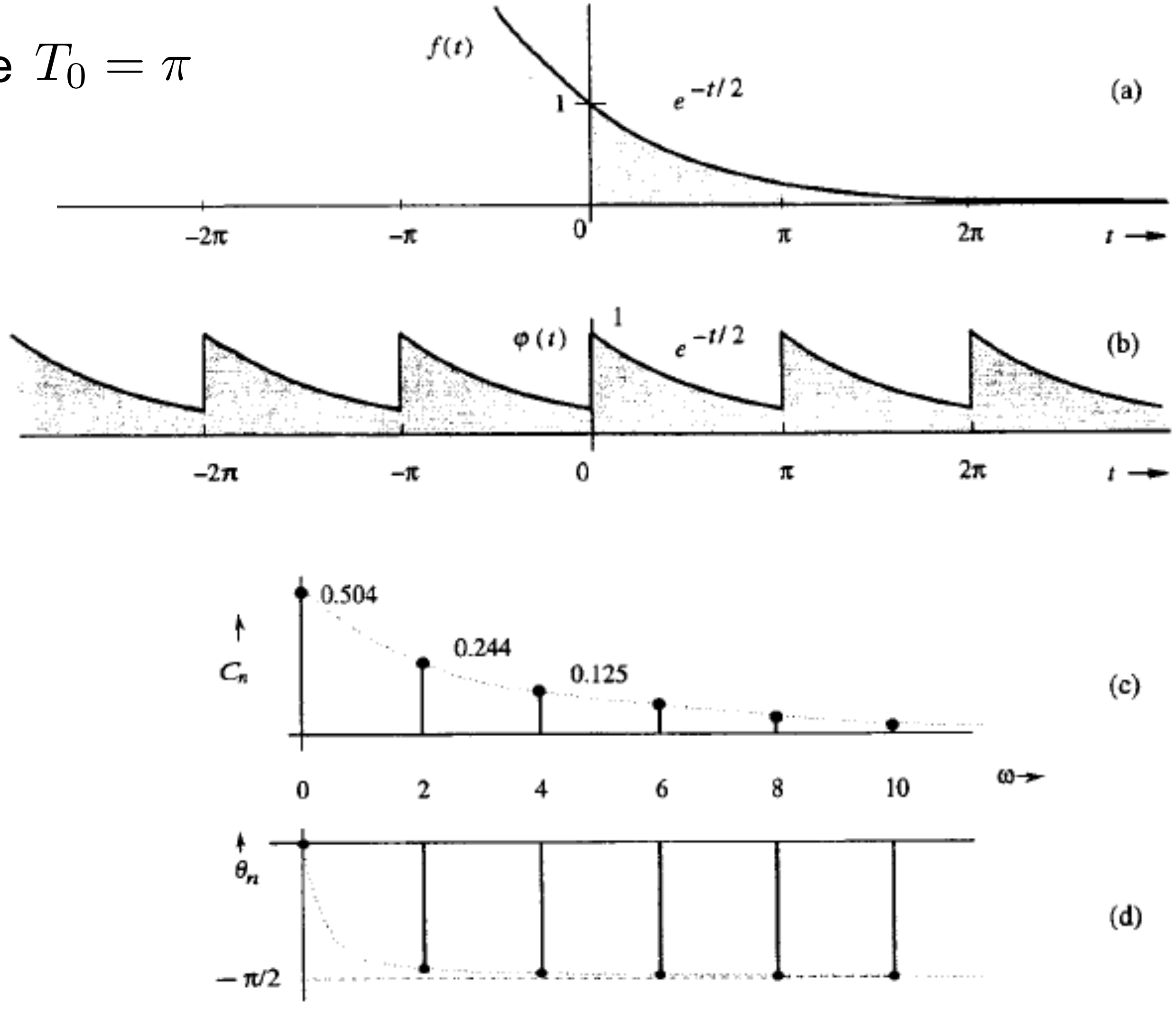


Fig. 3.7 A periodic signal and its Fourier spectra.

Periodicity

- The FS is a periodic function with period T_0
- Consider the function $\phi(t)$

$$\omega_0 = \frac{2\pi}{T_0} = 2$$

$$\varphi(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n) \qquad \text{for all } t$$

and

$$\begin{aligned}
\varphi(t + T_0) &= C_0 + \sum_{n=1}^{\infty} C_n \cos[n\omega_0(t + T_0) + \theta_n] \\
&= C_0 + \sum_{n=1}^{\infty} C_n \cos[(n\omega_0 t + 2n\pi) + \theta_n] \\
&= C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n) \\
&= \varphi(t) \qquad \text{for all } t
\end{aligned} \tag{3.57}$$

FS of periodic signals

- For periodic signals, the FS represents the signal over the whole time axis
- A periodic signal can be seen as the repetition of a signal segment over the entire temporal axis $\rightarrow$ the FS represents the periodic signal f(t) irrespectively of the time point chosen as starting point of the segment
- Fourier coefficients

$$a_0 = \frac{1}{T_0} \int_{T_0} f(t)\, dt$$

$$a_n = \frac{2}{T_0} \int_{T_0} f(t) \cos n\omega_0 t\, dt \qquad n = 1, 2, 3, \ldots \qquad (3.58\text{b})$$

and

$$b_n = \frac{2}{T_0} \int_{T_0} f(t) \sin n\omega_0 t\, dt \qquad n = 1, 2, 3, \ldots \qquad (3.58\text{c})$$

Fourier spectrum

Amplitude spectrum

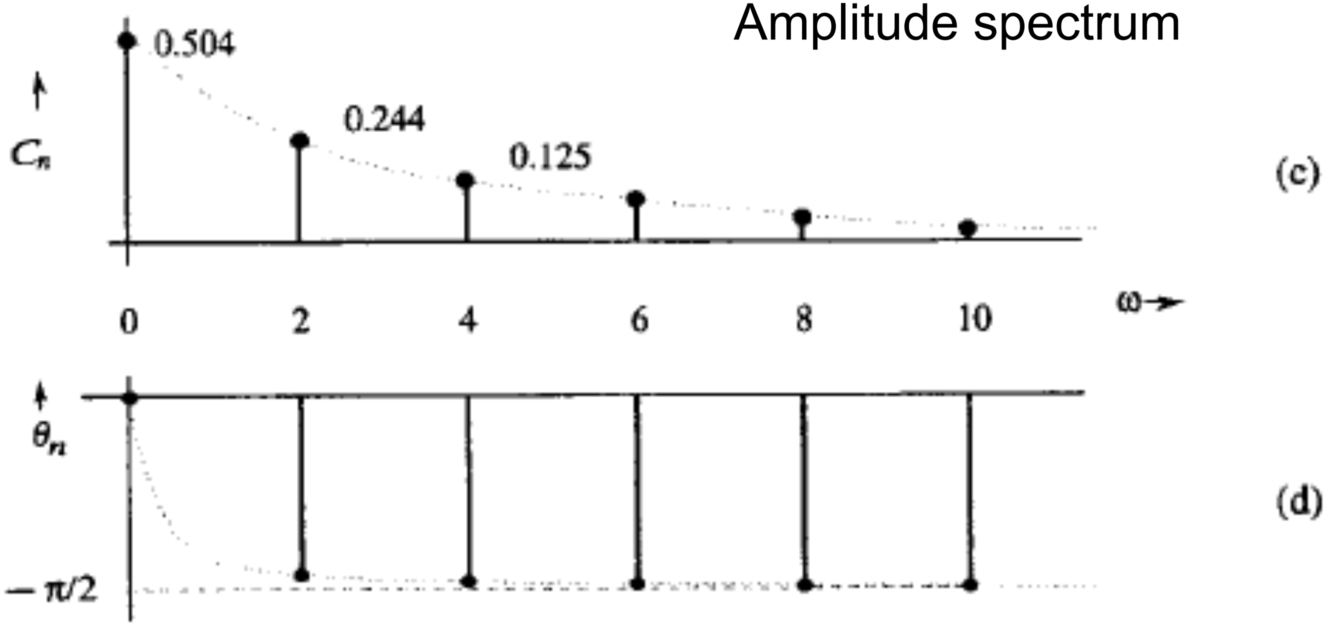


Fig. 3.7 A periodic signal and its Fourier spectra.

Phase spectrum

Table 3.1

n	0	1	2	3	4	5	6	7
C_n	0.504	0.244	0.125	0.084	0.063	0.0504	0.042	0.036
θ_n	0	−75.96	−82.87	−85.24	−86.42	−87.14	−87.61	−87.95

The double signal identity

Fourier domain

Time-domain

Other examples: box

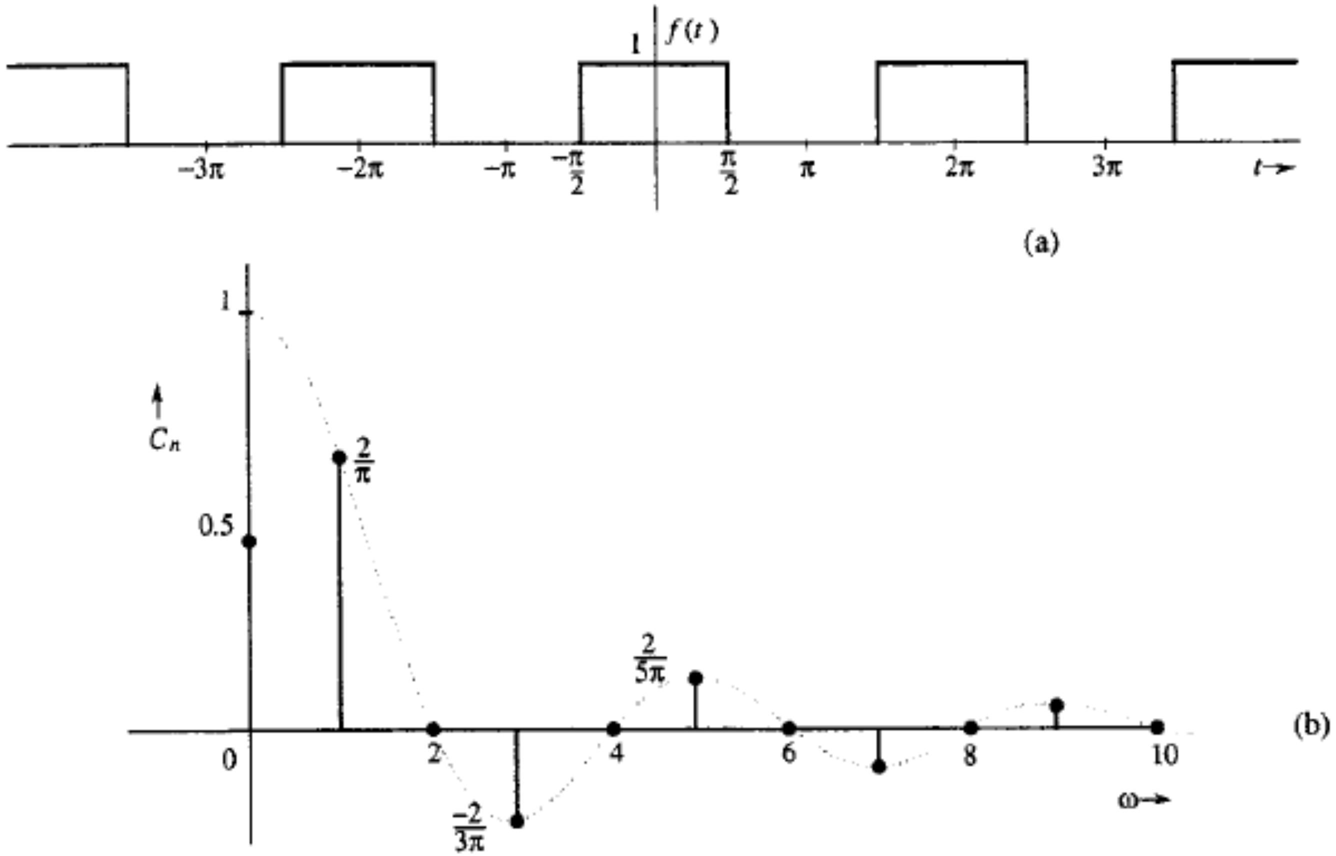


Fig. 3.8 A square pulse periodic signal and its Fourier spectra.

Other examples: triangle

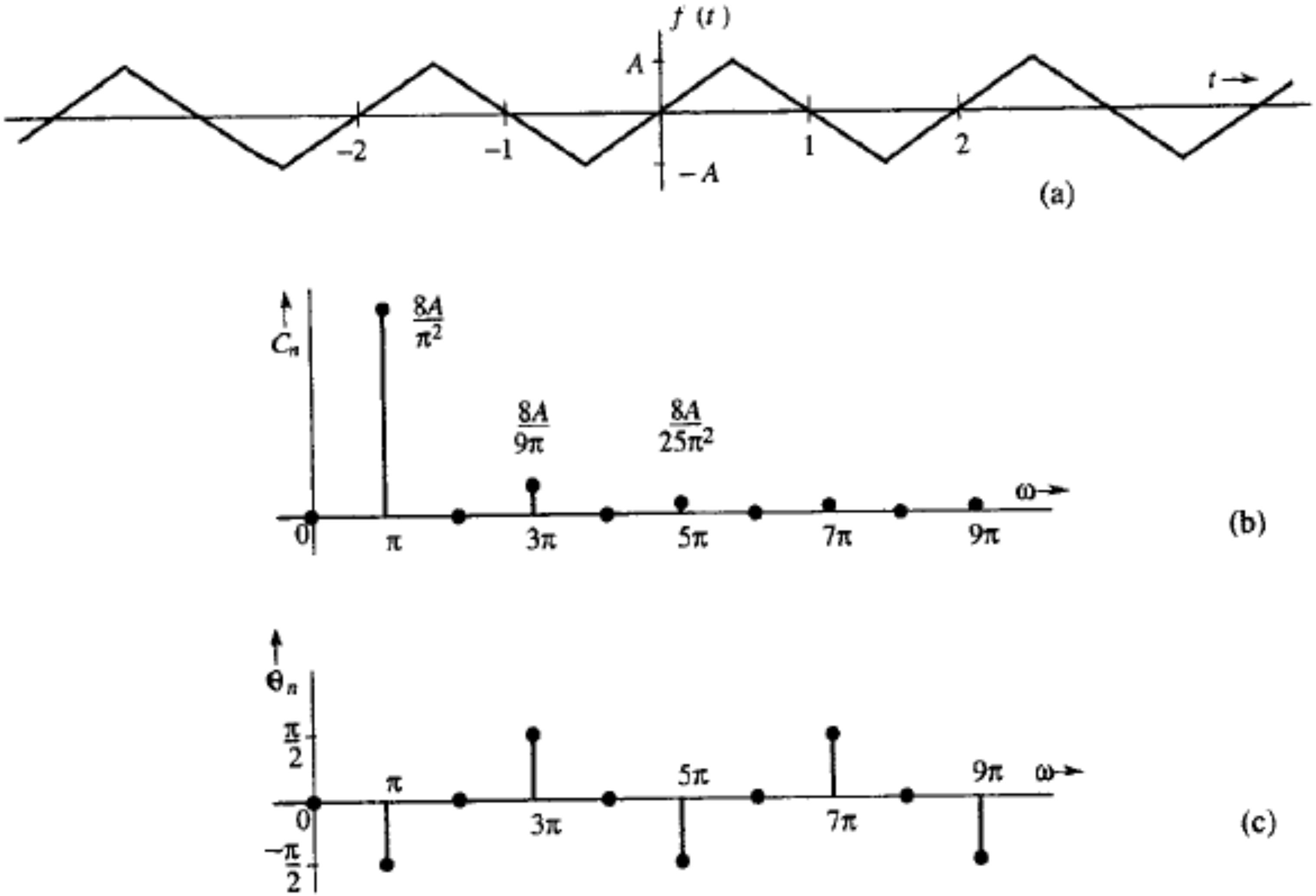


Fig. 3.9 A triangular periodic signal and its Fourier spectra.

The effect of symmetry

- Signals with even symmetry can be expressed using cosinusoids only and
- Signals with odd symmetry can be expressed using sinusoids only
- In consequence, integration can be performed over half the period only

How to determine the fundamental frequency?

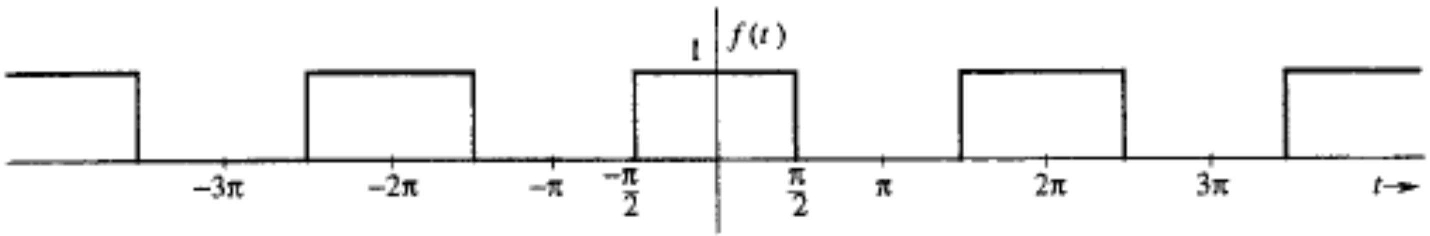
- For a periodic signal expressed as the sum of trigonometric functions is periodic if the ratio of their frequencies is a rational number
- Fundamental frequency: highest positive number of which all the other frequencies are multiples

Examples

$$f_1(t) = 2 + 7\cos\left(\frac{1}{2}t + \theta_1\right) + 3\cos\left(\frac{2}{3}t + \theta_2\right) + 5\cos\left(\frac{7}{6}t + \theta_3\right) \qquad \to \frac{1}{6}$$

$$f_2(t) = 2\cos(2t + \theta_1) + 5\sin(\pi t + \theta_2)$$

- Example: square pulse



$$T_0 = 2\pi \to \omega_0 = \frac{2\pi}{T_0} = 1$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$$

$$a_0 = \frac{1}{T_0}\int_{T_0} f(t)\,dt \quad \to \quad a_0 = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} dt = \frac{1}{2}$$

Example: square pulse

$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos nt\, dt = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$= \begin{cases} 0 & n \text{ even} \\ \frac{2}{\pi n} & n = 1, 5, 9, 13, \cdots \\ -\frac{2}{\pi n} & n = 3, 7, 11, 15, \cdots \end{cases} \tag{3.60b}$$

$$b_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \sin nt\, dt = 0 \tag{3.60c}$$

$$\boxed{f(t) = \frac{1}{2} + \frac{2}{\pi}\left(\cos t - \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t - \frac{1}{7}\cos 7t + \cdots\right)}$$

Only cosines

Alternating positive and negative signs

Example: square pulse

- The negative signes can be accommodated by a phase shift

$$-\cos x = cos(x - \pi)$$

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\left[\cos t + \frac{1}{3}\cos(3t - \pi) + \frac{1}{5}\cos 5t + \frac{1}{7}\cos(7t - \pi) + \frac{1}{9}\cos 9t + \cdots\right]$$

- Fourier series in the compact trigonometric form

$$C_0 = \tfrac{1}{2}$$

$$C_n = \begin{cases} 0 & n \text{ even} \\ \frac{2}{\pi n} & n \text{ odd} \end{cases}$$

$$\theta_n = \begin{cases} 0 & \text{for all } n \neq 3, 7, 11, 15, \cdots \\ -\pi & n = 3, 7, 11, 15, \cdots \end{cases}$$

Fourier series of the square pulse

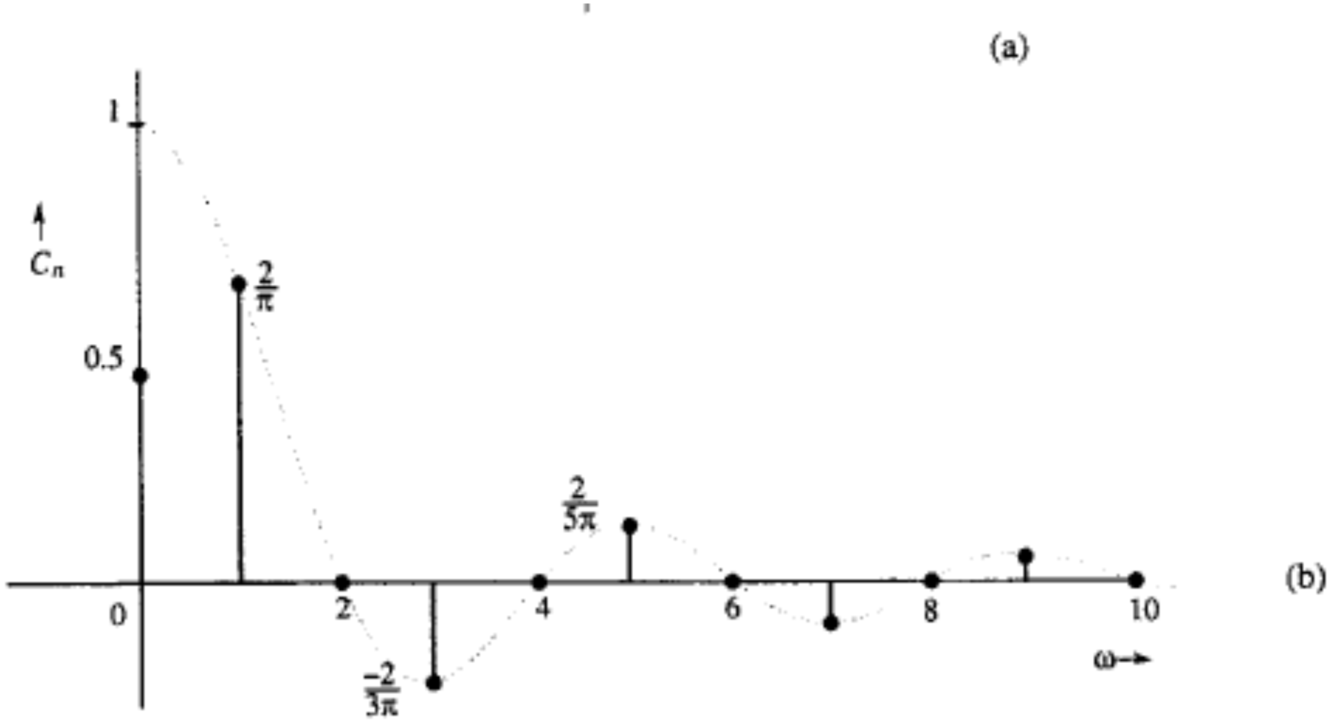


Fig. 3.8 A square pulse periodic signal and its Fourier spectra.

The effect of symmetry

- Even signals $\rightarrow$ cosines only
- Odd signals $\rightarrow$ sines only
- Proof

Even

$$a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\,dt$$

$$a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\cos n\omega_0 t\,dt$$

$$b_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\sin n\omega_0 t\,dt$$

Odd

$$a_0 = a_n = 0$$

$$b_n = \frac{4}{T_0}\int_{0}^{T_0/2} f(t)\sin n\omega_0 t\,dt$$

Trigonometric vs exponential Fourier series

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

$$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

$$D_n = \frac{1}{T_0} \int_{T_0} f(t) e^{-jn\omega_0 t}\, dt$$

$$\begin{aligned} C_n \cos(n\omega_0 t + \theta_n) &= \frac{C_n}{2}\left[e^{j(n\omega_0 t + \theta_n)} + e^{-j(n\omega_0 t + \theta_n)}\right] \\ &= \underbrace{\left(\frac{C_n}{2} e^{j\theta_n}\right)}_{D_n} e^{jn\omega_0 t} + \underbrace{\left(\frac{C_n}{2} e^{-j\theta_n}\right)}_{D_{-n}} e^{-jn\omega_0 t} \\ &= D_n e^{jn\omega_0 t} + D_{-n} e^{-jn\omega_0 t} \end{aligned}$$

$$\begin{aligned} f(t) &= D_0 + \sum_{n=1}^{\infty} D_n e^{jn\omega_0 t} + D_{-n} e^{-jn\omega_0 t} \\ &= \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \end{aligned}$$

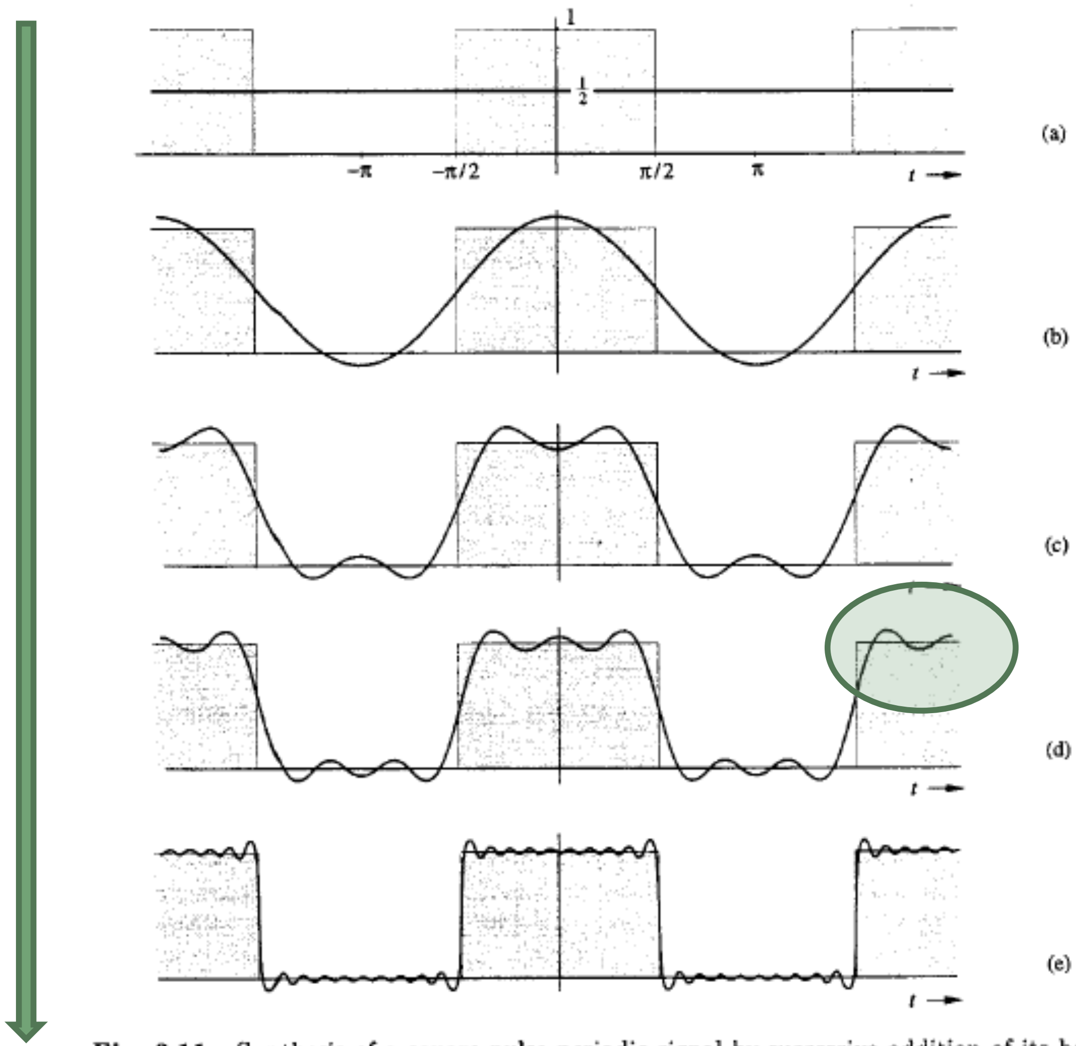
$$D_n = \tfrac{1}{2} C_n e^{j\theta_n}$$

$$D_{-n} = \tfrac{1}{2} C_n e^{-j\theta_n}$$

$$D_n = \frac{1}{2}(a_n - jb_n)$$

Partial signal reconstruction

Increasing the number of harmonic components



Gibbs phenomenon

Fig. 3.11 Synthesis of a square pulse periodic signal by successive addition of its harmonics.

Continuity and spectral decay

- If the first (k-1)-th derivatives of a signal f(t) are continuous and the k-th is discontinuous, the amplitude spectrum decays as $1/n^{k+1}$
  - Square wave: the signal is discontinuous -> k=0
  - Triangular signal: the first derivative is discontinuous -> k=1

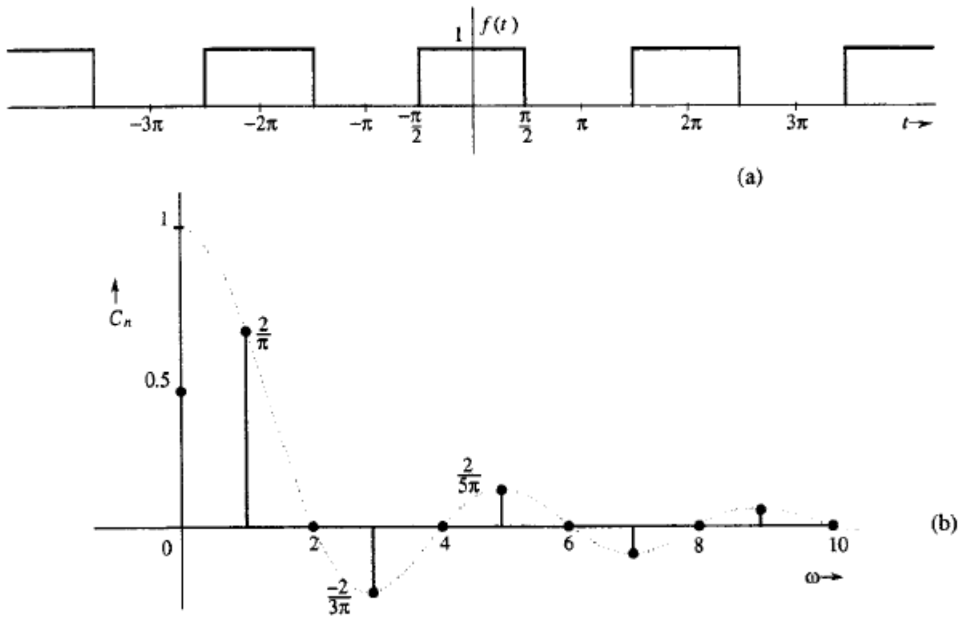


Fig. 3.8 A square pulse periodic signal and its Fourier spectra.

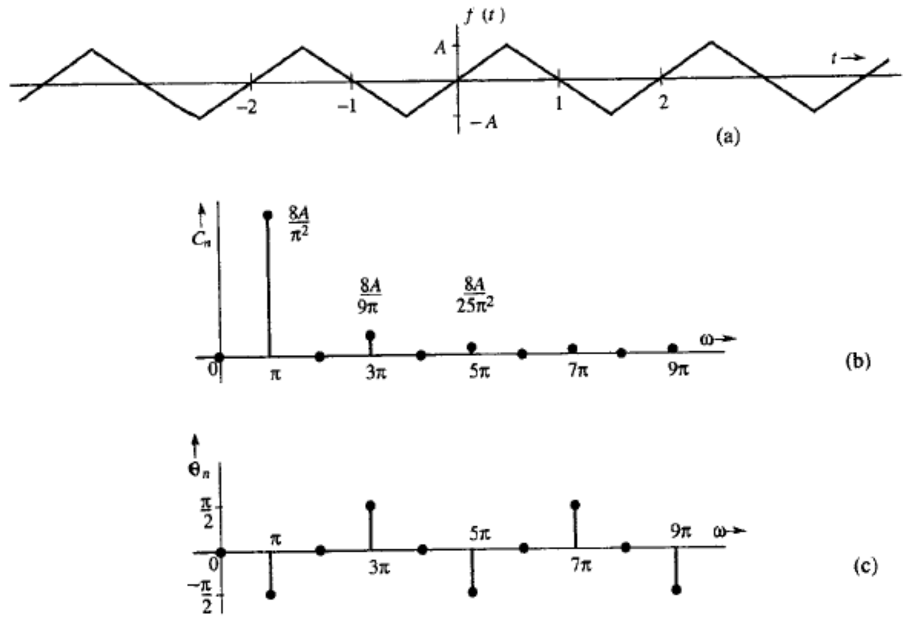


Fig. 3.9 A triangular periodic signal and its Fourier spectra.

Exponential Fourier series

- The exponential series $e^{jn\omega_0 t}, n = 0, \pm 1, \pm 2 \ldots$ is orthonormal over every interval of duration $T_0 = 2\pi/\omega_0$

$$\int_{T_0} e^{jm\omega_0 t}(e^{jn\omega_0 t})^* \, dt = \int_{T_0} e^{j(m-n)\omega_0 t} \, dt = \begin{cases} 0 & m \neq n \\ T_0 & m = n \end{cases} \qquad (3.69)$$

- And it is a complete set, thus any signal f(t) can be expressed over a period of duration T_0 as
-

$$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \qquad (3.70)$$

where [see Eq. (3.45)]

$$D_n = \frac{1}{T_0} \int_{T_0} f(t) e^{-jn\omega_0 t} \, dt \qquad (3.71)$$

Bandwidth of a signal

- Difference between the highest and the lowest frequencies of the spectral components of a signal

Parseval's theorem

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos\left(n\omega_0 t + \theta_n\right)$$

$$P_f = {C_0}^2 + \frac{1}{2}\sum_{n=1}^{\infty} {C_n}^2$$

$$P_f = \sum_{n=-\infty}^{\infty} |D_n|^2$$

LTIC response to periodic inputs

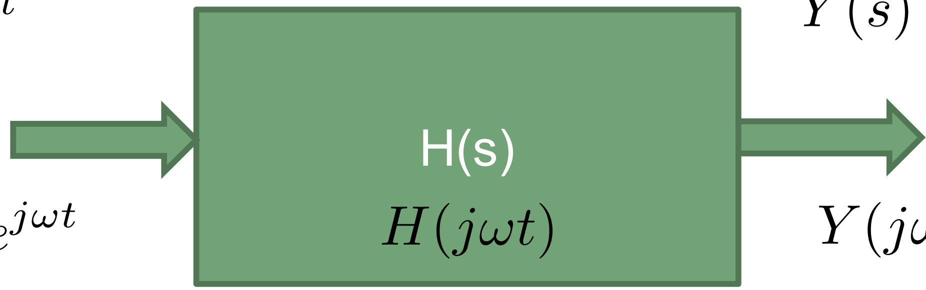
- A periodic signal can be expressed as the superposition of everlasting complex exponentials (or sinusoids)
- The response of the system to an everlasting exponential is H(s) thus, due to linearity

$$\underbrace{e^{j\omega t}}_{\text{input}} \Longrightarrow \underbrace{H(j\omega)e^{j\omega t}}_{\text{output}}$$

$$\boxed{\underbrace{\sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}}_{\text{input } f(t)} \Longrightarrow \underbrace{\sum_{n=-\infty}^{\infty} D_n H(jn\omega_0) e^{jn\omega_0 t}}_{\text{response } y(t)}}$$

$$F(s) = e^{st}$$

$$Y(s) = H(s)$$



$$F(j\omega) = e^{j\omega t}$$

$$Y(j\omega) = H(j\omega t)e^{j\omega t}$$

Limitations of the FS

- Can handle only periodic inputs $\rightarrow$ Continuous Time FT
- Cannot manage easily unstable or marginally stable systems $\rightarrow$ Laplace Transform