Differential geometry and topology – diary of topics covered

5/3/2014. Tensor algebra: vector space, dual vector space, p-tensors (example: dot product is a 2-tensor), tensor product, tensor algebra, bases and dimensions. Exterior algebra: alternating p-tensors, wedge product $\wedge$ and its properties, exterior algebra, bases and dimensions. Examples: determinants, pxp minors. Cross product.

7/3/2014. Volume forms, effect of linear map $A : V \to W$, pull-back maps on tensors. Determinant theorem: $\dim V = k$. Given a linear map $A : V \to V$, then for all $\omega \in \Lambda^k(V^*)$, $A^*(\omega) = (\det A)\omega$. Exterior algebra on $\mathbb{R}^k$: tangent bundle on $\mathbb{R}^k$, cotangent bundle on $\mathbb{R}^k$, tensor bundles, alternating tensor bundles. Differential forms, wedge product on forms. Exterior derivative d, properties of d, low-dimensional examples and relationship to gradient, curl divergence. Push-forward and pull-back maps.

12/3/2014. Examples: parametrized curves in $\mathbb{R}^2$ or $\mathbb{R}^3$, pull-back of 1-form to the curve; parametrized surfaces in $\mathbb{R}^3$, pull-back of a 2-form to the surface. Proposition: pull-back commutes with d. Differential forms on manifolds, independence of differential forms on coordinate system. Volume form on $\mathbb{R}^n$, volume form on a manifold. Orientable manifolds, orientability, Proposition: a n-dimensional manifold has an orientable atlas if and only if it has a non-vanishing n-form. Change of variables formula for integrals; partitions of unity, definition of integration on orientable manifolds.

14/3/2014. Independence of integral on atlas and partition of unity. Definition of manifolds-with-boundary, induced orientation on boundary; low dimensional examples. Stokes' theorem on $\mathbb{R}^n$, Stokes' theorem on half-space $\mathbb{H}^n = \mathbb{R}^{n-1} \times [0, \infty)$, Stokes' theorem for manifolds with boundary. Examples: fundamental theorem of calculus, classical Stokes' theorem in $\mathbb{R}^3$, divergence theorem, Green's formula.

19/3/2014. Application of Stokes' theorem: degree formula. Definition of local degree, local degree formula, degree formula, homotopy invariance of degree, independence of degree on the choice of regular point. De Rham theory: closed vs exact forms, the de Rham complex.

21/3/2014. Definition of de Rham cohomology groups. First example: $H^*_{DR}(\mathbb{R})$. $\dim H^0_{dR}(M) = \#$ components of M, definition: simply connected, Prop: M simply connected $\implies H^1_{dR}(M) = 0$.

26/3/2014. Homological algebra: cochain complex complex, differential, cohomology. (And dual: chain complex, boundary operator, homology) Definition of chain maps. Chain maps induce well-defined maps on cohomology. Example: pull-back $f^* : \Omega^n(N) \to \Omega^n(M)$ associated to a smooth map $f : M \to N$. Definition of null-homotopy for a chain map; null-homotopic maps induce the zero map on cohomology. Definition of homotopic chain maps, homotopic chain maps induce equal maps on cohomology. Thm: $H^*_{DR}(M \times \mathbb{R}) \cong H^*_{DR}(M)$. Corollary: Poincaré Lemma, $H^*_{DR}(\mathbb{R}^n) \cong H^*_{DR}(point)$.

28/3/2014. Corollary: Homotopic manifolds have the same de Rham cohomology. Exact sequences. Thm: A short exact sequence of chain maps induces a long exact sequence on cohomology. Diagram chasing: construction of the connecting homomorphism, proof of well-definedness on cohomology, proof of exactness of the long exact sequence.

2/4/2014. Application: Mayer-Vietoris long exact sequence. Proof of the Mayer-Vietoris short exact sequence for de Rham cohomology. Example calculations: S^2, S^1.

4/4/2014. Good covers, Prop: Every manifold has a good cover. Finite good covers and the Mayer-Vietoris argument. Thm: A manifold with a finite good cover has finite dimensional de Rham cohomology. Tensor product of vector spaces, statement of the Kunneth formula, examples.

9/4/2014. The Five Lemma, proof of the Kunneth formula using the Mayer-Vietoris argument. Statement of Poincarè duality for compact oriented manifolds. Practice exercises.

11/4/2014. Midterm.

16/4/2014. Compactly supported cohomology: definition, first example $H^*_c(\mathbb{R})$. Prop: $\dim H^0_c(M) = \#$ compact components, Thm: $H^{k+1}_c(M \times \mathbb{R}) \cong H^k_c(M)$. Corollary (Poincarè lemma for compact cohomology): $H^*_c(\mathbb{R}^k) \cong H^{*-k}_c(point)$. Finite dimensionality of compactly supported cohomology for manifolds with a finite good cover.

23/4/2014. Mayer-Vietoris short exact sequence and induced long exact sequence for compact cohomology. Proof by Mayer-Vietoris argument of Poincarè duality for oriented manifolds with a finite good cover. Example computations: genus g surface, genus g surface with punctures.

30/4/2014. Some Hodge theory. Linear algebra: symmetric bilinear pairing $\langle,\rangle$ on V, extension of $\langle,\rangle$ to alternating tensors $\Lambda^p(V)$, non-degenerate bilinear pairings determine an isomorphism between a vector space and its dual. Normalized volume form, definition of the Hodge star operator w.r.t. a non-degenerate symmetric bilinear form and a normalized volume form. Explicit computations: $\mathbb{R}^3$ w.r.t. standard inner product, $\mathbb{R}^4$ w.r.t. inner product with signs $(+,+,+,-)$. Example: the cross product. Prop: $* \circ * = (-1)^{p(k-p)} s_{\langle,\rangle}$ where $s_{\langle,\rangle}$ is the sign of the bilinear form.

7/5/2014. Class suspended (assemblea studentesca)

9/5/2014. On manifolds: Riemannian metrics, pseudo-Riemannian metrics, normalized volume forms, Hodge star on differential forms. Example: Maxwell's equations. Definition of the codifferential δ, Prop: $\delta \circ \delta = 0$, Laplace-Beltrami operators $\triangle$, explicit examples. L^2 inner product on $\Omega^k(M)$ for M compact and oriented, adjointness of d and δ, Lemma: $\triangle$ is symmetric and non-negative. Definition of harmonic forms, Prop: harmonic forms $= \ker \delta \cap \ker d$, uniqueness of harmonic representative of a cohomology class, existence of harmonic representative (harder – not fully proved in class, just sketched), Hodge Theorem.

14/5/2014. Alternative proof of Poincarè duality for oriented compact manifolds using Hodge theory. Back to de Rham theory: The Poincarè dual of a compact oriented submanifold, Thom forms, Thom form for a point, Thom form for the equator of a cylinder.

16/5/2014. Vector bundles, conormal bundle of a submanifold, tubular neighborhoods, transversely intersectly submanifolds and the wedge product of Thom forms, intersection numbers and self-intersection numbers, signs of intersection, low dimensional examples. Thm: Euler characteristic of M = self-intersection number of the diagonal $\triangle$ in $M \times M$.

23/5/2014. Vector fields and the local index of a zero, examples in $\mathbb{R}^2$. Exponentiating vector fields, relating the index of a vector field to intersection numbers, proof of the Poincarè-Hopf index theorem.

28/5/2014. Discussion of practice exercises. The Gauss-Bonnet theorem proved using the Poincarè-Hopf index theorem and the Degree formula.

30/5/2014. Simplicial homology, low-dimensional examples and computations of given $\triangle$-complexes (torus, circle, sphere, Klein bottle, $\mathbb{R}P^2$), manipulating quotient groups (for coefficients in $\mathbb{Z}$). Definition of singular homology, example: singular homology of a point.

4/6/2014. Homotopy invariance of singular homology. Reduced singular homology, relative homology. Mayer-Vietoris long exact sequence, Excision theorem (barycentric subdivision just sketched). Singular homology of S^n. Application: Brouwer fixed point theorem. Equivalence of simplicial and singular homology.

6/6/2014. Cohomology: definition of cochain complex, coboundary operator, ring structure: cup product, cap product, Poincarè duality (just sketched). Different coefficients: $\mathbb{Z}_p$ for p prime, $\mathbb{R}$. Statement of De Rham's theorem.

11/6/2014. Definition and construction of the fundamental group, Prop: $\pi_1(S^1) \cong \mathbb{Z}$, Borsuk-Ulam theorem, Kunneth formula. Discussion of exercises.

WARNING: This is not a complete transcript of the notes in class! And watch out for typos. This is intended only as a supplement to the notes in class and the references. If the material is well-covered in a reference, it is only sketched here. I give details here only when I haven't followed the references, notably the sections on Hodge theory, Thom forms, and the Poincarè-Hopf index theorem.
Reference for integration on manifolds and Gauss-Bonnet: *Differential topology* by Guillemin and Sternberg.
Reference for de Rham theory: *Differential forms in algebraic topology* by Bott and Tu, early chapters.
Reference for simplicial and singular homology, cohomology, fundamental group: *Algebraic Topology* by Allen Hatcher.

1.1. **Exterior Algebra.** Vector space $V = \text{span}\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_k}\right\}$, dual space $V^* = \text{span}\{dx_1, \ldots, dx_k\}$, where $dx_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{i,j}$. Vector space of p-tensors $\mathcal{T}^p(V^*)$, $\mathcal{T}^0(V^*) := \mathbb{R}$, $\mathcal{T}^1(V^*) = V^*$.
Examples: dot product, determinant
Tensor product $\otimes : \mathcal{T}^p(V^*) \times \mathcal{T}^q(V^*) \to \mathcal{T}^{p+q}(V^*), (S,T) \mapsto S \otimes T$. Tensor product distributive, associative, not commutative.

Aside: $\mathcal{T} = \bigoplus_{p\geq 0} \mathcal{T}^p(V^*)$. $(\mathcal{T}, +, \otimes)$ is called the tensor algebra: a graded algebra over $\mathbb{R}$, additive unit 0, multiplicative identity $1 \in \mathcal{T}^0(V^*)$, $S \otimes 1 = S = 1 \otimes S$, and it is infinite-dimensional.

Theorem. *$\mathcal{T}^p(V^*) = \text{span}\left\{dx_{i_1} \otimes \ldots \otimes dx_{i_p} \middle| 1 \leq i_1, \ldots, i_p \leq k\right\}$, so $\dim \mathcal{T}^p(V^*) = k^p$*

Exercise 1. *Show that the p-tensors $dx_{i_1} \otimes \ldots \otimes dx_{i_p}$ for $1 \leq i_1, \ldots, i_p \leq k$ are linearly independent.*

Vector space of alternating p-tensors $\Lambda^p(V^*)$, with $\Lambda^0(V^*) = \mathbb{R}, \Lambda^1(V^*) = V^*$. Alternating p-tensors are also called p-forms. If T is a p-form, p is called the *degree* of T.
Definition of linear map $\text{Alt} : \mathcal{T}^p(V^*) \to \Lambda^p(V^*)$.

Remark. The factor $1/p!$ makes it satisfy $\text{Alt}|_{\Lambda^p(V^*)} \equiv \text{Id}$, so Alt is a *projection* i.e. $\text{Alt} \circ \text{Alt} = \text{Alt}$. Be aware that some authors use another convention that omits the factor $1/p!$. Then Alt restricted to $\Lambda^p(V^*)$ is equivalent to scalar multiplication by $p!$ so Alt is not a projection, as $\text{Alt} \circ \text{Alt} \neq \text{Alt}$. But one advantage of not dividing by $p!$ is if you want to work over a field of characteristic greater than 0.

Definition of wedge product $\wedge : \Lambda^p(V^*) \times \Lambda^q(V^*), (S,T) \mapsto S \wedge T := \text{Alt}(S \otimes T)$.

Lemma. *Properties of $\wedge$.*
(1) *$\wedge$ is* distributive.
(2) *$(S \wedge T) \wedge U = \text{Alt}(S \otimes T \otimes U) = S \wedge (T \wedge U)$, so $\wedge$ is* associative.
(3) *For $S \in \Lambda^p(V^*), T \in \Lambda^q(V^*), S \wedge T = (-1)^{pq} T \wedge S$. In different contexts this is called different things: in the context of algebras, one says that $\wedge$ is* anti-commutative*, or* skew-commutative*, or* graded anti-commutative*, but in the context of superalgebras one says that $\wedge$ is* graded commutative *or* supercommutative*. (Yes, confusing!)*
(4) *$T \wedge T = 0$ if T is a form of* odd *degree*

Theorem. *$\Lambda^p(V^*) = \text{span}\{dx_{i_1} \wedge \ldots \wedge dx_{i_p} | 1 \leq i_1 < i_2 < \ldots < i_p \leq k\}$, so $\dim \Lambda^p(V^*) = \binom{k}{p} = \frac{k!}{p!(k-p)!}$*

Aside: Put $\Lambda = \bigoplus_{p\geq 0} \Lambda^p(V^*)$. $(\Lambda, +, \wedge)$ called the exterior algebra: a graded algebra over $\mathbb{R}$, additive unit 0, multiplicative identity $1 \in \Lambda^0(V^*)$, $S \wedge 1 = S = 1 \wedge S$. Called the *exterior algebra*, or sometimes called the *Grassmann algebra*. But it's *not* infinite dimensional, because $\Lambda^p(V^*) = 0$ for all $p > k$ (see exercises below). As a vector space, Λ has dimension $\sum_{p=0}^{k} \binom{k}{p} = 2^k$.

Exercise 2. *Show that the p-forms $dx_{i_1} \wedge \ldots \wedge dx_{i_p}$ for $1 \leq i_1 < i_2 < \ldots < i_p \leq k$ are linearly independent.*

Exercise 3. *Show that if $v_1, \ldots, v_p$ are linearly dependent in V, then $T(v_1, \ldots, v_p) = 0$ for any $T \in \Lambda^p(V^*)$.*

Exercise 4. *Show that all p-forms with $p > k$ are zero. (Use previous exercise.)*

Remark. Some authors, e.g. Spivak, define Alt as we have but use another convention for defining the wedge product, $S \wedge T := \frac{(p+q)!}{p!q!} \text{Alt}(S \otimes T)$. The *normalization factor* of $\frac{(p+q)!}{p!q!}$ implies that $dx_1 \wedge \ldots \wedge dx_k(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}) = 1$, which can be convenient (for example, it means that $dx_1 \wedge \ldots \wedge dx_k$ is equal to the usual matrix determinant).

Relationship between basis elements $dx_{i_1} \wedge \ldots \wedge dx_{i_p}$ and determinants of $p \times p$ matrix minors.

Let $\mathbf{v}_1 = (v_{11}, v_{12}, \ldots, v_{1n}), \ldots, \mathbf{v}_p = (v_{p1}, v_{p2}, \ldots, v_{pn})$. Show that

$$dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_p}(\mathbf{v}_1, \ldots, \mathbf{v}_p) = \frac{1}{p!} \det \begin{pmatrix} v_{1i_1} & v_{1i_2} & \ldots & v_{1i_p} \\ v_{2i_1} & v_{2i_2} & \ldots & v_{2i_p} \\ & & \ldots & \\ v_{pi_1} & v_{pi_2} & \ldots & v_{pi_p} \end{pmatrix}$$

i.e. it's a multiple of the $p \times p$ minor determined by the p columns $i_1, i_2, \ldots, i_p$.

The $p!$ only comes in because of how we chose to define the wedge product, because in our definition

$$\begin{aligned} dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_p} &= Alt(dx_{i_1} \otimes \ldots \otimes dx_{i_p}) \\ &= \frac{1}{p!} \sum_{\pi \in S_p} (-1)^\pi dx_{i_{\pi(1)}} \otimes \ldots \otimes dx_{i_{\pi(p)}} \end{aligned}$$

If we had defined the wedge product $S \wedge T$ as $\frac{(p+q)!}{p!q!} Alt(S \otimes T)$ we would have ended up without the extra factor of $1/p!$.

Example 1. The cross product $\mathbf{a} \times \mathbf{b}$ in $\mathbb{R}^3$ c.f. the wedge product of $a_1 dx_1 + a_2 dx_2 + a_3 dx_3$ with $b_1 dx_1 + b_2 dx_2 + b_3 dx_3$.

1.2. **Volume forms.** A k-form that is non-zero is called a *volume form.* All volume forms are scalar multiples of each other. They're all scalar multiples, for example, of $dx_1 \wedge dx_2 \wedge \ldots \wedge dx_k$.

A choice of volume form A determines a non-degenerate bilinear map $\langle , \rangle_A : \Lambda^p(V^*) \times \Lambda^{k-p}(V^*) \to \mathbb{R}$: for a p-form T, and $(k-p)$-form S, $\langle T, S \rangle_A := \alpha$ where α is uniquely determined by the condition $T \wedge S = \alpha A$. This bilinear map identifies $\Lambda^p \cong \left(\Lambda^{k-p}\right)^*$.

Non-degenerate means that for all $T \in \Lambda^p$, $T \neq 0$, there exists an $S \in \Lambda^{k-p}$ such that $\langle T, S \rangle_A \neq 0$. Example: Let T be any p form, and suppose that its coordinate $T_{i_1 i_2 \ldots i_p}$ is non-zero. Then if $\{j_1, \ldots, j_{k-p}\} = \{1, \ldots, k\} \setminus \{i_1, \ldots, i_p\}$, then putting $S = dx_{j_1} \wedge \ldots \wedge dx_{j_{k-p}}$ we have $\langle T, S \rangle_A \neq 0$.

Theorem (Determinant theorem)**.** $\dim V = k$. *Given a linear map $A : V \to V$, the dual map (i.e. transpose) $A^* : V^* \to V^*$ induces a pull-back map $A^* : \Lambda^p(V^*) \to \Lambda^p(V^*)$ for all p. Let $\omega \in \Lambda^k(V^*)$. Then $A^*(\omega) = (\det A)\omega$.*

1.3. **Differential forms on $\mathbb{R}^k$.** Coordinates $(x_1, \ldots, x_k)$ on $\mathbb{R}^k$.

1.3.1. *Tangent bundle.* Tangent bundle $T\mathbb{R}^k \cong \mathbb{R}^k \times \mathbb{R}^k$ is a manifold with coordinates
$(\mathbf{x}, \mathbf{v}) = (x_1, \ldots, x_k, v_1, \ldots, v_k)$, where $v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} + \ldots v_k \frac{\partial}{\partial x_k} \in T_{\mathbf{x}}\mathbb{R}^k$.
A point of the tangent bundle corresponds to a point $\mathbf{x}$ in $\mathbb{R}^k$, and a tangent vector to $\mathbb{R}^k$ at that point.

1.3.2. *Cotangent bundle.* Cotangent bundle $T^*\mathbb{R}^k \cong \mathbb{R}^k \times \mathbb{R}^k$ is a manifold with coordinates
$(\mathbf{x}, \mathbf{v}) = (x_1, \ldots, x_k, p_1, \ldots, p_k)$, where $p_1 dx_1 + p_2 dx_2 + \ldots p_k dx_k \in T^*_{\mathbf{x}}\mathbb{R}^k$.

1.3.3. *Alternating p-tensor bundle.* The *alternating p-tensor bundle* $\Lambda^p(T^*\mathbb{R}^k) \cong \mathbb{R}^k \times \mathbb{R}^{\binom{k}{p}}$ is a manifold with coordinates
$(\mathbf{x}, \mathbf{a})$ for $\mathbf{a} = (\{a_{i_1, \ldots, i_p}\}_{1 \le i_1 < i_2 < \ldots < i_p \le k})$, where $\sum_{1 \le i_1 < \ldots < i_p \le n} a_{i_1, \ldots, i_p} dx_{i_1} \wedge \ldots \wedge dx_{i_p} \in \Lambda^p(T^*_{\mathbf{x}}\mathbb{R}^k)$.

Definition 1. *A differential p-form is a smooth map $\phi : \mathbb{R}^k \to \Lambda^p(T^*\mathbb{R}^k)$ of the form $\phi(\mathbf{x}) = (\mathbf{x}, \mathbf{a}(\mathbf{x}))$. In fancier language, it's a smooth section of the alternating p-tensor bundle $\Lambda^p(T^*\mathbb{R}^k) \to \mathbb{R}^k$.*

The vector space of differential p-forms on $\mathbb{R}^k$ is written as $\Omega^p(\mathbb{R}^k)$. Every p-form on $\mathbb{R}^k$ can be uniquely written as a sum $\phi = \sum_{1 \le i_1 < \ldots < i_p \le k} a_{i_1, \ldots, i_p}(\mathbf{x}) dx_{i_1} \wedge \ldots \wedge dx_{i_p}$, for a collection of smooth functions $a_{i_1, \ldots, i_p} : \mathbb{R}^k \to \mathbb{R}$.

Example: differential 0-forms $\leftrightarrow$ smooth functions $a : \mathbb{R}^k \to \mathbb{R}$.

Wedge product on forms: $\wedge : \Omega^p(\mathbb{R}^k) \times \Omega^q(\mathbb{R}^k) \to \Omega^{p+q}(\mathbb{R}^k)$ is defined pointwise, i.e. at each point $\mathbf{x}$ take the wedge product of the alternating tensors at $\mathbf{x}$:

$$(\mathbf{x}, \mathbf{a}(\mathbf{x})) \wedge (\mathbf{x}, \widetilde{\mathbf{a}}(\mathbf{x})) := (\mathbf{x}, \mathbf{a}(\mathbf{x}) \wedge \widetilde{\mathbf{a}}(\mathbf{x})).$$

On monomials:

$\left(a_{i_1, \ldots, i_p}(\mathbf{x}) dx_{i_1} \wedge \ldots \wedge dx_{i_p}\right) \wedge \left(\widetilde{a}_{j_1, \ldots, j_q}(\mathbf{x}) dx_{j_1} \wedge \ldots dx_{j_q}\right) = a_{i_1, \ldots, i_p}(\mathbf{x}) \widetilde{a}_{j_1, \ldots, j_q}(\mathbf{x}) dx_{i_1} \wedge \ldots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \ldots dx_{j_q}$.

Definition 2. *The* exterior derivative $d : \Omega^p(\mathbb{R}^k) \to \Omega^{p+1}(\mathbb{R}^k)$ *is defined on monomials by*

$$a_{i_1, \ldots, i_p}(\mathbf{x}) dx_{i_1} \wedge \ldots \wedge dx_{i_p} \mapsto \sum_{j=1, \ldots, k} \frac{\partial a_{i_1, \ldots, i_p}(\mathbf{x})}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

(Note: RHS is effectively a sum over $j \in \{1, \ldots, k\} \setminus \{i_1, \ldots, i_p\}$).

Theorem. *Properties of d:*

(1) $d(\omega \wedge \sigma) = d\omega \wedge \sigma + (-1)^{\deg \omega} \omega \wedge d\sigma$ *i.e. d is an* anti-derivation.
(2) $d^2 = 0$

Examples in $\mathbb{R}^3$, relationship to gradient, curl, divergence.

1.4. **Push-forward, pull-back.** Smooth map $\phi : \mathbb{R}^k \to \mathbb{R}^l$, $\phi(\mathbf{x}) = (\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), \ldots, \phi_l(\mathbf{x}))$.

- The *push-forward map* $\phi_* : T\mathbb{R}^k \to T\mathbb{R}^l$ takes tangent vectors at $\mathbf{x}$ to tangent vectors at $\phi(\mathbf{x})$, $\mathbf{v} \mapsto D\phi(\mathbf{x})\mathbf{v}$ where $D\phi(\mathbf{x})$ is the Jacobian matrix at $\mathbf{x}$.
- The *pull-back map* $\phi^* : \Omega^p(\mathbb{R}^k) \to \Omega^p(\mathbb{R}^k)$ takes p-forms on $\mathbb{R}^k$ to p-forms on $\mathbb{R}^l$. On 0-forms, $\phi^* f := f \circ \phi$ for any $f : \mathbb{R}^l \to \mathbb{R}$. On p-forms,

$$\phi^* \left(f(y) dy_{i_1} \wedge \ldots \wedge dy_{i_p} \right) = (f \circ \phi)(\mathbf{x}) d\phi_{i_1} \wedge \ldots \wedge d\phi_{i_p}.$$

Example 2. $\gamma : \mathbb{R} \to \mathbb{R}^3$, $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$, parametrized curve. Let $\omega = f_1(x,y,z)dx + f_2(x,y,z)dy + f_3(x,y,z)dz \in \Omega^1(\mathbb{R}^3)$. Then $\gamma^*\omega = f_1(\gamma)\gamma_1'(t)dt + f_2(\gamma)\gamma_2'(t)dt + f_3(\gamma)\gamma_3'(t)dt = \mathbf{F}(\gamma(t)) \cdot \gamma'(t)dt$

Example 3. Parametrized surface $S : \mathbb{R}^2 \to \mathbb{R}^3$, and pull-back of 2-form $f_1 dy \wedge dz + f_2 dx \wedge dz + f_3 dx \wedge dy$ in $\mathbb{R}^3$.

Proposition. *Pull-back commutes with exterior derivative, i.e. $d\phi^* = \phi^* d$ as maps from $\Omega^p(\mathbb{R}^l) \to \Omega^{p+1}(\mathbb{R}^k)$.*

Independence of differential forms on the coordinate system; definition of differential forms on manifolds. All manifolds considered are Hausdorff and have a countable basis.

1.5. **Integration on manifolds.** Integrating forms with compact support on $\mathbb{R}^k$.

Definition 3. *Let $\omega = f dx_1 \wedge \ldots \wedge dx_k$ where $f : \mathbb{R}^k \to \mathbb{R}$ has compact support. Define*

$$\int_{\mathbb{R}^k} \omega := \int_{\mathbb{R}} \cdots \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x_1, \ldots, x_n) dx_1 \right) dx_2 \right) \ldots dx_n.$$

Change of variables formula for integrals; choice of an orientation on $\mathbb{R}^k$.
Orientable manifolds, orientability.

Proposition. *A manifold of dimension k has an oriented atlas $\iff$ it has a non-vanishing k-form.*

Change of variables formula for integrals; partitions of unity; definition of integration on manifolds.

Proposition. *Let M be a smooth oriented manifold of dimension k, and $\omega \in \Omega^k(M)$. Then the integral $\int_M \omega$ is independent of the choice of atlas of M and the choice of partition of unity subordinate to the atlas.*

A diffeomorphism preserves the integral only up to sign; only the orientation-preserving diffeomorphisms preserve the integral. Therefore integration is only defined on orientable manifolds. Any oriented manifold has exactly two orientations, so the value of the integral depends on the orientation chosen.

1.6. **Integration on manifolds-with-boundary.** Definition of manifold with boundary; note operation $\partial \circ \partial = \emptyset$.

Lemma. *The boundary of an oriented k-dimensional manifold-with-boundary is an oriented $k-1$ dimensional manifold with empty boundary.*

Definition: the *induced orientation* for ∂M, which depends on the outer normal to M.

Example 4. Induced orientation for boundary of an annular region in $\mathbb{R}^2$.

Theorem (Stokes' theorem on $\mathbb{R}^k$)**.**

Theorem (Stoke's theorem on half-space $\mathbb{R}^{k-1} \times [0, \infty)$)**.**

Theorem (Stokes' Theorem for manifolds with boundary)**.** *Let Ω be a k-dimensional manifold with boundary $\partial\Omega$, and ω a compactly supported $(k-1)$-form on Ω. Then*

$$\int_\Omega d\omega = \int_{\partial\Omega} \omega$$

C.f. 3-dimensional Stokes' theorem and Divergence theorem, 2-dimensional Green's formula, 1-dimensional fundamental theorem of calculus.

Exercise 5. *Show that the vector field $\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ has curl zero, but it is not the gradient of any function.*

1.7. **The degree formula.**

Theorem (Degree formula)**.** *Let $f : X^k \to Y^k$ be a smooth map. Then for any $\omega \in \Omega^k(Y^k)$,*

$$\int_X f^*\omega = (\deg f) \int_Y \omega$$

for some $\deg f \in \mathbb{Z}$ *called the* degree *of f.*

Proved via the following ingredients:

Proposition. *Local degree formula for a regular value of f.*

Theorem. *Homotopic maps $f_0 : X \to Y$ and $f_1 : X \to Y$ give rise to equal integrals $\int_X f_0^*\omega = \int_X f_1^*(\omega)$.*

Proof is by Stokes' theorem.

Theorem. *If $X = \partial W$ and $f : X \to Y$ extends to a smooth map $F : W \to Y$, then $\int_X f^*\omega = 0$ for any $\omega \in \Omega^k(Y)$.*

Proof is by Stokes' theorem again.

Lemma (Isotopy lemma)**.** *If Y is a connected manifold, $y, z \in Y$, then there exists a diffeo $h : Y \to Y$ s.t. $h(y) = z$, h is isotopic to the identity, and h is compactly supported.*

Remark. In the proof the degree seems to depend on the choice of neighborhood of a regular point. The final formula shows that that $\deg f$ is independent of this choice.

2. De Rham theory

Closed forms vs. exact forms.

Exercise 6. *Prove that a 1-form ω on S^1 is the differential of a function $\iff$ $\int_{S^1} \omega = 0$.*

Exercise 7. *Suppose that ω_1 and ω_2 are cohomologous p-forms on X, and Z is a compact oriented p-dimensional submanifold. Prove that $\int_Z \omega_1 = \int_Z \omega_2$. Conclude that integration over Z defines a map of the cohomology group $H^p(X)$ into $\mathbb{R}$, i.e. $\int_Z : H^p(X) \to \mathbb{R}$.*

Definition of the de Rham complex, and of the de Rham cohomology groups H^n_{dR}.

Example 5. $H^*_{DR}(\mathbb{R})$.

Remark: de Rham cohomology is defined for all smooth manifolds, oriented or otherwise.

Proposition. $\dim H^0_{dR}(M) = \#$*components of M.*

Definition 4. *M is* simply connected *if M is connected, and every smooth $f : S^1 \to M$ is homotopic to a constant map $f : S^1 \to \{x\}$, for some $x \in M$.*

If $f : S^1 \to M$ is homotopic to a constant map via a map $F : S^1 \times [0,1] \to M$, where $F|_{t=1} = f$ and $F|_{t=0} = x$, we can identify F with a smooth map $\widetilde{f} : D \to M$ by $\widetilde{f}(re^{i\theta}) = F_r(\theta)$; and vice-versa we can identify an extension $\widetilde{f} : D \to M$ with a homotopy F. In other words, an equivalent definition of simply connected is

Definition 5. *M is* simply connected *if M is connected, and for every smooth $f : S^1 \to M$, there exists a smooth extension $\widetilde{f} : D \to M$ such that $\widetilde{f}|_{\partial D} = f$.*

Proposition. *M simply connected $\implies$ $H^1_{dR}(M) = 0$.*

2.1. **Homological algebra.**

- Definition of a (co)chain complex, differential, (co)homology of the complex. Example: de Rham cohomology $H^*_{dR}(M)$.
- Definition of a chain map, chain map induces well-defined function on cohomology. Example: given map $f : M \to N$ between manifolds, get chain map $f^* : \Omega^*(N) \to \Omega^*(M)$, therefore get maps on cohomology $f^* : H^n(N) \to H^n(M)$ for all n.
- Definition of a null-homotopy for a chain map, null-homotopic chain maps induce the zero map on cohomology. Definition of a chain-homotopy between chain maps; chain homotopic chain maps induce the same map on cohomology.

Theorem. *$H^n_{dR}(M) \cong H^n_{dR}(M \times \mathbb{R})$ for all n.*

Proof. The projection $\pi : M \times \mathbb{R} \to M$ given by $(x,t) \mapsto x$, and the zero-section $\sigma : M \to M \times \mathbb{R}$ given by $x \mapsto (x,0)$ induce chain maps π^* and σ^*. One shows that they induce inverse maps on cohomology by showing that $\pi^* \circ \sigma^*$ and $\sigma^* \circ \pi^*$ are chain homotopic to the identity map. One has $\sigma^* \circ \pi^* = (\pi \circ \sigma)^* = (\mathrm{Id}_M)^* = \mathrm{Id}$ so $\sigma^* \circ \pi^*$ is the identity map already. To show that $\pi^* \circ \sigma^*$ is chain homotopic to the identity one constructions a homotopy operator $K^n : \Omega^n(M \times \mathbb{R}) \to \Omega^{n-1}(M)$ that is the composition of projection onto the subspace of n forms on $M \times \mathbb{R}$ that have a dt in them, followed by integration with respect to t. Compute $\pi^* \circ \sigma^* - Id = \pm dK^n \pm K^{n+1}$, therefore $\pi^* \circ \sigma^*$ induces the identity map on cohomology. ☐

Corollary (Poincaré lemma)**.** $H^n_{dR}(\mathbb{R}^k) \cong H^n_{dR}(\{pt\}) = \begin{cases} \mathbb{R}, & n = 0 \\ 0, & n \neq 0 \end{cases}$

Corollary. *Homotopic manifolds have the same de Rham cohomology (homotopy axiom of de Rham cohomology).*

- Definition of an exact sequence, definition of a short exact sequence of cochain complexes.

Theorem. *A short exact sequence $0 \to B^* \xrightarrow{f} C^* \xrightarrow{g} D^* \to 0$ of cochain complexes induces a long exact sequence*

$$\dots \xrightarrow{g} H^{n-1}(D) \xrightarrow{\partial^*} H^n(B) \xrightarrow{f} H^n(C) \xrightarrow{g} H^n(D) \xrightarrow{\partial^*} H^{n+1}(B) \xrightarrow{f} \dots$$

on cohomology.

Proof of the long exact sequence.

Step 1: Construction of the boundary connecting map $\partial^* : H^n(D) \to H^{n+1}(B)$.

Let $[\omega] \in H^n(D)$, where $\omega \in \ker d_D$. Exactness of the s.e.s. $\implies$ g is surjective, so let $\overline{\omega} \in C^n$ be such that $g(\overline{\omega}) = \omega$. Now consider $d\overline{\omega} \in C^{n+1}$. Then

$$\begin{aligned} g(d(\overline{\omega})) &= d(g(\overline{\omega})) \text{ (since } g \text{ is a chain map so } g \circ d_C = d_D \circ g) \\ &= d\omega \text{ (because by construction } g(\overline{\omega}) = \omega) \\ &= 0 \text{ (because } \omega \in \ker d_D). \end{aligned}$$

This means that $d\overline{\omega}$ is in $\ker g$, and by exactness of the short exact sequence $\ker g = \mathrm{im} f$, therefore there exists some $x \in B^{n+1}$ such that $f(x) = d\overline{\omega}$.

We need to show that $x \in \ker d_B$ so that it determines a cohomology class in $H^{n+1}(B)$: we have $f(d_B x) = d_C f(x) = d_C d_C \overline{\omega} = 0$ since $d_C \circ d_C = 0$, and then $f(d_B(x)) = 0 \implies d_B(x) \in \ker f \underset{\ker f = 0}{\implies} d_B(x) = 0$.

So we define a map $\partial^* : H^n(D) \to H^{n+1}(B)$ by $[\omega] \mapsto [x]$. We need to show that the map is well-defined, i.e., if $[\omega_1] = [\omega_2]$, then $[x_1] = [x_2]$ for any x_1, x_2 obtained from ω_1 and ω_2 as described above.

If $[\omega_1] = [\omega_2]$ then we have $\omega_1 - \omega_2 = d\sigma$ for some $\sigma \in D^{n-1}$. Given $\overline{\omega}_1$ and $\overline{\omega}_2$ such that $g(\overline{\omega}_1) = \omega_1, g(\overline{\omega}_2) = \omega_2$, we have $g(\overline{\omega}_1 - \overline{\omega}_2) = \omega_1 - \omega_2 = d\sigma$. Now by the surjectivity of $g : B^{n-1} \to C^{n-1}$ we can also find a $\overline{\sigma} \in B^{n-1}$

such that $g(\overline{\sigma}) = \sigma \in C^{n-1}$, and so we have that

$$\begin{aligned} g(\overline{\omega}_1 - \overline{\omega}_2 - d_C\overline{\sigma}) &= g(\overline{\omega}_1 - \overline{\omega}_2) - g(d_C\overline{\sigma}) \\ &= d_D\sigma - d_D g(\overline{\sigma}) \\ &= d_D\sigma - d_D\sigma \\ &= 0, \end{aligned}$$

so $\overline{\omega}_1 - \overline{\omega}_2 - d_D\overline{\sigma} \in \ker g = \mathrm{im} f$ (by exactness). So there esists $\xi \in B^n$ such that $f(\xi) = \overline{\omega}_1 - \overline{\omega}_2 - d\overline{\sigma}$, and therefore

$$\begin{aligned} d_C f(\xi) &= d_C(\overline{\omega}_1 - \overline{\omega}_2 - d_C\overline{\sigma}) \\ &= d_C(\overline{\omega}_1) - d_C(\overline{\omega}_2). \end{aligned}$$

Since f is injective there are unique x_1, x_2 such that $f(x_i) = d_C\overline{\omega}_i$. Then finally we have

$$\begin{aligned} f(d_B(\xi)) &= d_C(f(\xi)) \text{ because } f \text{ is a chain map} \\ &= d_C\overline{\omega}_1) - d_C(\overline{\omega}_2) \\ &= f(x_1) - f(x_2) \\ &= f(x_1 - x_2) \text{ by linearity of } f. \end{aligned}$$

So moving the right hand side to the left and using linearity of f, we get

$$f(d_B(\xi) - (x_1 - x_2)) = 0$$

and now, using the fact that $\ker f = 0$, we can say $d_B(\xi) - (x_1 - x_2) = 0$, i.e. $[x_1] = [x_2]$ in $H^{n+1}(B)$.
Step 2: Exactness of the long exact sequence

$$\ldots \longrightarrow H^{n-1}(D) \xrightarrow{\partial^*} H^n(B) \xrightarrow{f} H^n(C) \xrightarrow{g} H^n(D) \xrightarrow{\partial^*} H^{n+1}(B) \longrightarrow \ldots$$

- Exactness at $H^n(B)$: we need to show $\ker f = \mathrm{im}\partial^*$.

  ($\subseteq$) Let $[\omega] \in \ker f$. So $f([\omega]) = [f(\omega)] = 0$. This means that $f(\omega) = d\alpha$ for some $\alpha \in C^{n-1}$. Now consider $g(\alpha) \in D^{n-1}$. We have $d_D(g(\alpha)) = g(d\alpha) = g(f(\omega)) = 0$ since $f(\omega) \in \ker g$ by exactness. Therefore, $g(\alpha) \in \ker d_D$, so $[g(\alpha)] \in H^{n-1}(D)$ and so by our definition of ∂^*, we have $\partial^*[g(\alpha)] = [\omega]$, so $[\omega] \in \mathrm{im}\partial^*$.

  ($\supseteq$) Let $[\omega] \in \mathrm{im}\partial^*$. Remember this means that there's some $\alpha \in C^{n-1}$ such that $\partial^*[g(\alpha)] = [\omega]$, and such that $d\alpha = f(\omega)$. But this being the case, we have $f([\omega]) = [f(\omega)] = [d\alpha] = 0$ in cohomology, so $[\omega] \in \ker f$.
- Exactness at $H^n(C)$: we need to show $\ker g = \mathrm{im} f$.

  ($\subseteq$) Let $[\omega] \in \ker g$, where $\omega \in \ker d_C$. Then $g[\omega] = [g(\omega)] = 0$, so $g(\omega) = d\alpha$ for some $\alpha \in D^{n-1}$. By the surjectivity of $g : C^{n-1} \to D^{n-1}$ let $\overline{\alpha} \in C^{n-1}$ be such that $g(\overline{\alpha}) = \alpha$. Then we have

  $$\begin{aligned} g(d\overline{\alpha} - \omega) &= g(d(\overline{\alpha})) - g(\omega) \\ &= d(g(\overline{\alpha})) - g(\omega) \\ &= d\alpha - g(\omega) \\ &= d\alpha - d\alpha \\ &= 0 \end{aligned}$$

  and therefore $d\overline{\alpha} - \omega \in \ker g = \mathrm{im} f$ so there exists some $x \in B^n$ such that $f(x) = d\overline{\alpha} - \omega$. Moreover $f(d_B(x)) = d_C(f(x)) = d_C(d_C\overline{\alpha} - \omega) = d_C(d_C\overline{\alpha}) - d_C\omega = 0$, so $d_C x \in \ker f$. Since $\ker f = 0$ this means $d_C x = 0$, so $x \in \ker d_C$ and therefore $[x]$ is a cohomology class in $H^n(C)$. So passing to cohomology we get that $f([x]) = [f(x)] = [d\overline{\alpha} - \omega] = [-\omega] = -[\omega]$. Therefore $f([-x]) = [\omega]$ so $[\omega] \in \mathrm{im} f$.

  ($\supseteq$) Let $[\alpha] \in \mathrm{im} f$, i.e. $[\alpha] = [f(\omega)]$ for some $\omega \in B^n \cap \ker d_B$. Then $g([\alpha]) = g([f(\omega)]) = [g(f(\omega))] = [0]$ as $\mathrm{im} f = \ker g$. Therefore $[\alpha] \in \ker g$.
- Exactness at $H^n(D)$: we need to show $\ker \partial^* = \mathrm{im} g$.

  ($\subseteq$) Let $[\omega] \in \ker \partial^*$, i.e. $\partial^*[\omega] = 0$ where $\omega \in \ker d_D$. Remember that the construction sets

$\partial^*[\omega] = [x]$ where $x \in B^n \cap \ker d_B$ satisfies $f(x) = d\overline{\omega}$ and $\overline{\omega}$ is some element of C^n such that $g(\overline{\omega}) = \omega$. So if $\partial^*[\omega] = 0$ it means that $[x] = 0$ so $x = d_B\alpha$ for some $\alpha \in B^n$. We have

$$\begin{aligned} d_C\overline{\omega} &= f(x) \text{ by construction} \\ &= f(d_B(\alpha)) \\ &= d_C(f(\alpha)) \end{aligned}$$

and therefore $d_C(\overline{\omega} - f(\alpha)) \in \ker d_C$ so determines a cohomology class in $H^n(C)$, and now it follows that

$$\begin{aligned} g([\overline{\omega} - f(\alpha)]) &= [g(\overline{\omega}) - \underbrace{g(f(\alpha))}_{=0}] \\ &= [g(\overline{\omega})] \\ &= [\omega] \end{aligned}$$

and therefore $[\omega] \in \operatorname{img}$.

($\supseteq$) Let $[\omega] \in \operatorname{img}$, so there exists some $\overline{\omega} \in \ker d_C$ such that $g[\overline{\omega}] = [g(\overline{\omega})] = [\omega]$. The definition of $\partial^*[\omega]$ is $[x]$ where $x \in B^{n+1}$ satisfies $f(x) = d_C\overline{\omega}$. However by supposition we have $d_C\overline{\omega} = 0$ and therefore $f(x) = 0$, so by the injectivity of f we have $x = 0$, so $[x] = 0$, and therefore $\partial^*[\omega] = 0$ so $[\omega] \in \ker \partial^*$.

□

Example 6. Mayer-Vietoris exact sequence. We have natural inclusion maps $U \cap V \overset{i_U}{\hookrightarrow} U, U \cap V \overset{i_V}{\hookrightarrow} V, U \overset{j_U}{\hookrightarrow} U \cup V, V \overset{j_V}{\hookrightarrow} U \cup V$, and their associated pull-back maps (which reverse the direction of the arrows!) give a sequence of maps

$$\Omega^n(U \cup V) \overset{(j_U^*, j_V^*)}{\longrightarrow} \Omega^n(U) \oplus \Omega^n(V) \overset{i_V^* - i_U^*}{\longrightarrow} \Omega^n(U \cap V)$$

Practically, the maps $j_U^*, j_V^*, i_V^*, i_U^*$ correspond to "restriction of n-form to an open subset". For instance, for $\omega \in \Omega^n(U \cup V)$ an n-form on $U \cup V$, the n-form $j_U^*(\omega)$ on U really means the n-form $\omega|_U$.

Proposition. *The Mayer-Vietoris sequence*

$$0 \longrightarrow \Omega^n(U \cup V) \overset{(j_U^*, j_V^*)}{\longrightarrow} \Omega^n(U) \oplus \Omega^n(V) \overset{i_U^* - i_V^*}{\longrightarrow} \Omega^n(U \cap V) \longrightarrow 0$$

is a short exact sequence.

Proof. Need to prove:

- : (j_U^*, j_V^*) is injective: if $\omega \in \Omega^n(U \cup V)$ gets mapped to 0 then $\omega|_U = 0$ and $\omega|_V = 0 \implies \omega = \omega|_{U \cup V} = 0$.
- $\ker(i_U^* - i_V^*) = \operatorname{im}(j_U^*, j_V^*)$: let $(\sigma, \tau) \in \Omega^n(U) \oplus \Omega^n(V)$, then
$$(i_U^* - i_V^*)(\sigma, \tau) = 0 \quad \iff \quad \sigma|_{U \cap V} - \tau|_{U \cap V}$$
which is equivalent to σ and τ being restrictions of a single form ω on $U \cup V$, and so $(\sigma, \tau) = (j_U^*, j_V^*)(\omega)$.
- $i_V^* - i_U^*$ is surjective. Let $\sigma \in \Omega^n(U \cap V)$. Take a partition of unity $\rho_U + \rho_V = 1$ such that the support of ρ_U is in U, and the support of ρ_V is in V. Then, $\rho_U\sigma$ is an n-form on V and $\rho_V\sigma$ is an n-form on U, and $(i_U^* - i_V^*)(\rho_V\sigma, -\rho_U\sigma) = \rho_V\sigma + \rho_U\sigma = \sigma$.

□

Corollary. *There is a long exact sequence of cohomology groups*

$$\ldots \longrightarrow H^n_{dR}(U \cup V) \longrightarrow H^n_{dR}(U) \oplus H^n_{dR}(V) \longrightarrow H^n_{dR}(U \cap V) \longrightarrow H^{n+1}_{dR}(U \cup V) \longrightarrow \ldots$$

This l.e.s. is called the Mayer-Vietoris long exact sequence.

Example 7. Computing the cohomology of S^1; cover S^1 with two open sets $U \cup V$ whose intersection is a pair of disjoint intervals, $U \cap V = I_1 \sqcup I_2$. In particular we have $\mathbb{R} \cong U \cong V \cong I_1 \cong I_2$. Note that we also know that $\dim H^0_{dR}(M) =$ number of connected components of M.

Example 8. cohomology of S^2: cover S^2 with a pair of open disks $U \cup V$ whose intersection is diffeomorphic to a cylinder $S^1 \times \mathbb{R}$.

2.2. **Finite good covers and finite dimensionality of de Rham cohomology.**

Definition 6. *An open cover* $\{(U_\alpha, \phi_\alpha), \alpha \in \mathcal{A}\}$ *is called a* good cover *if all non-empty intersections* $\bigcap_{i=1}^{n} U_{\alpha_i}$ *are diffeomorphic to* $\mathbb{R}^k$.

Example 9. Good cover for S^1, good cover for S^2.

Theorem. *Every manifold has a good cover.*

Proof. (Have they seen before how a Riemannian metric can be put on any smooth manifold?) Put a Riemannian metric on the manifold, and then use geodesically convex neighborhoods of each point as the open sets in the cover. A geodesically convex neighborhood is automatically diffeomorphic to $\mathbb{R}^k$. And the intersection of geodesically convex neighborhoods is again a geodesically convex open set, so they make a good cover. □

Theorem. *A manifold M with a finite good cover has finite dimensional de Rham cohomology.*

Proof. Fix a dimension $k \geq 0$. We will prove that all k-dimensional manifolds with a finite good cover have finite dimensional de Rham cohomology by induction on the cardinality n of the cover.

Base step: If $n = 1$ then $M \cong \mathbb{R}^k$. We know from the Poincaré Lemma that $H^0_{dR}(\mathbb{R}^k) = \mathbb{R}$ and $H^0_{dR}(\mathbb{R}^k) = 0$ for $n \neq 0$. So M has finite dimensional de Rham cohomology.

Inductive hypothesis: suppose that every manifold with a good cover by n open sets has finite dimensional de Rham cohomology.

Inductive step: Suppose that M is a manifold with a good cover by $n + 1$ open sets $U_1, \dots, U_n, U_{n+1}$. Then we may write $M = U_{n+1} \cup \overline{M}$ where $\overline{M} = \bigcup_{i=1}^{n} U_i$. Thus $\overline{M}$ is a manifold of dimension k with a finite good cover by n open sets, so by inductive hypothesis $\overline{M}$ has finite dimensional de Rham cohomology. And $U_{n+1} \cong \mathbb{R}^k$ also has finite dimensional de Rham cohomology. Also, $U_{n+1} \cap \overline{M} = \bigcup_{i=1}^{n} (U_{n+1} \cap U_i)$ so it also has a good cover of cardinality n, so by inductive hypothesis $U_{n+1} \cap \overline{M}$ has finite dimensional de Rham cohomology.

Now by the Mayer-Vietoris long exact sequence, we can write

$$\longrightarrow H^{n-1}_{dR}(\overline{M} \cap U_{n+1}) \xrightarrow{\alpha} H^n_{dR}(M) \xrightarrow{\beta} H^n_{dR}(\overline{M}) \oplus H^n_{dR}(U_{n+1}) \xrightarrow{\gamma}$$

By exactness we have that $H^n_{dR}(M)/\operatorname{im}\ \alpha \cong \operatorname{im}\ \beta = \ker\gamma$, so that $\dim H^n_{dR}(M) = \dim\ker\gamma + \dim\operatorname{im}\ \alpha < \infty$. Therefore M also has finite dimensional de Rham cohomology.

□

2.3. **Integer invariants.**

Definition 7. *Betti numbers:* $B_i(M) := \dim H^i_{dR}(M)$.

Definition 8. *Euler characteristic:* $\chi(M) := \sum_{i=0}^{k}(-1)^i B_i(M)$

Example 10.
- $\chi(S^2) = 1 - 0 + 1 = 2$
- $\chi(T^2) = 1 - 2 + 1 = 0$
- $\chi(\mathbb{R} \times S^1) = \chi(S^1) = \chi(\text{Möbius strip}) = 0$.

2.4. **The five lemma.**

Lemma. *Let*

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "\alpha"] & B \arrow[r, "f_2"] \arrow[d, "\beta"] & C \arrow[r, "f_3"] \arrow[d, "\gamma"] & D \arrow[r, "f_4"] \arrow[d, "\delta"] & E \arrow[d, "\epsilon"] \\
A' \arrow[r, "g_1"] & B' \arrow[r, "g_2"] & C' \arrow[r, "g_3"] & D' \arrow[r, "g_4"] & E'
\end{tikzcd}$$

be a commutative diagram of abelian groups and homomorphisms, in which the rows are exact. If the four outer maps α, β, δ and ϵ are isomorphisms, then so is γ.

Proof. We need to show that γ is injective and surjective.

Let $c \in \ker\gamma$, so $\gamma(c) = 0$. Then $0 = g_3 \circ \gamma(c) = \delta \circ f_3(c)$, so $f_3(c) \in \ker\delta$, but by hypothesis δ is an isomorphism, so $\ker\delta = 0$; therefore $f_3(c) = 0$, so $c \in \ker f_3$. By the exactness of the rows, this means $c \in \operatorname{im} f_2$, so there exists a $b \in B$ such that $f_2(b) = c$. And $0 = \gamma(c) = \gamma \circ f_2(b) = g_2 \circ \beta(b)$ therefore $\beta(b) \in \ker g_2 = \operatorname{im} g_1$ so $\beta(b) = g_1(a')$ for some $a' \in A'$. Since α is an isomorphism, put $a = \alpha^{-1}(a') \in A$. Then $\beta \circ f_1(a) = g_1 \circ \alpha(a) = g_1(a') = \beta(b)$, therefore since β is an isomorphism, $b = f_1(a)$. But this means that $c = f_2(b) = f_2 \circ f_1(a) = 0$ since by exactness $f_2 \circ f_1 = 0$. Hence, $\ker\gamma = 0$ so γ is injective.

Let $c' \in C'$. Let $d' = g_3(c')$. Then $d' \in \ker g_4$. Using the fact that δ is an isomorphism, let $d = \delta^{-1}(d')$. Then $0 = g_4 \circ \delta(d) = \epsilon \circ f_4(d)$ implies $f_4(d) \in \ker\epsilon = 0$, so $f_4(d) = 0$ so $d \in \ker f_4$. By exactness then this means there exists a $c \in C$ such that $d = f_3(c)$. Now, $g_3 \circ \gamma(c) = \delta \circ f_3(c) = \delta(d) = d' = g_3(c') \implies g_3(\gamma(c) - c') = 0 \implies \gamma(c) - c' \in \ker g_3$. Thus by exactness there exists a $b' \in B'$ such that $g_2(b') = \gamma(c) - c'$, and putting $b = \beta^{-1}(b')$ we have $g_2 \circ \beta(b) = \gamma(c) - c'$. Now by commutativity we have $g_2 \circ \beta(b) = \gamma \circ f_2(b)$, so altogether we have $\gamma \circ f_2(b) = \gamma(c) - c'$, and therefore $c' = \gamma(c - f_2(b))$. Thus $c' \in \operatorname{im}\gamma$, so γ is surjective. $\square$

Remark. For finite dimensional vector spaces over $\mathbb{R}$ it's enough to prove injectivity of γ, as surjectivity follows from knowing that $\dim C = \dim C'$.

2.5. **Künneth formula.**

Definition 9. *Tensor product of vector spaces.*

Theorem (Kunneth formula)**.** *If M and N have finite good covers, then $H^*_{dR}(M \times N) = H^*_{dR}(M) \otimes H^*_{dR}(N)$.*

Proof. We have the maps $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$, which induce a map

$$\begin{aligned} \varphi : H^*_{dR}(M) \otimes H^*_{dR}(N) &\to H^*(M \times N) \\ ([\omega], [\tau]) &\mapsto [\pi_M^*(\omega) \wedge \pi_N^*(\tau)] \end{aligned}$$

The statement of the Kunneth formula is that this map φ is an isomorphism. The proof is by induction on the finite cardinality n of a good cover for M.

Base step: If $n = 1$, then $M = \mathbb{R}^k$ and the isomorphism follows from the Poincaré lemma.

Inductive hypothesis: Suppose it holds whenever M has a good cover with cardinality n.

Inductive step: Suppose that M has a good cover of cardinality $n+1$. Then we can write $M = U_{n+1} \cup \overline{M}$ where $\overline{M} = \bigcup_{i=1}^{n} U_i$, and $\overline{M} \cap U_{n+1}$ also has a finite good cover of cardinality n. The Mayer-Vietoris sequence is the long exact sequence

$$\longrightarrow H^p(U_{n+1}) \oplus H^p(\overline{M}) \to H^p(\overline{M} \cap U_{n+1}) \longrightarrow H^{p+1}(M) \longrightarrow H^{p+1}(U_{n+1}) \oplus H^{p+1}(\overline{M}) \to H^{p+1}(\overline{M}) \to H^{p+1}(\overline{M} \cap U_{n+1})$$

The exactness is preserved if we tensor everything by $H^{m-p}(N)$ and take the sum over all integers p, then we get the long exact sequence

$$\begin{tikzcd}
{[H^*(M) \otimes H^*(N)]^{m+1}} \arrow[r] & \dots & \\
{[H^*(M) \otimes H^*(N)]^{m}} \arrow[r] & {\left[H^*(\overline{M}) \otimes H^*(N)\right]^m \oplus [H^*(U_{n+1}) \otimes H^*(N)]^m} \arrow[r] & {\left[H^*(\overline{M} \cap U_{n+1}) \otimes H^*(N)\right]^m} \arrow[ull] \\
 & \dots \arrow[r] & {\left[H^*(\overline{M} \cap U_{n+1}) \otimes H^*(N)\right]^{m-1}} \arrow[ull]
\end{tikzcd}$$

We also have the Mayer-Vietoris sequence

$$\dots \longrightarrow H^m(M \times N) \longrightarrow H^m(\overline{M} \times N) \oplus H^m(U_{n+1} \times N) \longrightarrow H^m((\overline{M} \cap U_{n+1}) \times N) \longrightarrow H^{m+1}(M \times N) \longrightarrow \dots$$

Now we include the maps between the corresponding terms of the Mayer-Vietoris sequence in each of these two long exact sequences defined as in the map φ above,

$$\begin{tikzcd}
\dots \arrow[r] & {[H^*(M)\otimes H^*(N)]^m} \arrow[r] \arrow[d, "\varphi_M"] & \left[H^*(\overline{M})\otimes H^*(N)\right]^m \oplus [H^*(U_{N+1})\otimes H^*(N)]^m \arrow[r] \arrow[d, "\varphi_{\overline{M}}\oplus\varphi_{U_{n+1}}"] & \dots \\
\dots \arrow[r] & H^m(M\times N) \arrow[r] & H^m(\overline{M}\times N)\oplus H^m(U_{n+1}\times N) \arrow[r] & \dots
\end{tikzcd}$$

Claim: the diagram commutes. Once we show that the diagram commutes, it follows from the inductive hypothesis that the vertical arrows are all isomorphisms except for the vertical arrows φ_M, but then it follows by the 5 lemma that this map too is an isomorphism.

So let us check that the diagram commutes. There are three squares to check. Two squares are very straightforward (the square pictured above and the square to its right).

Exercise 8. *Check that the square pictured above and the square to its right are commutative.*

Let us check the commutativity of the third square, which involves the connecting maps ∂^* in the Mayer-Vietoris long exact sequence:

$$\begin{tikzcd}
{[H^*(\overline{M}\cap U)\otimes H^*(N)]^m} \arrow[r, "\partial^*\otimes\mathrm{Id}"] \arrow[d] & {[H^*(M)\otimes H^*(N)]^{m+1}} \arrow[d] \\
H^m((\overline{M}\cap U)\times N) \arrow[r, "\partial^*"] & H^{m+1}(M\times N)
\end{tikzcd}$$

where the vertical arrows correspond to the maps $\sum\limits_{\alpha}[\omega_\alpha]\otimes[\tau_\alpha]\mapsto\sum\limits_{\alpha}[\omega_\alpha]\wedge[\tau_\alpha]$.

By linearity it's enough to show that the square commutes for individual elements of the form $[\omega]\otimes[\tau]$ with $[\omega]\in H^p(\overline{M}\cap U)$ and $[\tau]\in H^q(N)$ with $p+q=m$.

Then we have, following the top horizontal arrow,

$$\partial^*\otimes\mathrm{Id}([\omega]\otimes[\tau])=[x]\times[\tau]$$

where $x\in\Omega^{m+1}(M)$ is a closed form such that $(x|_{\overline{M}},x|_U)=(d\alpha,d\beta)$ for some $(\alpha,\beta)\in\Omega^p(\overline{M})\oplus\Omega^p(U)$ such that $\omega=\alpha|_{\overline{M}\cap U}-\beta|_{\overline{M}\cap U}$.

So then, the image of $[\omega]\otimes[\tau]$ in the bottom right square is $[x]\wedge[\tau]$.

For the other side of the square, we need to compute $\partial^*([\omega]\wedge[\tau])=\partial^*[\omega\wedge\tau]$. By definition of the connecting map, $\partial^*[\omega\wedge\tau]=[y]$ where $y\in\Omega^{m+1}(M\times N)$ is any form for which $(y|_{\overline{M}\times N},y|_{\overline{U}\times N})=(d\gamma,d\epsilon)$ for some $(\gamma,\epsilon)\in H^m(\overline{M}\times N)\oplus H^m(U\times N)$ such that $(\gamma-\epsilon)|_{(\overline{M}\cap U)\times N}=\omega\wedge\tau$. So it is enough to check that $y=x\wedge\tau$ satisfies these conditions. We have

$$\begin{aligned}
(x\wedge\tau)|_{\overline{M}\times N} &= x|_{\overline{M}}\wedge\tau|_N\\
&= (d\alpha)\wedge\tau\\
&= d(\underbrace{\alpha\wedge\tau}_{\gamma})
\end{aligned}$$

where the last equality is because $d(\alpha\wedge\tau)=d\alpha\wedge\tau+\alpha\wedge\underbrace{d\tau}_{=0}=d\alpha\wedge\tau$. Similarly

$$\begin{aligned}
(x\wedge\tau)|_{\overline{M}\times N} &= x|_U\wedge\tau|_N\\
&= (d\beta)\wedge\tau\\
&= d(\underbrace{\beta\wedge\tau}_{\epsilon})
\end{aligned}$$

and finally,

$$\begin{aligned}
(\gamma-\epsilon)|_{(\overline{M}\cap U)\times N} &= (\alpha\wedge\tau-\beta\wedge\tau)|_{(\overline{M}\cap U)}\times N\\
&= (\alpha-\beta)|_{(\overline{M}\cap U)}\wedge\tau\\
&= \omega\wedge\tau,
\end{aligned}$$

as desired.

□

2.6. **Poincaré duality.** Let M be an oriented and compact manifold of dimension k. In particular $\partial M = \emptyset$. Then we can integrate k-forms on M. And integration is well-defined up to exact forms, for if $\omega_1 - \omega_2 = d\sigma$, then by Stokes' theorem

$$\int_M \omega_1 - \int_M \omega_2 = \int_M d\sigma = \int_{\partial M} \sigma = 0.$$

Now we can also take a p form and a $k-p$ form and wedge them to get a k-form, and can integrate that. Suppose that ω_1 is a closed p-form and ω_2 is a closed $k-p$ form. Then

$$\begin{aligned}(\omega_1 + d\sigma_1) \wedge (\omega_2 + d\sigma_2) &= \omega_1 \wedge \omega_2 + \underbrace{d\sigma_1 \wedge \omega_2}_{=d(\sigma_1 \wedge \omega_2)} + \underbrace{\omega_1 \wedge d\sigma_2}_{=\pm d(\omega_1 \wedge \sigma_2)} + \underbrace{d\sigma_1 \wedge d\sigma_2}_{=d(\sigma_1 \wedge d\sigma_2)} \\ &= \omega_1 \wedge \omega_2 + d(\sigma_1 \wedge \omega_2 \pm \omega_1 \wedge \sigma_2 + \sigma_1 \wedge d\sigma_2)\end{aligned}$$

where we have used the Leibniz rule for d. Putting everything together, we get that integration produces a well-defined bilinear map

$$\begin{aligned}\int_M : H^p_{dR}(M) \times H^{k-p}_{dR}(M) &\longrightarrow \mathbb{R}, \\ ([\omega_1], [\omega_2]) &\mapsto \int_M \omega_1 \wedge \omega_2.\end{aligned}$$

Theorem (Poincaré duality)**.** *Suppose that M is an oriented, compact k dimensional manifold. Then the bilinear pairing above is* non-degenerate. *Consequently, $H^p_{dR}(M) \cong H^{k-p}_{dR}(M)$ for all $p = 0, \ldots, k$.*

This theorem is actually a special case of Poincaré duality; there is also a version of Poincaré duality for oriented M with a finite good cover (i.e. M is not necessarily compact). Suppose that M is only an oriented k dimensional manifold, i.e. not necessarily compact. Then we can only integrate compactly supported k forms on M. If we wedge any p form with a *compactly supported* $k-p$ form, we get a compactly supported k form that we can integrate. The more general statement of Poincaré duality is that there is an isomorphism between $H^p_{dR}(M)$ and $H^{k-p}_c(M)$ which is the *compactly supported cohomology* of M.

Theorem. *Suppose that M is an oriented k dimensional manifold, with a finite good cover. Then the bilinear pairing*

$$\begin{aligned}\int_M : H^p_{dR}(M) \times H^{k-p}_c(M) &\longrightarrow \mathbb{R}, \\ ([\omega_1], [\omega_2]) &\mapsto \int_M \omega_1 \wedge \omega_2\end{aligned}$$

is non-degenerate, and $H^p_{dR}(M) \cong H^{k-p}_c(M)$.

So before proving Poincaré duality we need to define and describe compactly supported cohomology.

2.7. **Compactly supported cohomology $H^*_c(M)$.** Let M be a k-dimensional manifold. The space of *compactly supported* n-forms on M is the subspace of $\Omega^n_c(M) \subset \Omega^n(M)$

$$\Omega^n_c(M) = \{\omega \in \Omega^n_{dR}(M) \big| \ \omega \text{ has compact support in } M\}.$$

The associated cochain complex is then

$$0 \longrightarrow \Omega^0_c(M) \xrightarrow{d} \Omega^1_c(M) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^{k-1}_c(M) \xrightarrow{d} \Omega^k_c(M) \xrightarrow{d} 0$$

where $d : \Omega^n_c(M) \to \Omega^{n+1}_c(M)$ is the usual exterior derivative on forms. Compactly supported cohomology is then the cohomology of this chain complex, $H^n_c(M) = \ker d / \operatorname{im} d$.

Example 11. $H^*_c(\mathbb{R})$: $H^0_c(\mathbb{R}) = 0, H^1_c(\mathbb{R}) = \mathbb{R}$.

Proposition. $\dim H^0_c(M) = \#$ *compact components of M. In particular, if M is connected and non-compact, then $H^0_c(M) = 0$.*

Proof. By definition $H^0_c(M) = \ker d = \{f : M \to \mathbb{R} \big| f \text{ has compact support in } M \text{ and } df = 0\}$, i.e. the locally constant functions with compact support. For each compact component of M there is a one-dimensional vector space of constant functions with compact support, and for each non-compact component of M the only constant function with compact support is $\{0\}$. □

There is a version of the Poincaré lemma for compactly supported cohomology – but note the shift in degree.

Theorem. $H_c^{n+1}(M\times\mathbb{R})\cong H_c^n(M)$

Proof. Follow same lines of proof as the analogous theorem for de Rham cohomology. Define

$$\begin{aligned}\alpha:\Omega_c^{n+1}(M\times\mathbb{R}) &\to \Omega_c^n(M)\\ \sum_{|I|=n} f_I(\mathbf{x},t)dt\wedge dx_I+\sum_{|J|=n+1} g_J(\mathbf{x},t)dx_J &\mapsto \sum_{|I|=n}(\int_{\mathbb{R}} f_I(\mathbf{x},t)dt)dx_J\end{aligned}$$

Now we define a map in the opposite direction. Let $s:\mathbb{R}\to\mathbb{R}$ be any smooth bump function with compact support such that $\int\limits_{\mathbb{R}} s(t)dt=1$. Then we set

$$\begin{aligned}\beta:\Omega_c^n(M) &\to \Omega_c^{n+1}(M\times\mathbb{R})\\ \sum_{|J|=n} h_J(\mathbf{x})dx_J &\mapsto \sum_{|J|=n} h_J(\mathbf{x})s(t)dt\wedge dx_J\end{aligned}$$

Note that $\alpha\circ\beta=Id$, so $\alpha\circ\beta:H^n(M)\to H^n(M)$ is the identity map. It remains to show that $\beta\circ\alpha$ is chain homotopic to the identity. One defines a chain homotopy map

$$\begin{aligned}K^n:\Omega_c^n(M\times\mathbb{R}) &\longrightarrow \Omega_c^{n-1}(M\times\mathbb{R})\\ \sum_{|I|=n} f_I(\mathbf{x},t)dt\wedge dx_I+\sum_{|J|=n+1} g_J(\mathbf{x},t)dx_J &\mapsto \sum_{|I|=n}\left[\int_{-\infty}^t f_I(\mathbf{x},\tau)-s(\tau)\left(\int_{\mathbb{R}} f_I(\mathbf{x},\widetilde{\tau})d\widetilde{\tau}\right)d\tau\right]dx_I\end{aligned}$$

Exercise 9. *Verify that*

(1) $\int_{-\infty}^t f_I(\mathbf{x},\tau)-s(\tau)\left(\int_{\mathbb{R}} f_I(\mathbf{x},\widetilde{\tau})d\widetilde{\tau}\right)d\tau$ *really is a compactly supported function in the variables* $\mathbf{x},t$.

(2) $\beta\circ\alpha-\mathrm{Id}=\pm dK^n\pm K^{n+1}d$.

From 2. it follows that $\beta\circ\alpha$ is equal to the identity map on cohomology, and hence β and α are isomorphisms on cohomology.

□

Corollary (Poincaré Lemma for compactly supported cohomology)**.** $H_c^{*+k}(\mathbb{R}^k)\cong H_c^*(pt)$. $H_c^k(\mathbb{R}^k)\cong\mathbb{R}$, $H_c^n(\mathbb{R}^k)=0$ *for* $n\neq k$.

Note that this means that compactly supported cohomology is *not* invariant under homotopy.

2.8. **Mayer-Vietoris for compactly supported cohomology.** Let ω be a differential form with compact support inside an open set U. Then for any other open set V containing U, we can identify ω with a compactly supported differential form on V just by extending ω by zero to all of V. Thus we have an inclusion map

$$\Omega_c^p(U)\hookrightarrow\Omega^p(V).$$

Applying this to the Mayer-Vietoris sequence of open sets, we get a sequence

$$\Omega_c^n(U\cap V)\overset{(i_U,i_V)}{\hookrightarrow}\Omega_c^n(U)\oplus\Omega_c^n(V)\overset{j_V-j_U}{\hookrightarrow}\Omega_c^n(U\cup V)$$

Proposition (Mayer-Vietoris s.e.s. for compactly supported cohomology)**.** *The Mayer-Vietoris sequence*

$$0\longrightarrow\Omega_c^n(U\cap V)\overset{(i_U,i_V)}{\hookrightarrow}\Omega_c^n(U)\oplus\Omega_c^n(V)\overset{j_V-j_U}{\hookrightarrow}\Omega_c^n(U\cup V)\longrightarrow 0$$

is exact.

Proof.

- (i_U,i_V) is injective: if $i_U(\omega)=0$ and $i_V(\omega)=0$ then $\omega\equiv 0$ on $U\cap V$.
- $\mathrm{im}(i_U,i_V)=\ker(j_V-j_U)$: We have $(j_V-j_U)(\sigma,\tau)=0$ iff the support of both σ and τ is in $U\cap V$ and $\sigma=\tau$, i.e. $(i_U,i_V)(\sigma)=(\sigma,\tau)$.
- (j_V-j_U) is surjective: Let $\omega\in\Omega_c^n(U\cup V)$, and take a partition of unity $\rho_U+\rho_V=1$ where ρ_U is supported on U and ρ_V is supported on V. Then $\rho_U\omega$ is a compactly supported n-form with support contained in U, and $\rho_V\omega$ is a compactly supported n-form with support contained in V. Then we have $\omega=\rho_U\omega+\rho_V\omega=(j_V-j_U)(-\rho_U\omega,\rho_V\omega)$.

□

Corollary. *There is a Mayer-Vietoris l.e.s. for compactly supported cohomology,*

$$\ldots\longrightarrow H_c^{n-1}(U\cup V)\longrightarrow H_c^n(U\cap V)\longrightarrow H_c^n(U)\oplus H_c^n(V)\longrightarrow H_c^n(U\cup V)\longrightarrow H^{n+1}(U\cap V)\longrightarrow\ldots$$

2.9. **Proof of Poincaré duality.**

Proof. The idea is to do induction on the cardinality of a finite good cover for M.

Exercise 10. *Verify the base case, $M = \mathbb{R}^k$.*

For the inductive step, we start out with the two Mayer-Vietoris sequences

$$\ldots \longrightarrow H^p_{dR}(M) \longrightarrow H^p_{dR}(\overline{M}) \oplus H^p_{dR}(U) \longrightarrow H^p_{dR}(\overline{M} \cap U) \longrightarrow H^{p+1}_{dR}(M) \longrightarrow \ldots$$

and

$$\ldots \longrightarrow H^{p-1}_c(M) \longrightarrow H^p_c(\overline{M} \cap U) \longrightarrow H^p_c(\overline{M}) \oplus H^p_c(U) \longrightarrow H^p_c(M) \longrightarrow H^{p+1}_c(U \cap V) \longrightarrow \ldots$$

Taking duals of the second sequence flips the direction of all the arrows and gives a l.e.s.

$$\ldots \longrightarrow H^p_c(M)^* \longrightarrow H^p_c(\overline{M})^* \oplus H^p_c(U)^* \longrightarrow H^p_c(\overline{M} \cap U)^* \longrightarrow H^{p-1}_c(M)^* \longrightarrow \ldots$$

Now form the double complex

$$\begin{tikzcd}
\ldots H^p_{dR}(M) \arrow[r] \arrow[d, "\varphi_M"] & H^p_{dR}(\overline{M}) \oplus H^p_{dR}(U) \arrow[r] \arrow[d, "\varphi_{\overline{M}} \oplus \varphi_{\overline{U}}"] & H^p_{dR}(\overline{M} \cap U) \arrow[r] \arrow[d, "\varphi_{\overline{M} \cap U}"] & H^{p+1}_{dR}(M) \ldots \arrow[d, "\varphi_M"] \\
\ldots H^{k-p}_c(M)^* \arrow[r] & H^{k-p}_c(\overline{M})^* \oplus H^{k-p}_c(U)^* \arrow[r] & H^{k-p}_c(\overline{M} \cap U)^* \arrow[r] & H^{k-p-1}_c(M)^* \ldots
\end{tikzcd}$$

where the vertical maps are those given by $\omega \mapsto \int_M \omega \wedge \cdot$. If we know that the diagram commutes, then the inductive hypothesis and the 5 lemma show that the maps φ_M are isomorphisms, completing the induction. So we just need to check that each of the squares in the diagram above commutes. (All squares correspond to one of the three above, for different values of p.)

Let's check the commutativity of the last square,

$$\begin{tikzcd}
H^p_{dR}(\overline{M} \cap U) \arrow[r, "\partial^*"] \arrow[d, "\varphi_{\overline{M} \cap U}"] & H^{p+1}_{dR}(M) \arrow[d, "\varphi_M"] \\
H^{k-p}_c(\overline{M} \cap U)^* \arrow[r, "\widehat{\partial}"] & H^{k-p-1}_c(M)^*
\end{tikzcd}$$

by working the maps out out explicitly.

Let $[\omega] \in H^p_{dR}(\overline{M} \cap U_{n+1})$ for some closed form $\omega \in \Omega^p(\overline{M} \cap U_{n+1})$. In the definition of the connecting map ∂^* in the long exact sequence, $\partial^*[\omega] = [x]$ where x is a closed form such that $x|_{\overline{M}} = d\alpha, x|_U = d\beta$ for $\alpha \in \Omega^p(\overline{M})$ and $\beta \in \Omega^p(U)$ such that $(\alpha - \beta)|_{\overline{M} \cap U} = \omega$. Hence, $\varphi_M \partial^*[\omega]\,([\eta]) := \int_M x \wedge \eta$ for $[\eta] \in H^{k-p-1}_c(M)$.

On the other hand, $\widehat{\partial}\phi_{\overline{M} \cap U}[\omega]\,(\eta) := \phi_{\overline{M} \cap U}\omega\,(\partial^* \eta)$, where $\partial^* : H^{k-p-1}_c(M) \to H^{k-p}_c(\overline{M} \cap U)$ is the connecting map in the Mayer-Vietoris l.e.s. for compactly supported cohomology.

The definition of the connecting map is $\partial^*[\eta] = [y]$ where $y \in \Omega^{k-p}_c(\overline{M} \cap U)$ is a closed form such that $(y, y) \in \Omega^{k-p}_c(\overline{M}) \oplus \Omega^{k-p}_c(U)$ satisfies $(y, y) = (d\gamma, d\epsilon)$ for some $(\gamma, \epsilon) \in \Omega^{k-p-1}_c(\overline{M}) \oplus \Omega^{k-p-1}_c(U)$ such that $\gamma - \epsilon = \eta$.

Let us now put things together.

$$\begin{aligned}
\int_{\overline{M}\cap U} \omega \wedge \partial^* \eta &= \int_{\overline{M}\cap U} \omega \wedge y \\
&= \int_{\overline{M}\cap U} (\alpha - \beta) \wedge y \\
&= \int_{\overline{M}\cap U} \alpha \wedge y - \int_{\overline{M}\cap U} \beta \wedge y \\
&= \int_{\overline{M}} \alpha \wedge y - \int_{U} \beta \wedge y \\
&= \int_{\overline{M}} \alpha \wedge d\gamma - \int_{U} \beta \wedge d\epsilon \\
&= \int_{\overline{M}} x|_{\overline{M}} \wedge \gamma - \int_{U} x|_{U} \wedge \epsilon \\
&= \int_{M} x \wedge \gamma - \int_{M} x \wedge \epsilon \\
&= \int_{M} x \wedge (\gamma - \epsilon) \\
&= \int_{M} x \wedge \eta
\end{aligned}$$

as desired.

Exercise 11. *Check that the first two squares in the diagram above commute.*

□

2.10. **Some Hodge theory.**

2.10.1. *Linear algebra.* Let V be a finite dimensional vector space of dimension k. Suppose that we have a symmetric, bilinear, non-degenerate pairing $\langle , \rangle : V \times V \to \mathbb{R}$.

(A symmetric bilinear pairing is equivalently defined in terms of a $k \times k$ symmetric matrix A such that the bilinear pairing is given by $\langle v, w \rangle = v^T A w$. Remember that all symmetric matrices are diagonalizable, i.e. conjugate to a diagonal matrix with eigenvalues all down the diagonal. So non-degenerate means that the eigenvalues of A are all non-zero. If the eigenvalues are all positive, the matrix is called positive definite and the pairing is called an inner product.)

A non-degenerate bilinear pairing $\langle , \rangle$ on V determines an explicit isomorphism $V \xrightarrow{\cong} V^*$ given by $v \mapsto \langle v, \cdot \rangle$. (Because the non-degeneracy condition says that the kernel of this map is 0.)

The bilinear pairing $\langle , \rangle$ on V extends to a symmetric bilinear pairing on the alternating tensor products, $\Lambda^p(V)$ by defining it on basic elements by

$$\langle v_1 \wedge v_2 \wedge \ldots \wedge v_p, w_1 \wedge w_2 \wedge \ldots \wedge w_p \rangle = \det\left(\langle v_i, w_j \rangle\right)$$

and extending the definition to sums of such terms by linearity.

In terms of an orthogonal basis of eigenvectors $e_1, \ldots, e_k$ of A, one can write every alternating p-tensor as a linear combination of $e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_p}$ for $1 \le i_1 < i_2 < \ldots < i_p \le k$. Since $\langle e_i, e_j \rangle = \lambda_i \delta_{ij}$ where λ_i is the eigenvalue associated to the eigenvector e_i, one has that

$$\langle e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_p}, e_{j_1} \wedge e_{j_2} \wedge \ldots \wedge e_{j_p} \rangle = \det\left(\langle e_{i_k}, e_{j_l} \rangle\right) = \begin{cases} 0 & \text{if} \quad i_k \neq j_k \text{for some } k \\ \prod_{k=1}^{p} \lambda_{i_k} & \text{if} \quad i_k = j_k \forall k. \end{cases}$$

All of which implies that the $e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_p}$ are an orthogonal basis of eigenvectors for the bilinear form $\langle , \rangle$ on $\Lambda^p(V)$, and as the eigenvalues are products of the eigenvalues of A, they are also non-zero, therefore the bilinear pairing $\langle , \rangle$ on $\Lambda^p(V)$ is non-degenerate. So the bilinear pairing gives us an explicit isomorphism

$$\begin{array}{rcl} \Phi_{\langle , \rangle} : \Lambda^p(V) & \longrightarrow & (\Lambda^p(V))^* \\ \tau & \mapsto & \langle \cdot, \tau \rangle \end{array}$$

Now let $\widetilde{\Omega} \in \Lambda^k(V)$ be any volume form, which we use to fix an orientation of $\Lambda^k(V)$. There is a unique volume form $\Omega \in \Lambda^k(V)$ having the same orientation as $\widetilde{\Omega}$, and normalized with respect to the bilinear pairing on $\Lambda^k(V)$, meaning that $\langle \Omega, \Omega \rangle = \pm 1$. So, since the wedge product of alternating p-tensors gives a non-degenerate pairing

$$\begin{array}{rcl} \Lambda^p(V) \times \Lambda^{k-p}(V) & \longrightarrow & \Lambda^k(V) \\ (\omega, \tau) & \mapsto & \omega \wedge \tau \end{array}$$

we can define a non-degenerate bilinear pairing

$$\begin{array}{rcl} b_\Omega : \Lambda^p(V) \times \Lambda^{k-p}(V) & \longrightarrow & \mathbb{R} \\ (\omega, \tau) & \mapsto & b_\Omega(\omega, \tau) \end{array}$$

by the condition that $b(\omega, \tau)\Omega = \omega \wedge \tau$. This gives us an explicit isomorphism

$$\begin{array}{r} \Psi_\Omega : \Lambda^{k-p}(V) \longrightarrow (\Lambda^p(V))^* \\ \tau \mapsto b_\Omega(\cdot, \tau) \end{array}$$

Hence,

$$\begin{array}{rcl} * : \Lambda^p(V) & \longrightarrow & \Lambda^{k-p}(V) \\ \tau & \mapsto & \Psi_\Omega^{-1} \circ \Phi_{\langle,\rangle}(\tau) \end{array}$$

is an explicit isomorphism, determined by the bilinear pairing $\langle, \rangle$ and the volume form Ω, which is called the Hodge star operator.

Definition 10. *The Hodge star operator $* : \Lambda^p(V) \to \Lambda^{k-p}(V)$, with respect to a non-degenerate bilinear pairing $\langle, \rangle$ on V and a normalized volume form $\Omega \in \Lambda^k(V)$, is the linear isomorphism uniquely determined by the condition that*

$$\omega \wedge (*\tau) = \langle \omega, \tau \rangle \Omega.$$

Remark. We insist on Ω being normalized in order for the resulting map $*$ to satisfy $* \circ * = \pm 1$, i.e. the identity up to sign. If we don't insist on Ω being normalized, then we can still define an isomorphism $* : \Lambda^p(V) \to \Lambda^{k-p}(V)$ by the condition $\omega \wedge (*\tau) = \langle \omega, \tau \rangle \Omega$, but the difference will be that this isomorphism will not satisfy $** = \pm 1$. For instance if we take the inner product on $\mathbb{R}^2$ defined by the matrix $2\,\mathrm{Id}$, then $\Omega = e_1 \wedge e_2$ is not normalized, as $\langle \Omega, \Omega \rangle = \det \begin{pmatrix} \langle e_1, e_1 \rangle & \langle e_1, e_2 \rangle \\ \langle e_2, e_1 \rangle & \langle e_2, e_2 \rangle \end{pmatrix} = \det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4$. So in this case, for example, $*e_1 = 2e_2$, and $*e_2 = -2e_1$, so $* * e_1 = -4e_1$.

Example 12. Let $\langle, \rangle : \mathbb{R}^3 \to \mathbb{R}$ be the standard dot product of vectors, with the standard orthonormal basis e_1, e_2, e_3, and let $e_1 \wedge e_2 \wedge e_3$ be the standard orientation. Then

$$\begin{aligned} 1 * (1) = e_1 \wedge e_2 \wedge e_3 &\implies *(1) = e_1 \wedge e_2 \wedge e_3 \\ e_j \wedge (*e_1) = \delta_{j1} e_1 \wedge e_2 \wedge e_3 &\implies *e_1 = e_2 \wedge e_3, \\ e_j \wedge (*e_2) = \delta_{j2} e_1 \wedge e_2 \wedge e_3 &\implies *e_2 = -e_1 \wedge e_3, \\ e_j \wedge (*e_3) = \delta_{j3} e_1 \wedge e_2 \wedge e_3 &\implies *e_3 = e_1 \wedge e_2 \\ e_i \wedge e_k \wedge *(e_1 \wedge e_2) = \delta_{i1}\delta_{k2} e_1 \wedge e_2 \wedge e_3 &\implies *(e_1 \wedge e_2) = e_3 \\ e_i \wedge e_k \wedge *(e_1 \wedge e_3) = \delta_{i1}\delta_{k3} e_1 \wedge e_2 \wedge e_3 &\implies *(e_1 \wedge e_3) = -e_2 \\ e_i \wedge e_k \wedge *(e_2 \wedge e_3) = \delta_{i2}\delta_{k3} e_1 \wedge e_2 \wedge e_3 &\implies *(e_2 \wedge e_3) = e_1 \\ e_i \wedge e_j \wedge e_k \wedge *(e_1 \wedge e_2 \wedge e_3) = \delta_{i1}\delta_{j2}\delta_{k3} e_1 \wedge e_2 \wedge e_3 &\implies *(e_1 \wedge e_2 \wedge e_3) = 1. \end{aligned}$$

The cross product is best described in terms of the wedge product and the Hodge star operator: namely, if $v, u \in \mathbb{R}^3$, then $*(v \times u) = v \wedge u$.

Example 13. Let $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ be the symmetric matrix determining a bilinear pairing on the vector space $\mathbb{R}^4$, and let $e_1 \wedge e_2 \wedge e_3 \wedge e_4$ be the standard orientation on $\mathbb{R}^4$. Hodge star operator on 0-tensors: i.e. $* : \mathbb{R} \to \Lambda^4(\mathbb{R}^4)$. The Hodge condition is that $1 \wedge (*1) = \langle 1, 1 \rangle e_1 \wedge e_2 \wedge e_3 \wedge e_4$. Now by definition $1 \wedge \omega = 1 \cdot \omega = \omega$

and $\langle 1,1\rangle = 1\cdot 1 = 1$. So we see that $*1 = e_1\wedge e_2\wedge e_3\wedge e_4$.
Hodge star operator on alternating 2-tensors, i.e. $*:\Lambda^2(\mathbb{R}^4)\to\Lambda^2(\mathbb{R}^4)$.

$$(e_j\wedge e_k)\wedge *(e_1\wedge e_2) = \langle e_j\wedge e_k, e_1\wedge e_2\rangle e_1\wedge e_2\wedge e_3\wedge e_4 = \delta_{j1}\delta_{k2}\lambda_1\lambda_2 e_1\wedge e_2\wedge e_3\wedge e_4 = \delta_{j1}\delta_{k2}e_1\wedge e_2\wedge e_3\wedge e_4$$
$$\implies *(e_1\wedge e_2) = e_3\wedge e_4$$
$$(e_j\wedge e_k)\wedge *(e_1\wedge e_3) = \langle e_j\wedge e_k, e_1\wedge e_3\rangle e_1\wedge e_2\wedge e_3\wedge e_4 = \delta_{j1}\delta_{k3}\lambda_1\lambda_3 e_1\wedge e_2\wedge e_3\wedge e_4 = \delta_{j1}\delta_{k3}e_1\wedge e_2\wedge e_3\wedge e_4$$
$$\implies *(e_1\wedge e_3) = -e_2\wedge e_4$$
$$(e_j\wedge e_k)\wedge *(e_1\wedge e_4) = \langle e_j\wedge e_k, e_1\wedge e_4\rangle e_1\wedge e_2\wedge e_3\wedge e_4 = \delta_{j1}\delta_{k4}\lambda_1\lambda_4 e_1\wedge e_2\wedge e_3\wedge e_4 = -\delta_{j1}\delta_{k3}e_1\wedge e_2\wedge e_3\wedge e_4$$
$$\implies *(e_1\wedge e_4) = -e_2\wedge e_3$$
$$(e_j\wedge e_k)\wedge *(e_2\wedge e_3) = \langle e_j\wedge e_k, e_2\wedge e_3\rangle e_1\wedge e_2\wedge e_3\wedge e_4 = \delta_{j2}\delta_{k3}\lambda_2\lambda_3 e_1\wedge e_2\wedge e_3\wedge e_4 = \delta_{j2}\delta_{k3}e_1\wedge e_2\wedge e_3\wedge e_4$$
$$\implies *(e_2\wedge e_3) = e_1\wedge e_4$$
$$(e_j\wedge e_k)\wedge *(e_2\wedge e_4) = \langle e_j\wedge e_k, e_2\wedge e_4\rangle e_1\wedge e_2\wedge e_3\wedge e_4 = \delta_{j2}\delta_{k4}\lambda_2\lambda_4 e_1\wedge e_2\wedge e_3\wedge e_4 = -\delta_{j2}\delta_{k4}e_1\wedge e_2\wedge e_3\wedge e_4$$
$$\implies *(e_2\wedge e_4) = e_1\wedge e_3$$
$$(e_j\wedge e_k)\wedge *(e_3\wedge e_4) = \langle e_j\wedge e_k, e_3\wedge e_4\rangle e_1\wedge e_2\wedge e_3\wedge e_4 = \delta_{j3}\delta_{k4}\lambda_3\lambda_4 e_1\wedge e_2\wedge e_3\wedge e_4 = -\delta_{j3}\delta_{k4}e_1\wedge e_2\wedge e_3\wedge e_4$$
$$\implies *(e_3\wedge e_4) = -e_1\wedge e_2.$$

Hodge star operator $*:\Lambda^4(\mathbb{R}^4)\to\Lambda^0(\mathbb{R}^4)=\mathbb{R}$.
By the Hodge star condition we need to satisfy, for $\Omega = e_1\wedge e_2\wedge e_3\wedge e_4$,

$$\Omega\wedge *\Omega = \langle\Omega,\Omega\rangle\Omega.$$

Given that

$$\begin{aligned}
\langle\Omega,\Omega\rangle &= \langle e_1\wedge e_2\wedge e_3\wedge e_4, e_1\wedge e_2\wedge e_3\wedge e_4\rangle\\
&= \det\begin{pmatrix}\langle e_1,e_1\rangle & \langle e_1,e_2\rangle & \langle e_1,e_3\rangle & \langle e_1,e_4\rangle\\ \langle e_2,e_1\rangle & \langle e_2,e_2\rangle & \langle e_2,e_3\rangle & \langle e_2,e_4\rangle\\ \langle e_3,e_1\rangle & \langle e_3,e_2\rangle & \langle e_3,e_3\rangle & \langle e_3,e_4\rangle\\ \langle e_4,e_1\rangle & \langle e_4,e_2\rangle & \langle e_4,e_3\rangle & \langle e_4,e_4\rangle\end{pmatrix}\\
&= \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}\\
&= -1
\end{aligned}$$

and therefore $*\Omega = -1$.

Proposition. *The Hodge star operators*

$$*:\Lambda^p(V)\to\Lambda^{k-p}(V),\quad *:\Lambda^{k-p}(V)\to\Lambda^p(V)$$

satisfy $** = (-1)^{p(k-p)}s_{\langle,\rangle}$ *where* $s_{\langle,\rangle}$ *is the sign of the bilinear pairing, given by*

$$s_{\langle,\rangle} = (-1)^{\#\textit{negative eigenvalues of the bilinear pairing}}$$

so in particular, if the bilinear pairing is positive definite, we have $*\circ * = (-1)^{n(k-n)}$.

2.11. **Hodge star on differential forms.** All of the above can be repeated pointwise for differential forms on a manifold M, given a symmetric non-degenerate bilinear pairing $\langle,\rangle_p : T_pM\times T_pM\to\mathbb{R}$ of the tangent spaces at each point $p\in M$, and a normalized volume form $\Omega\in\Omega^k(M)$. A bilinear pairing on the tangent space T_pM at each point p of M that varies smoothly in p is called a 2-tensor on M. In a local chart U of M, with local coordinates $x_1,\dots,x_k$ on M, a 2-tensor can be thought of as a function $A:U\to M_{k\times k}(\mathbb{R})$ where $A(x_1,\dots,x_k) = A(p)$, where $\langle v_p, w_p\rangle_p = v_p^T A(p)w(p)$. Or in tensor notation, a 2-tensor can be expressed as a sum $\sum\limits_{i,j} f_{i,j}(x_1,\dots,x_k)dx_i\otimes dx_j$ for functions $f_{i,j}:U\to\mathbb{R}$.

So a symmetric, non-degenerate bilinear pairing at each point is a symmetric non-degenerate 2-tensor. If it is *positive definite* at each point it is called a Riemannian metric. Otherwise it's called a *pseudo-Riemannian* metric. In a local chart U with local coordinates $x_1,\dots,x_k$ on M, a symmetric non-degenerate 2-tensor corresponds to a function $A:U\to M_{k\times k}(\mathbb{R})$ where $A(x_1,\dots,x_k)$ is a symmetric matrix with non-zero eigenvalues for every $x_1,\dots,x_k\in U$.

Given a non-degenerate symmetric 2-tensor g on M, and a normalized volume form $\Omega \in \Omega^k(M)$, then the Hodge star operator is the isomorphism $* : \Omega^n(M) \to \Omega^{k-n}(M)$ defined point-wise in M, i.e. for $\omega \in \Omega^n(M)$, its Hodge star $*\omega \in \Omega^{k-n}(M)$ is determined point-wise in M by

$$(*\omega)_p := *(\omega_p)$$

where the Hodge star $*$ on the right hand side is the Hodge star operator $* : \Lambda^n((T_p(M))^*) \to \Lambda^{k-n}((T_p(M))^*)$ defined at the point p with respect to $\Omega(p) \in \Lambda^k((T_pM)^*)$ and the symmetric non-degenerate bilinear pairing $g_p : T_pM \to T_pM \to \mathbb{R}$.

Example 14. An important example in physics is Maxwell's equations. In this case one considers 4 dimensional manifolds with local coordinates x, y, z, t. The non-degenerate bilinear form is the *Lorentzian metric*, $dx \otimes dx + dy \otimes dy + dz \otimes dz - dt \otimes dt$, or in matrix notation $A(x, y, z, t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, which is a pseudo-Riemmannian metric. The volume form is $dx \wedge dy \wedge dz \wedge dt$. In a vacuum, the electromagnetic field is represented by a 2-form

$$\omega = E_x dx \wedge dt + E_y dy \wedge dt + E_z dz \wedge dt + H_x dy \wedge dz - H_y dx \wedge dz + H_z dx \wedge dy$$

and its Hodge star $*\omega$ is also a 2-form, given explicitly (by the calculations of Exercise 13) by

$$*\omega = -E_x dy \wedge dz + E_y dx \wedge dz - E_z dy \wedge dx + H_x dx \wedge dt + H_y dy \wedge dt + H_z dz \wedge dt$$

and Maxwell's equations say that $d\omega = 0, d * \omega = 0$. See Introduction to Bott & Tu (p.8) (although this is w.r.t. the Lorentzian metric having signature $(+, -, -, -)$ so the Hodge star has the opposite sign), or Section 20 of Introduction to Guillemin & Sternberg (p. 136).

2.12. **Laplace-Beltrami operators.** Let us now consider a manifold with a Riemannian metric on it (so the bilinear form at each point is positive definite) as well as a volume form. Combining the exterior derivative d with the Hodge star operator, we define a *codifferential*

$$\begin{aligned} \delta : \Omega^{n+1}(M) &\longrightarrow \Omega^n(M) \\ \omega &\mapsto (-1)^{kn+1} * d * (\omega) \end{aligned}$$

Proposition. $\delta \circ \delta = 0$

Proof. $\delta \circ \delta = \pm * d * *d* = \pm * d^2 * = 0$ since $** = \pm 1$ and $d^2 = 0$. ☐

Definition 11. *The* Hodge Laplacian *or* Laplace-Beltrami operator *is the differential operator*

$$\triangle = (d + \delta)^2 = d\delta + \delta d : \Omega^n(M) \to \Omega^n(M)$$

Example 15. Consider $\mathbb{R}^3$ with the standard metric $dx \otimes dx + dy \otimes dy + dz \otimes dz$, and the standard volume form $dx \wedge dy \wedge dz$. By the calculations of Example 12, the Hodge star operators are

$$\begin{aligned} * : \Omega^0(\mathbb{R}^3) &\longrightarrow \Omega^3(\mathbb{R}^3) \\ f(x, y, z) &\mapsto f(x, y, z) dx \wedge dy \wedge dz \\ * : \Omega^1(\mathbb{R}^3) &\longrightarrow \Omega^2(\mathbb{R}^3) \\ f\ dx + g\ dy + h\ dz &\mapsto f\ dy \wedge dz - g\ dx \wedge dz + h\ dx \wedge dy \\ * : \Omega^2(\mathbb{R}^3) &\longrightarrow \Omega^1(\mathbb{R}^3) \\ f\ dx \wedge dy + g\ dx \wedge dz + h\ dy \wedge dz &\mapsto f\ dz - g\ dy + h\ dx \\ * : \Omega^3(\mathbb{R}^3) &\longrightarrow \Omega^0(\mathbb{R}^3) \\ f(x, y, z) dx \wedge dy \wedge dz &\mapsto f(x, y, z) \end{aligned}$$

So the codifferential δ is given by

$$\begin{array}{rcl}
\delta : \Omega^0(\mathbb{R}^3) & \to & \Omega^{-1}(\mathbb{R}^3) \\
f & \mapsto & 0 \\
\delta : \Omega^1(\mathbb{R}^3) & \to & \Omega^0(\mathbb{R}^3) \\
fdx + gdy + hdz & \mapsto & (-1)^{0\cdot 3+1} * d(fdy \wedge dz - gdx \wedge dz + hdx \wedge dy) \\
& & = - * (f_x + g_y + h_z) dx \wedge dy \wedge dz = -(f_x + g_y + h_z) \\
\delta : \Omega^2(\mathbb{R}^3) & \to & \Omega^1(\mathbb{R}^3) \\
fdx \wedge dy - gdx \wedge dz + hdy \wedge dz & \mapsto & (-1)^{1\cdot 3+1} * d(fdz + gdy + hdx) \\
& & = *[(f_x - h_z)dx \wedge dz + (f_y - g_z)dy \wedge dz + (g_x - h_y)dx \wedge dy] \\
& & = (h_z - f_x)dy + (f_y - g_z)dx + (g_x - h_y)dz \\
\delta : \Omega^3(\mathbb{R}^3) & \to & \Omega^2(\mathbb{R}^3) \\
fdx \wedge dy \wedge dz & \mapsto & *d(f) = (-1)^{2\cdot 3+1} * (f_x dx + f_y dy + f_z dz) = -(f_x dy \wedge dz - f_y dx \wedge dz + f_z dx \wedge dy.)
\end{array}$$

Let's compute some Laplace-Beltrami operators:

$$\begin{array}{rcl}
\triangle : \Omega^0(\mathbb{R}^3) & \to & \Omega^0(\mathbb{R}^3) \\
f(x, y, z) & \mapsto & \delta d(f) + d\delta(f) = \delta df = -(\partial_{xx} f + \partial_{yy} f + \partial_{zz} f),
\end{array}$$

c.f. the usual Laplacian $\triangle = \partial_{xx} + \partial_{yy} + \partial_{zz}$ on functions. The Laplace-Beltrami operator on 1-forms:

$$\begin{array}{rcl}
\triangle : \Omega^1(\mathbb{R}^3) & \to & \Omega^1(\mathbb{R}^3) \\
fdx + gdy + hdz & \mapsto & \delta[(g_x - f_y)dx \wedge dy - (f_z - h_x)dx \wedge dz + (h_y - g_z)dy \wedge dz] + d(f_x + g_y + h_z) \\
& & = [g_{yx} - f_{yy} - (f_{zz} - h_{zx})]dx + [h_{zy} - g_{zz} - (g_{xx} - f_{xy})]dy + [f_{xz} - h_{xx} - (h_{yy} - g_{yz})]dz \\
& & -[f_{xx} + g_{xy} + h_{xz}]dx - [f_{yx} + g_{yy} + h_{yz}]dy - [f_{zx} + g_{zy} + h_{zz}]dz \\
& & = [-f_{xx} - f_{yy} - f_{zz}]dx + [-g_{xx} - g_{yy} - g_{zz}]dy + [-h_{xx} - h_{yy} - h_{zz}]dz
\end{array}$$

Exercise 12. *Let $H \subset \mathbb{R}^2$ be the upper half plane, i.e. $\{(x, y) \in \mathbb{R}^2 | y > 0\}$, equipped with the metric $g(x, y) = (dx \otimes dx + dy \otimes dy)/y^2$ and the normalized volume form $\Omega(x, y) = y^2 dx \wedge dy$. Find explicit expressions for the Hodge star operators $*$, the codifferential operators δ, and the Laplace-Beltrami operators $\triangle$.*

2.13. **Poincaré duality and Hodge duality.** Let us suppose that M is an orientable and compact k-dimensional manifold, with no boundary. Then we can integrate k-forms on it. We suppose that M is equipped with a Riemannian metric $\langle , \rangle$, and a compatible volume form Ω. Then we can define the L^2 inner product on each vector space $\Omega^p_{dR}(M)$ via

$$(\omega, \tau) := \int_M \langle \omega, \tau \rangle \Omega.$$

Note that, by definition of the Hodge star operator, this means $(\omega, \tau) = \int_M \omega \wedge (*\tau)$.

Lemma. *With respect to the above inner product, d and δ are adjoint operators, i.e. $(d\omega, \tau) = (\omega, \delta\tau)$.*

Proof. Let $\omega \in \Omega^p(M)$, $\tau \in \Omega^{p+1}(M)$. Now $\omega \wedge *\tau$ is a $k-1$ form, so

$$\begin{array}{rcl}
d(\omega \wedge *\tau) & = & d\omega \wedge *\tau + (-1)^p \omega \wedge d(*\tau) \\
& = & d\omega \wedge *\tau + (-1)^p (-1)^{p(k-p)} \omega \wedge * * d(*\tau) \\
& = & d\omega \wedge *\tau + (-1)^p (-1)^{p(k-p)} (-1)^{pk+1} (\omega \wedge *\delta\tau) \\
& = & d\omega \wedge *\tau - \omega \wedge *\delta\tau
\end{array}$$

is a k-form. By Stokes' theorem (since $\partial M = \emptyset$),

$$0 = \int_M d\omega \wedge *\tau - \int_M \omega \wedge *\delta\tau \overset{def}{\Longrightarrow} 0 = (d\omega, \tau) - (\omega, \delta\tau).$$

□

Lemma. *The Laplace-Beltrami operator $\triangle : \Omega^p_{dR}(M) \to \Omega^p_{dR}(M)$ is symmetric with respect to the inner product, i.e. $(\omega, \triangle\tau) = (\triangle\omega, \tau)$, and non-negative, i.e. $(\triangle\omega, \omega) \geq 0$ for every $\omega \in \Omega^p(M)$.*

Proof. Symmetry and non-negativity follow immediately from the adjointness of d and δ:

$$(\omega, \triangle\tau) = (\omega, \delta d\tau) + (\omega, d\delta\tau) = (d\delta\omega, \tau) + (\delta d\omega, \tau) = (\triangle\omega, \tau)$$
$$(\triangle\omega, \omega) = (\omega, \delta d\omega) + (\omega, d\delta\omega) = (d\omega, d\omega) + (\delta\omega, \delta\omega) \geq 0.$$

□

Definition 12. *A differential form $\omega \in \Omega^p(M)$ is called* harmonic *if $\triangle\omega = 0$.*

Exercise 13. *Show that the Hodge star operator sends harmonic forms to harmonic forms.*
Hint: show that if $d\omega = 0$, then $\delta * \omega = 0$, and if $\delta\omega = 0$, then $d * \omega = 0$.

$$(\delta * \omega, \delta * \omega) = (\delta * \omega, \pm * d * *\omega) = (\delta * \omega, \pm * d\omega)$$
$$(d * \omega, d * \omega) = (d * \omega, \pm * \delta * *\omega) = (d * \omega, \pm * \delta\omega).$$

Lemma. *A differential form $\omega \in \Omega^p(M)$ is harmonic if and only if $d\omega = 0$ and $\delta\omega = 0$.*

Proof. ($\implies$:) If $\triangle\omega = 0$ then $0 = (\triangle\omega, \omega) = (d\omega, d\omega) + (\delta\omega, \delta\omega) \implies d\omega = 0$ and $\delta\omega = 0$.

($\impliedby$:) If $d\omega = 0$ and $\delta\omega = 0$ then $\triangle\omega = \delta d\omega + d\delta\omega = 0$. □

If $\omega \in \Omega^p(M)$ is harmonic, then it is closed ($d\omega = 0$) and therefore it determines a cohomology class $[\omega] \in H^p(M)$.

Lemma. *Suppose that $\omega, \tau \in \Omega^p(M)$ are two harmonic p-forms such that $[\omega] = [\tau]$. Then $\omega = \tau$. In other words, there is at most one harmonic form in a given cohomology class.*

Proof. By assumption, $\omega - \tau = d\sigma$ for some $\sigma \in \Omega^{p-1}(M)$. So since $\delta\omega = 0$ and $\delta\tau = 0$, we get that $\delta d\sigma = 0$. But then we have

$$(\omega - \tau, \omega - \tau) = (d\sigma, d\sigma) = (\sigma, \delta d\sigma) = (\sigma, 0) = 0$$

and therefore, $\omega - \tau = 0$. □

Theorem. *Every cohomology class $[\omega] \in H^p(M)$ contains a harmonic representative.*

Proof. This is an existence theorem, which is harder to prove. It is proved rigorously with tools from analysis. The idea is as follows: let $[\omega_0]$ be a cohomology class in $H^p(M)$. Then any other representative of the same cohomology class is of the form

$$\omega = \omega_0 + d\alpha$$

where $\alpha \in \Omega^{p-1}(M)$. Consider the functional

$$\begin{aligned} \Phi : \Omega^{p-1}(M) &\to [0, \infty) \\ \alpha &\mapsto (\omega_0 + d\alpha, \omega_0 + d\alpha) = \int_M \langle \omega_0 + d\alpha, \omega_0 + d\alpha \rangle \Omega \end{aligned}$$

that takes the L^2 norm of a given representative of the cohomology class. Suppose that this functional attains a minimum, say at $\omega_0 + d\alpha_0$. Then it means that for all $\beta \in \Omega^{p-1}(M)$,

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0} (\omega_0 + d\alpha_0 + td\beta, \omega_0 + d\alpha_0 + td\beta) \\ &= 2(d\beta, \omega_0 + d\alpha_0) \\ &= 2(\beta, \delta(\omega_0 + d\alpha_0)) \end{aligned}$$

and therefore $\delta(\omega_0 + d\alpha_0) = 0$. Since it is already the case that $d(\omega_0 + d\alpha_0) = 0$, we see that $\omega_0 + d\alpha_0$ is a harmonic form. So we need an argument as to why the functional should attain a minimum.

As the functional is bounded below, it certainly has an infimum, so we can find a sequence of forms in $\Omega^{p-1}(M)$ for which the functional converges to the infimum. However this convergence is with respect to the L^2 norm on $\Omega^{p-1}(M)$. An L^2-limit is not necessarily smooth, and the analysis part of the proof (which is the hard part) is required to show that the minimum is achieved by a *smooth* differential form. □

The previous two results are known as the Hodge Theorem, or Hodge-Weyl Theorem:

Theorem (Hodge Theorem)**.** *Let M be a compact, oriented manifold. Then every cohomology class can be represented by a unique harmonic form.*

The Hodge theorem also gives another proof of Poincaré duality for compact orientable manifolds:

Theorem (Poincaré duality)**.** *Let M be a compact orientable manifold of dimension k. Then the bilinear map*

$$\begin{aligned} \int_M : H^p_{dR}(M) \times H^{k-p}_{dR}(M) &\longrightarrow \mathbb{R} \\ ([\omega],[\tau]) &\mapsto \int_M \omega \wedge \tau \end{aligned}$$

is non-degenerate, and therefore $H^p_{dR}(M) \cong H^{k-p}_{dR}(M)$.

Proof. Let ω be a representative of the cohomology class $[\omega]$. To show that the bilinear pairing is non-degenerate we need to show that if $[\omega] \neq 0$, then there exists a $[\tau] \in H^{k-p}_{dR}(M)$ such that $\int_M \omega \wedge \tau \neq 0$. Every manifold admits a Riemannian metric, let us fix some Riemannian metric on M. And let Ω be a volume form on M. Then we can do Hodge theory. By the Hodge theorem, we can take ω to be the unique harmonic representative of the cohomology class $[\omega]$. Since we are assuming that the cohomology class is non-zero, $\omega \neq 0$. Now let us take $\eta = *\omega$. Then, by an exercise above, η is also harmonic, therefore η is closed and determines a cohomology class. Moreover, we have

$$\int_M \omega \wedge \tau = \int_M \omega \wedge *\omega = (\omega, \omega) \neq 0.$$

□

3. The Poincaré dual of a compact, oriented submanifold

For simplicity we'll suppose throughout this section that M is a compact oriented and connected k-dimensional manifold. Remember from the previous section that the bilinear pairing of Poincarè duality implies an isomorphism between the dual space $H^p_{dR}(M)^*$ and the vector space $H^{k-p}_{dR}(M)$. In other words, for every linear functional $f : H^p_{dR}(M) \to \mathbb{R}$, there is a unique cohomology class $\eta_f \in H^{k-p}_{dR}(M)$ such that $f(\omega) = \int_M \omega \wedge \eta_f$ for all $\omega \in H^k_{dR}(M)$.

One source of linear functionals comes from integration over submanifolds of M. Let $S \subset M$ be an oriented p-dimensional submanifold of M. The inclusion map $i : S \hookrightarrow M$ induces a pull-back map on p-forms and hence on the p-th cohomology, $i^* : H^p(M) \to H^p(S)$. So integration of the pull-back of p cohomology classes over S therefore defines a linear functional

$$\begin{aligned} f_S : H^p_{dR}(M) &\longrightarrow \mathbb{R} \\ \omega &\mapsto \int_S i^*\omega \end{aligned}$$

Therefore, there is a unique cohomology class $\eta_S \in H^{k-p}_{dR}(M)$ such that

$$\int_S i^*\omega = \int_M \omega \wedge \eta_S$$

for all $\omega \in H^p_{dR}(M)$. This cohomology class is called the *Poincaré dual* of S, and a closed differential form τ representing ω is often called a *Thom form* for S.

What does a Thom form look like, practically? We look at some examples.

Example 16. Let S be a point $p \in M$, i.e. a 0 dimensional submanifold. Then an orientation on S is a choice of sign, $+$ or $-$, and integration of the pull-back of a zero form on M is evaluation of the 0-form (i.e. function $f : M \to \mathbb{R}$) at p with the appropriate sign from the orientation. Since a 0-cohomology class is represented by a closed 0-form on M, i.e. a function $f : M \to \mathbb{R}$ that's constant on each component of M, we see that

$$\int_S i^* f = f(p) = f(p) \cdot 1 = \int_M f \wedge V$$

where V is *any* k-form on M such that $\int_M V = 1$. (Since $H^k(M)$ is one-dimensional, all cohomology classes are represented by multiples of V, so they're distinguished from each other precisely by the value of their integral over M.)

Example 17. Let M be the 2 dimensional cylinder $S^1 \times \mathbb{R}$, with cylindrical coordinates (θ, y). Let S be the circle given by the zero-section, i.e. $S = \{(\theta, 0) | \theta \in S^1\}$. Let us find a Thom form for S. Given a 1-form $\omega = f(\theta, y)d\theta + g(\theta, y)dy \in \Omega^1(M)$, ω is closed if and only if $d\omega = 0$, in other words $0 = (\frac{\partial g}{\partial \theta} - \frac{\partial f}{\partial y})d\theta \wedge dy$, hence if and only if $\frac{\partial g}{\partial \theta} = \frac{\partial f}{\partial y}$. Let us consider a 1-form of the type $\tau = t(y)dy$ for some compactly supported function $t : \mathbb{R} \to \mathbb{R}$. Then, we have

$$\begin{aligned}
\int_M \omega \wedge \tau &= \int_M (f(\theta, y)d\theta + g(\theta, y)dy) \wedge (t(y)dy) \\
&= \int_M f(\theta, y)t(y)d\theta \wedge dy \\
&= \int_{\mathbb{R}} \underbrace{\left(\int_{S^1} f(\theta, y)d\theta\right)}_{:=F(y)} t(y)dy
\end{aligned}$$

Since $\frac{dF}{dy} = \frac{d}{dy}\left(\int_{S^1} f(\theta, y)d\theta\right) = \int_{S^1} \frac{\partial f}{\partial y}(\theta, y)d\theta = \int_{S^1} \frac{\partial g}{\partial \theta}(\theta, y)d\theta = g(2\pi, y) - g(0, y) = 0$, this shows that $F(y) = c$ for some constant c, so in particular $F(y) = F(0)$ for all y, and that therefore

$$\int_M \omega \wedge \tau = c \int_{\mathbb{R}} t(y)dy = \left(\int_{S^1} f(\theta, 0)d\theta\right) \int_{\mathbb{R}} t(y)dy = \left(\int_S i^*\omega\right) \left(\int_{\mathbb{R}} t(y)dy\right),$$

so provided $t(y)$ is any compactly supported function such that $\int_{\mathbb{R}} t(y)dy = 1$, the 1-form $\tau = t(y)dy$ is a Thom form for S.

3.1. **Tubular neighborhood theorem.** First a definition of vector bundles:

Definition 13. *Let M be a smooth manifold, and let r be a positive integer. A* vector bundle of rank k over M *is a triple (E, M, π) where E is a smooth manifold, and $\pi : E \to M$ is a smooth map with the following structure:*

1. *$\forall\, x \in M$, $\pi^{-1}(x)$ is a vector space of dimension r (i.e. a vector space isomorphic to $\mathbb{R}^r$),*
2. *there is an open cover $\bigcup_\alpha U_\alpha$ of M such that for every α, there is a diffeomorphism $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$ such that for every $x \in U_\alpha$, the restriction $\Phi_\alpha : \pi^{-1}(x) \to \{x\} \times \mathbb{R}^k$ is a linear isomorphism of vector spaces.*

Example 18. The tangent bundle TM of a smooth manifold M of dimension k is a vector bundle of rank k over M. The cotangent bundle T^*M is another vector bundle of rank k over M.

The pairs (U_α, Φ_α) are called *local trivializations*, the preimage $\pi^{-1}(x) \subset E$, often written E_x, is called the *fiber* of E over x. Any vector bundle has a submanifold $M_0 \subset E$ called the *zero-section*, which is diffeomorphic to M; in each fiber E_x there is a well-defined zero $0_x \in E_x$ (and this zero does not depend on the local trivialization Φ_α at x since linear isomorphisms always send zero to zero). In other words there is an injective map $s : M \to E$ which sends x to the element $0_x \in E_x$, whose image $M_0 = s(M)$ is the zero-section of E, and $s : M \to M_0$ is a diffeomorphism with inverse $\pi|_{M_0} : M_0 \to M$.

Let M be an oriented manifold of dimension k, and let $S \subset M$ be an oriented and compact submanifold of M, of dimension p. The restriction of the tangent bundle TM to S is a vector bundle over S, and a subbundle of this vector bundle is the tangent bundle to S, TS. That is, at each point $s \in S$, we have $\mathbb{R}^p \cong T_sS \subset T_sM \cong \mathbb{R}^k$.

For any inclusion of finite dimensional vector spaces $V \subset W$ their quotient is the vector space V/W whose elements are the equivalence classes of the equivalence relation $a \sim b \iff a - b \in W$. If you want to be fancy and use the language of exact sequences you can say that it is the vector space determined by the condition that the sequence

$$0 \to W \xrightarrow{i} V \xrightarrow{p} V/W \to 0$$

is exact, where i is the inclusion map. If we fix an inner product $\langle , \rangle$ on V then V splits into $W \oplus W^\perp$ and there is a natural isomorphism of vector spaces $W^\perp \cong V/W$.

Given a p-dimensional submanifold S of a k-dimensional manifold M, there is an inclusion of vector bundles $TS \subset TM$, i.e. at each point $s \in S$ there is an inclusion of vector bundles $T_sS \subset T_sM$ where $T_sS \cong \mathbb{R}^p$ and $T_sM \cong \mathbb{R}^k$. The normal bundle to S is the vector bundle $N \to S$ whose fiber at each point $s \in S$ is the quotient vector space $N_s = T_sM/T_sS$. If M is equipped with a Riemannian metric then the normal bundle NS is identified with the orthogonal complement $(TS)^\perp$.

Definition 14. *Let $S \subset M$ be a submanifold of M. A* tubular neighborhood *of S in M is an open subset U containing S such that there is a diffeomorphism $\Psi : N \to U$ whose restriction to the zero section is the identity map on S.*

Theorem (Tubular neighborhood theorem)**.** *Any compact submanifold S of M has a tubular neighborhood.*

Proof outline. In fact the compactness assumption on S isn't necessary, but it's all we need for our purposes. First put a Riemannian metric on M (any smooth manifold has a Riemannian metric). The metric then identifies the normal bundle to S with the vector bundle $(TS)^\perp$ on S, and exponential maps that give a diffeomorphism between a neighborhood of each point in M and a neighborhood of the origin in the tangent space to each point in M. So one can construct a diffeomorphism between the normal bundle and the neighborhood U using the exponential map in the directions that are normal (i.e. orthogonal) to S. $\square$

Lemma. *Let τ be a Thom form for S considered as the zero-section of the normal bundle $N \xrightarrow{\pi} S$. Let U be a tubular neighborhood of S for $\Phi : U \xrightarrow{\cong} N$. Then the pull-back $\Phi^*\tau$ is compactly supported in U, and if we extend it by zero to all of M, it is a Thom form for S in M.*

Proof. Let ω be a p form on M. Let $\rho : M \to [0,1]$ be a function with compact support in U such that $\rho = 1$ on the support of $\Phi^*\tau$. Then $\rho\omega$ is a p-form on U, and $(\Phi^*)^{-1}(\rho\omega)$ is a p-form on N, and the restriction of ω to S in M is equal to the restriction of $(\Phi^*)^{-1}(\rho\omega)$ to S in N, so

$$\int_S \omega = \int_S (\Phi^*)^{-1}(\rho\omega) = \int_N (\Phi^*)^{-1}(\rho\omega) \wedge \tau = \int_U \rho\omega \wedge \Phi^*(\tau) = \int_U \omega \wedge \Phi^*(\tau)$$

showing that $\Phi^*(\tau)$ is a Thom form for S in U. Then, since $\rho\omega \wedge \Phi^*\tau = \omega \wedge \Phi^*\tau$, we have

$$\int_S \omega = \int_S \rho\omega = \int_U \rho\omega \wedge \Phi^*\tau = \int_U \omega \wedge \Phi^*\tau = \int_M \omega \wedge \Phi^*\tau$$

and therefore $\Phi^*\tau$ is a Thom form for S in M. $\square$

Since this is true for any tubular neighborhood of S, this also shows that one can always find a Thom form for S that is supported arbitrarily close to S.

3.2. **Poincaré duals and intersection.** Two submanifolds S and $\widetilde{S}$ are said to intersect transversely if, at each point $p \in S \cap \widetilde{S}$, one has $T_pM = T_pS + T_p\widetilde{S}$, i.e. the tangent spaces to S and $\widetilde{S}$ span the whole tangent space. Note that a necessary condition is that $\dim S + \dim \widetilde{S} \geq n$.

Let's recall the implicit function theorem.

Theorem. *Let $U \subset \mathbb{R}^m$, $0 \in U$, and suppose that $F : U \to \mathbb{R}^n$ is a smooth function such that $F(0) = 0$ and 0 is a regular value of F. Then $F^{-1}(0)$ is a smooth $m - n$ dimensional submanifold of U.*

Lemma. *If a pair of submanifolds intersects transversely, then their intersection is also a smooth submanifold.*

Proof. We just need to come up with local charts for $S \cap \widetilde{S}$ at each point. Let $x \in S \cap \widetilde{S}$ be a point in the intersection, and let U be a neighborhood of x in M. Let $x_1, \ldots, x_k$ be local coordinates on U. Let $c_1 = k - \dim S$ be the codimension of S, and $c_2 = k - \dim \widetilde{S}$ the codimension of $\widetilde{S}$. Since they are submanifolds, we can find local defining functions $F_1 : U \to \mathbb{R}^{c_1}$ and $F_2 : U \to \mathbb{R}^{c_2}$ for the two submanifolds, such that $F_1^{-1}(0)$ gives $S \cap U$ and $F_2^{-1}(0) = \widetilde{S} \cap U$. We then have that $\ker(dF_1)_x = T_xS$ and $\ker(dF_2)_x = T_x\widetilde{S}$, and both dF_1 and dF_2 are surjective at x. Transversality of the intersection implies that $T_xM = T_xS + T_x\widetilde{S}$. Notice that this means that $dF_1 : T_x\widetilde{S} \to \mathbb{R}^{c^1}$ must also be surjective, since we can find a complement to it that is contained in the kernel of dF_1. And similarly $dF_2 : T_xS \to \mathbb{R}^{c_2}$ is surjective for the same reason. So now set $F : U \to \mathbb{R}^{c_1+c_2}$ where $F(v) = \begin{bmatrix} F_1(v) \\ F_2(v)) \end{bmatrix} \in \mathbb{R}^{c_1+c_2}$. Then $dF = \begin{bmatrix} dF_1 \\ dF_2 \end{bmatrix}$. It is easy to see that $F^{-1}(0) = F_1^{-1}(0) \cap F_2^{-1}(0) = S \cap \widetilde{S} \cap U$, so by the implicit function theorem we just need to check that 0 is a regular value of F, i.e. dF is surjective at all $x \in F^{-1}(0)$. For this use our earlier observations. Let $\mathbf{b} = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^{c_1+c_2}$. Now let $v \in T_x\widetilde{S}$ be such that $dF_1(v) = a$, and let $w \in T_xS$ be such that $dF_2(v) = b$. Then $dF(v+w) = \begin{bmatrix} dF_1(v+w) \\ dF_2(v+w) \end{bmatrix} = \begin{bmatrix} dF_1(v) \\ dF_2(w) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$.

$\square$

Relationship between intersection and wedge product:

Proposition. *Let S and $\widetilde{S}$ be smooth submanifolds of M* **that intersect transversely**, *and $S \cap \widetilde{S}$ their intersection, which is also a smooth submanifold of M. Then*

$$\eta_{S\cap\widetilde{S}} = \eta_S \wedge \eta_{\widetilde{S}}.$$

Proof. We have that $T_pM = T_pS + T_p\widetilde{S}$ at each $p \in S \cap \widetilde{S}$. Let $N_S \to S$ be the normal bundle to S, and $N_{\widetilde{S}} \to \widetilde{S}$ be the normal bundle to $\widetilde{S}$. The normal bundle to $S \cap \widetilde{S}$ has as its fiber the quotient $T_pM/T_p(S \cap \widetilde{S})$.

Exercise 14. *Show that $T_pM/T_p(S \cap \widetilde{S}) \cong T_pM/T_pS \oplus T_pM/T_pS$ by defining a linear map from $T_pM \to T_pM/T_pS \oplus T_pM/T_pS$ whose kernel is precisely $T_p(S \cap \widetilde{S})$.*

In other words, the normal bundle to $S \cap \widetilde{S}$ is the direct sum of the normal bundles to S and $\widetilde{S}$, i.e., $N_{S\cap\widetilde{S}} = N_S \oplus N_{\widetilde{S}}$. The proof follows from the general case of oriented vector bundles, which we outline below. □

Definition 15. *The vector bundle $E \longrightarrow S$ is orientable if there is a system of local trivializations $\{(U_\alpha, \phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r)\}_{\alpha\in\mathcal{A}}$ for which the linear isomorphisms on the fibers, $\phi_\alpha \circ \phi_\beta^{-1} : \mathbb{R}^r \to \mathbb{R}^r$, preserve orientation, i.e. are elements of $GL_r(\mathbb{R})$ with positive determinant.*

We can even suppose that the linear isomorphisms on the fibers are given by elements of $SO(r, \mathbb{R})$ (e.g. by doing Gram-Schmidt to the columns we get a map from $GL_r(\mathbb{R}) \to SO_r(\mathbb{R})$.)

Let us first describe what a Thom form is for the zero-section of an oriented vector bundle over a compact oriented base.

Let $E \to M$ be an oriented vector bundle of rank r, where M is compact of dimension m. We have a commuting triangle of maps

$$\begin{tikzcd} M_0 \arrow[r, "i"] & E \arrow[d, "\pi"] \\ & M \arrow[ul, "s"] \end{tikzcd}$$

where π is the bundle projection, s is the map given by the zero section and i is inclusion of the image of the zero section $s(M) = M_0$ in the total space E. Moreover $s \circ \pi : E \to M_0$ is a deformation retract of E onto the zero section M_0 (because the fibers are vector spaces, which are contractible), so both π^* and s^* are isomorphisms, with $\pi^* s^* = \mathrm{Id}_{H^*_{dR}(E)}$ and $s^*\pi^* = \mathrm{Id}_{H^*_{dR}(M)}$.

Proposition. *Let $E \to M$ be an oriented vector bundle of rank r, where M is compact of dimension m. A compactly supported closed form $\tau \in \Omega^r_c(E)$ represents the Poincaré dual of the zero-section M_0 in E if and only if τ restricted to each fiber $\mathbb{R}^r$ satisfies $\int_{\mathbb{R}^r} \tau = 1$.*

Proof. ($\Longleftarrow$:) Suppose τ is a compactly supported closed form satisfying $\int_{\mathbb{R}^r} \tau = 1$. Now let $\xi \in H^m_{dR}(E)$. Given the isomorphism provided by π^*, we can suppose without loss of generality that $\xi = [\pi^*(\omega)]$ for some closed $\omega \in \Omega^m(M)$. Let $\{U_\alpha\}$ be an open cover of M such that $\pi^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{R}^r$, and let $1 = \sum_\alpha \rho_\alpha$ be a partition of unity on M subordinate to the cover $\{U_\alpha\}$. Then we have

$$\begin{aligned} \int_E \pi^*(\omega) \wedge \tau &= \int_E \pi^*(\sum_\alpha \rho_\alpha \omega) \wedge \tau \\ &= \sum_\alpha \int_E \pi^*(\rho_\alpha \omega) \wedge \tau. \end{aligned}$$

Now the $m + k$ form $\pi^*(\rho_\alpha\omega) \wedge \tau$ is supported on $\pi^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{R}^r$, so the integral can be computed on $U_\alpha \times \mathbb{R}^r$. Since $\pi^*(\rho_\alpha\omega)$ is constant in the $\mathbb{R}^r$ coordinates, we can compute the integral as an iterated integral, first integrating $\int_{\mathbb{R}^r} \tau$ with respect to the variables in the $\mathbb{R}^r$ direction to get a function in the U_α variables. But by assumption this integral is 1, so altogether we get that

$$\begin{aligned} \int_E \pi^*(\rho_\alpha\omega) \wedge \tau &= \int_{U_\alpha \times \mathbb{R}^r} \pi^*(\rho_\alpha\omega) \wedge \tau \\ &= \int_{U_\alpha} \left(\int_{\mathbb{R}^r} \tau \right) i^*\pi^*(\rho_\alpha\omega) \\ &= \int_{U_\alpha} i^*\pi^*(\rho_\alpha\omega) \\ &= \int_{U_\alpha} \rho_\alpha\omega. \end{aligned}$$

Therefore

$$\begin{aligned}\int_E \xi\wedge\tau = \int_E \pi^*(\omega)\wedge\tau &= \sum_\alpha \int_E \pi^*(\rho_\alpha\omega)\wedge\tau \\ &= \sum_\alpha \int_{U_\alpha} \rho_\alpha\omega \\ &= \int_M \omega \\ &= \int_{M_0} i^*\pi^*\omega = \int_{M_0} i^*\xi.\end{aligned}$$

($\Longrightarrow$:) Suppose $\tau\in\Omega^r_c(E)$ is a closed form representing the Poincaré dual of the zero section M_0 in E. For each $x\in M$, let $j_x: E_x\to E$ be the inclusion of the fiber. Define a smooth function $T: M\to\mathbb{R}$ by $T(x)=\int_{E_x} j_x^*\tau$. Claim: T is constant. Given the claim, we can compute what this constant must be. By Poincaré duality let $\xi\in H^m_{dR}(E)$ be the dual of $[\tau]\in H^r_c(E)$, i.e. $\int_E \xi\wedge\tau=1$. As before we can suppose that $\xi=[\pi^*\omega]$ for some closed $\omega\in\Omega^m(M)$, so we have $1=\int_E\pi^*\omega\wedge\tau=\int_{M_0} i^*\pi^*\omega=\int_M\omega$. We can again take a partition of unity $\{\rho_\alpha\}$ on M subject to an open cover with local trivializations, to write

$$1=\int_E \pi^*\omega\wedge\tau=\sum_\alpha\int_{\pi^{-1}(U_\alpha)}\pi^*(\rho_\alpha\omega)\wedge\tau=\sum_\alpha c\int_{U_\alpha}\rho_\alpha\omega=c\int_M\omega,$$

where $c=\int_{E_x}\tau$. Thus, from $1=\int_M\omega$ and $1=c\int_M\omega$, we get $c=1$. Let's now check the claim that the fiberwise integration function $T: M\to\mathbb{R}$ is constant. We can do this locally, in a local trivialization $\pi^{-1}(U_\alpha)\cong U_\alpha\times\mathbb{R}^r$. Let's say that $\mathbf{x}=(x_1,\dots,x_m)$ are local coordinates on U_α and $\mathbf{t}=(t_1,\dots,t_r)$ are local coordinates on $\mathbb{R}^r$. Then we can write $\tau=\sum\limits_{|I|+|J|=r} g_{I,J}(\mathbf{x},\mathbf{t})dx_I\wedge dt_J$ for some functions $g_{I,J}$ that are compactly supported in the $\mathbf{t}$ directions. In particular, we have that $T(\mathbf{x})=\int_{\mathbb{R}^r} j^*\tau=\int_{\mathbb{R}^r} g_{\emptyset,(t_1,\dots,t_r)}(\mathbf{x},\mathbf{t})dt_1\wedge\dots dt_r$. On the other hand, we know that $d\tau=0$, and so we have

$$\begin{aligned}0 &= d\left(\sum_{|I|+|J|=r} g_{I,J}(\mathbf{x},\mathbf{t})dx_I\wedge dt_J\right)\\ &= \sum_{|I|+|J|=r}(dg_{I,J}(\mathbf{x},\mathbf{t}))\wedge dx_I\wedge dt_J.\end{aligned}$$

By linear independence it follows that the coefficient of $dx_k\wedge dt_1\wedge\dots\wedge dt_r$ in the above expression has to be zero, i.e.

$$\begin{aligned}0 &= \frac{\partial g_{\emptyset,(t_1,\dots,t_r)}(\mathbf{x},\mathbf{t})}{\partial x_k}dx_k\wedge dt_1\wedge\dots\wedge dt_r+\sum_{j=1}^r\frac{\partial g_{(x_k),(t_1,\dots,\widehat{t_j},\dots,t_r)}(\mathbf{x},\mathbf{t})}{\partial t_j}dt_j\wedge dx_k\wedge dt_1\wedge\dots\wedge\widehat{dt_j}\wedge\dots\wedge dt_r\\ &= \underbrace{\left(\frac{\partial g_{\emptyset,(t_1,\dots,t_r)}(\mathbf{x},\mathbf{t})}{\partial x_k}+\sum_{j=1}^r(-1)^j\frac{\partial g_{(x_k),(t_1,\dots,\widehat{t_j},\dots,t_r)}(\mathbf{x},\mathbf{t})}{\partial t_j}\right)}_{=0}dx_k\wedge dt_1\wedge\dots\wedge dt_r\end{aligned}$$

Now we can show that $T(\mathbf{x})$ is constant in these local coordinates. We have, for each direction x_k, the partial derivative

$$\begin{aligned}\frac{\partial T}{\partial x_k} &= \frac{\partial}{\partial x_k}\int_{\mathbb{R}^r} g_{\emptyset,(t_1,\dots,t_r)}(\mathbf{x},\mathbf{t})dt_1\dots dt_r\\ &= \int_{\mathbb{R}^r}\frac{\partial g_{\emptyset,(t_1,\dots,t_r)}(\mathbf{x},\mathbf{t})}{\partial x_k}dt_1\dots dt_r\\ &= \sum_{j=1}^r\pm1\int_{\mathbb{R}^r}\frac{\partial g_{(x_k),(t_1,\dots,\widehat{t_j},\dots,t_r)}(\mathbf{x},\mathbf{t})}{\partial t_j}dt_1\dots dt_r\\ &= 0\end{aligned}$$

The last equality is just the fundamental theorem of calculus – all the functions $g_{(x_k),(t_1,\ldots,\widehat{t_j},\ldots,t_r)}(\mathbf{x},\mathbf{t})$ are compactly supported in the t_i directions. So for each j, $\int_{\mathbb{R}} \frac{\partial g_{(x_k),(t_1,\ldots,\widehat{t_j},\ldots,t_r)}(\mathbf{x},\mathbf{t})}{\partial t_j} dt_j = g_{(x_k),(t_1,\ldots,\widehat{t_j},\ldots,t_r)}(\mathbf{x},\mathbf{t})\big|_{-\infty}^{+\infty} = 0 - 0 = 0$, and it follows that the iterated integral $\int_{\mathbb{R}^r} \frac{\partial g_{(x_k),(t_1,\ldots,\widehat{t_j},\ldots,t_r)}(\mathbf{x},\mathbf{t})}{\partial t_j} dt_1 \ldots dt_r$ is also zero. $\square$

Lemma. *Let $X \to S$, $Y \to S$ be two vector bundles, where X, Y, S are all orientable manifolds and S is also compact. Let $X \oplus Y \xrightarrow{\pi} S$ be their direct sum, i.e. for each $p \in S, \pi^{-1}(p) = X_p \oplus Y_p$. Let τ_1 be a Thom form for S in X, and τ_2 a Thom form for S in Y. Then $\tau_1 \wedge \tau_2$ is a Thom form for S in $X \oplus Y$.*

Proof. Given the previous lemma it's enough to check that $\tau_1 \wedge \tau_2$ is a compactly supported closed form and that $\int_{\mathbb{R}^{r_1+r_2}} \tau_1 \wedge \tau_2 = 1$. But the support of $\tau_1 \wedge \tau_2$ is the intersection of two compact sets, hence compact, and $d(\tau_1 \wedge \tau_2) = d\tau_1 \wedge \tau_2 \pm \tau_1 \wedge d\tau_2 = 0$ so it's closed. Moreover, $\int_{\mathbb{R}^{r_1+r_2}} \tau_1 \wedge \tau_2 = \left(\int_{\mathbb{R}^{r_1}} \tau_1\right)\left(\int_{\mathbb{R}^{r_2}} \tau_2\right) = 1 \cdot 1 = 1$. $\square$

3.3. **Intersection numbers.** In particular, if S and $\widetilde{S}$ are compact and have complementary dimension in M and intersect each other transversely, their intersection $S \cap \widetilde{S}$ is a compact zero dimensional manifold, i.e. a finite collection of points in M. If both S and $\widetilde{S}$ are oriented, then we can define an orientation at each intersection point (i.e. a plus or minus sign) by taking, at each $x \in S \cap \widetilde{S}$, a positively oriented basis β for $T_x S$ and a positively oriented basis $\widetilde{\beta}$ for $\widetilde{S}$, and taking the sign of the ordered basis $\{\beta, \widetilde{\beta}\}$ of $T_x M$. Therefore, for $\omega = 1 \in \Omega^0(M)$, we have

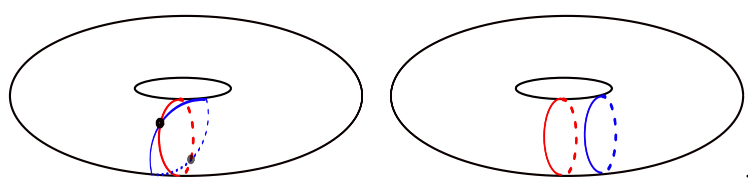
$$\int_M \eta_S \wedge \eta_{\widetilde{S}} = \int_M \eta_{S \cap \widetilde{S}} = \int_{S \cap \widetilde{S}} 1 = \sum_{x \in S \cap \widetilde{S}} \pm 1$$

where the sign is determined by the orientation of $x \in S \cap \widetilde{S}$. In particular, this quantity is an integer, and moreover, since the quantity $\int_M \eta_S \wedge \eta_{\widetilde{S}}$ is invariant under isotopies of S and $\widetilde{S}$, so is the right hand side.

Definition 16. *The* intersection number $I(S, \widetilde{S})$ *of two compact, oriented submanifolds of M with complementary dimension in M is defined as follows: take any isotopy ϕ of $\widetilde{S}$ such that the intersection $S \cap \widetilde{S}$ is transversal, and set $I(S, \widetilde{S}) = \sum\limits_{x \in S \cap \phi(\widetilde{S})} \pm 1$ with the sign determined by the oriented intersection.*

This definition allows us to make sense of $I(S, \widetilde{S})$ for any pair of compact oriented submanifolds of complementary dimension in M, i.e. whether or not they intersect transversely. For example, we can define the self-intersection number $I(S, S)$ for a compact oriented submanifold of exactly half the dimension of M.

Example 19. Let S be the meridian on the torus (in red below). We compute $I(S, S)$ (with respect to some orientation on S) by doing a small perturbing isotopy $\phi(S)$ of S so that $\phi(S)$ intersects S transversely, and counting oriented intersection numbers of $S \cap \phi(S)$. We have drawn below two possible isotopies of S, in blue. Verify that they give the same result, $I(S, S) = 0$.



Exercise 15. *Show that $I(S, \widetilde{S}) = (-1)^{\dim S \dim \widetilde{S}} I(\widetilde{S}, \widetilde{S})$.*

3.4. **Euler number and Euler characteristic.** Given a compact, oriented manifold M, Let $M \times M$ be the product manifold. The product contains a submanifold $\triangle = \{(x, x) | x \in M\}$ called the *diagonal*, which is canonically isomorphic to M. Let $\eta_\triangle \in H^k_{dR}(M \times M)$ be the Poincaré dual of $\triangle$ in $M \times M$.

Definition 17. *The integer $I(\triangle, \triangle)$ is called the* Euler number *of M.*

Proposition (Euler number = Euler characteristic)**.** *$I(\triangle, \triangle) = \chi(M) = \sum\limits_q (-1)^q \dim H^q_{dR}(M)$.*

Proof. We have $I(\triangle, \triangle) = \int_{M \times M} \eta_\triangle \wedge \eta_\triangle = \int_\triangle \eta_\triangle$, and we will directly compute this final integral. Let $\pi_1 : M \times M \to M$ and $\pi_2 : M \times M \to M$ be the two projections from $M \times M$ to the first and second factor respectively. Let us first work out what $\eta_\triangle \in H^k_{dR}(M \times M)$ is, by combining what we know from the Kunneth theorem, $H^k_{dR}(M \times M) = \bigoplus\limits_{a=0}^{k} H^a_{dR}(M) \otimes H^{k-a}_{dR}(M)$, and what we know from Poincaré duality,

$H^{k-a}(M) = (H^a(M))^*$. Let $\{\omega_{a,i}\}$ be a basis for the vector space $H^a_{dR}(M)$, and let $\{\tau_{k-a,i}\}$ be a basis for the vector space $H^{k-a}_{dR}(M))$ that is dual to the basis $\{\omega_{a,i}\}$ i.e. $\int_M \omega_{a,i} \wedge \tau_{k-a,j} = \delta_{ij}$. So by the Kunneth theorem a basis for $H^k_{dR}(M \times M)$ is $\{\pi_1^*\omega_{a,i} \wedge \pi_2^*\tau_{k-a,j}\}_{a,i,j}$, so $\eta_\triangle = \sum\limits_{a,i,j} c_{a,i,j}\pi_1^*\omega_{a,i} \wedge \pi_2^*\tau_{k-a,j}$ for coefficients $c_{a,i,j} \in \mathbb{R}$.

To work out what the coefficients $c_{a,i,j}$ we essentially do a projection onto the factor $\pi_1^*\omega_{a,i} \wedge \pi_2^*\tau_{k-a,j}$ by using the bilinear pairing with its dual, $\pi_1^*\tau_{k-a,i} \wedge \pi_2^*\omega_{k-a,j}$.

On the one hand,

$$\int_\triangle \pi_1^*\tau_{k-a,i} \wedge \pi_2^*\omega_{a,j} = (-1)^{a(k-a)} \int_M \omega_{a,j} \wedge \tau_{k-a,i} = (-1)^{a(k-a)}\delta_{ij}$$

where the first equality is possible because the projections $\pi_1 : \triangle \to M$ and $\pi_2 : \triangle \to M$ are isomorphisms fitting into a commutative diagram

$$\begin{tikzcd} & \triangle \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ M \arrow[rr, "\mathrm{Id}"] & & M \end{tikzcd}$$

and therefore, $\int_\triangle \pi_1^*\tau_{k-a,i} \wedge \pi_2^*\omega_{a,j} = \int_M (\pi_2^{-1})^*\pi_1^*\tau_{k-a,i} \wedge (\pi_2^{-1})^*\pi_2^*\omega_{a,j} = \int_M \tau_{k-a,i} \wedge \omega_{a,j}$.

On the other hand,

$$\begin{aligned} \int_\triangle \pi_1^*\tau_{k-a,i} \wedge \pi_2^*\omega_{a,j} &= \int_{M\times M} \pi_1^*\tau_{k-a,i} \wedge \pi_2^*\omega_{a,j} \wedge \eta_\triangle \\ &= \sum_{\alpha,m,n} c_{\alpha,m,n} \int_{M\times M} \pi_1^*\tau_{k-a,i} \wedge \pi_2^*\omega_{a,j} \wedge \pi_1^*\omega_{\alpha,m} \wedge \pi_2^*\tau_{k-\alpha,n} \\ &= \sum_{\alpha,m,n} c_{\alpha,m,n}(-1)^{a\alpha} \int_{M\times M} \pi_1^*\tau_{a,i} \wedge \pi_1^*\omega_{\alpha,m} \wedge \pi_2^*\omega_{k-a,j} \wedge \pi_2^*\tau_{k-\alpha,n} \\ &= \sum_{\alpha,m,n} c_{\alpha,m,n}(-1)^{a\alpha} \int_{M\times M} \pi_1^*(\tau_{k-a,i} \wedge \omega_{\alpha,m}) \wedge \pi_2^*(\omega_{a,j} \wedge \tau_{k-\alpha,n}) \end{aligned}$$

Now for $\tau_{k-a,i} \wedge \omega_{\alpha,m} \neq 0$ we need $k - a + \alpha \leq k$, and for $\omega_{a,j} \wedge \tau_{k-\alpha,n} \neq 0$ we need $a + k - \alpha \leq k$, from which we see that the only way for both to be non-zero is for $\alpha = a$. So continuing from above, we get

$$\begin{aligned} &= \sum_{m,n} c_{a,m,n}(-1)^{a^2} \int_{M\times M} \pi_1^*(\tau_{k-a,i} \wedge \omega_{a,m}) \wedge \pi_2^*(\omega_{a,j} \wedge \tau_{k-a,n}) \\ &= \sum_{m,n} c_{a,m,n}(-1)^{a^2} \left(\int_M \tau_{k-a,i} \wedge \omega_{a,m}\right)\left(\int_M \omega_{a,j} \wedge \tau_{k-a,n}\right) \\ &= \sum_{m,n} c_{a,m,n}(-1)^{a^2}(-1)^{a(k-a)}\delta_{im}\delta_{jn} \\ &= c_{a,i,j}(-1)^{a^2}(-1)^{a(k-a)}. \end{aligned}$$

so combining we have $(-1)^{a(k-a)}\delta_{ij} = c_{a,i,j}(-1)^{a^2}(-1)^{a(k-a)} \implies c_{a,i,j} = (-1)^{a^2}\delta_{ij} = (-1)^a\delta_{ij}$ since $a^2 \mod 2 = a \mod 2$.

So now we can compute

$$\begin{aligned}
I(\triangle,\triangle) &= \int_{M\times M} \eta_\triangle \wedge \eta_\triangle \\
&= \int_\triangle \eta_\triangle \\
&= \int_\triangle \sum_{a,i} (-1)^a \pi_1^*(\omega_{a,i}) \wedge \pi_2^*(\tau_{k-a,i}) \\
&= \sum_{a,i} (-1)^a \int_\triangle \pi_1^*(\omega_{a,i}) \wedge \pi_2^*(\tau_{k-a,i}) \\
&= \sum_{a,i} (-1)^a \int_M \omega_{a,i} \wedge \tau_{k-a,i} \\
&= \sum_a (-1)^a \sum_i 1 \\
&= \sum_a (-1)^a \dim H^a_{dR}(M) \\
&= \chi(M).
\end{aligned}$$

□

3.5. **Vector fields and the Poincaré-Hopf index theorem.**

Definition 18. *A vector field on M is a section of the tangent bundle $TM \xrightarrow{\pi} M$. In other words, it's a map $\Gamma : M \to TM$ such that $\pi \circ \Gamma$ is the identity map on M.*

Or you can think of it as a smooth function $\Gamma : M \to TM$ of the form $x \mapsto (x, v(x))$ for some $v(x) \in T_xM$.

Let M be a compact oriented manifold, and Γ a vector field on M. We suppose that Γ has isolated zeros, i.e. for all $x \in M$ such that $\Gamma(x) = (x, 0)$, there is a small neighborhood of x in which the only 0 of Γ is x. Therefore, for a sufficiently small ball B containing x, $\Gamma|_{\partial B}$ is not zero, so for each $p \in \partial B$ we can define a unit vector $\Gamma(p)/\|\Gamma(p)\| \in S^{k-1}$. Since $\partial B \cong S^{k-1}$, we get a map $f : S^{k-1} \to S^{k-1}$. This map f has a degree (defined earlier in the course – pick a regular value in S^{k-1}, and count the points in $f^{-1}(a)$ with signs according to whether f is orientation preserving or reversing at that point) .

Definition 19. *The local index of the zero x is defined to be the integer* $\operatorname{ind}(x) := \deg f$ *computed for a sufficiently small ball B containing x.*

Exercise 16. *Why is this local index number well-defined?*

Definition 20. *Define the index of Γ by* $\operatorname{ind}(\Gamma) := \sum_x \operatorname{ind}(x)$ *where the sum is over all zeros x of* Γ.

Theorem (Poincaré-Hopf index theorem)**.** *Let M be compact and oriented, and let $\Gamma : M \to TM$ be a vector field on M with isolated zeros. Then* $\operatorname{ind}(\Gamma) = \chi(M)$.

Proof. This is the outline of the ingredients in the proof:

(1) Show that Γ can be replaced with another vector field $\widehat{\Gamma}$ such that $\operatorname{ind}(\widehat{\Gamma}) = \operatorname{ind}(\Gamma)$, and $\widehat{\Gamma}$ has isolated zeros and $d\widetilde{\Gamma}_x$ is invertible at each zero, and the index at each zero is either 1 or -1 depending on whether $d\widetilde{\Gamma}_x$ preserves orientation or changes orientation.
(2) Show that for such a $\widehat{\Gamma}$, if we fix a Riemannian metric on M and use it to exponentiate the vector field, i.e. define an isomorphism $\Phi_{\widehat{\Gamma}} : M \to M$ by $\Phi_{\widehat{\Gamma}}(p) = \exp_p(\widehat{\Gamma}(p))$, then $\Phi_{\widehat{\Gamma}}$'s fixed points are precisely at the zeros of $\widehat{\Gamma}$, and the intersection $\triangle \cap \operatorname{Gr} \Phi_{\widehat{\Gamma}}$ is transverse.
(3) Show that the sign of the intersection $T_{(x,x)}\triangle \oplus T_{(x,x)} \operatorname{Gr}(\Phi_{\widehat{\Gamma}})$ at the point (x, x) is precisely the index of x in $\widehat{\Gamma}$.

Once we have these steps, we get the result, for $\operatorname{ind}\Gamma = \operatorname{ind}\widehat{\Gamma} = I(\triangle, \Phi_{\widehat{\Gamma}}) = I(\triangle,\triangle) = \chi(M)$, where $I(\triangle, \operatorname{Gr}(\Phi_{\widehat{\Gamma}})) = I(\triangle,\triangle)$ by isotopy invariance, using the fact that $\triangle = \operatorname{Gr}(\mathrm{Id})$ and $\Phi_{\widehat{\Gamma}}$ is isotopic to the identity map on M by the isotopy $\Phi^t_{\widehat{\Gamma}}(p) = \exp_p(t\Gamma(p))$.

□

Here is a proof of the individual ingredients.

Given a vector field Γ with isolated zeros, the idea is that we can make changes very locally, in a small neighborhood of each zero, and not change the index. Let x be an isolated zero of Γ. First let us characterize what it means for the index of Γ to be ± 1.

Lemma. *Let p be a zero of Γ_p. Then if $d\Gamma_p$ is invertible at p, the index of Γ at p is $+1$ if $d\Gamma_p$ preserves orientation, and -1 if $d\Gamma_p$ reverses orientation.*

Proof. Working in a local chart about x, we can take Γ to be a vector field on a neighborhood of the origin in $\mathbb{R}^k$, with an isolated zero at the origin. By definition, the index is the degree of the map on a sufficiently small ball

$$f : \partial B^k \to S^k$$

given by $f(z) = \Gamma(z)/\|\Gamma(z)\|$. For small z we can write $\Gamma(z) = \Gamma(0) + d\Gamma_0 z + r(z) = d\Gamma_0 z + r(z)$ where $r(z)$ is a small quadratic term, and $\|r(z)\|/\|z\| \to 0$ as $z \to 0$. In particular, given any $\epsilon > 0$, for small enough z we can guarantee that $\|\Gamma(z) - d\Gamma_0 z\| = \|r(z)\| \leq \epsilon\|z\|$, so that in particular, for any $s \in [0,1]$ we have that $\|d\Gamma_0 z + sr(z)\| \geq \|d\Gamma_0 z\| - \|sr(z)\| \geq \|d\Gamma_0 z\| - \epsilon\|z\|$.

Suppose then that $d\Gamma_0$ is invertible. Then there exists some $\delta > 0$ such that $\|d\Gamma_0 z\| \geq \delta\|z\|$, so $\|d\Gamma_0 z\| - \epsilon\|z\| \geq \delta\|z\| - \epsilon\|z\|$ so as long as we choose ϵ smaller than δ, we get $\|d\Gamma_0 z + sr(z)\| > 0$ for $z \neq 0$, z sufficiently small.

This means that we can define a homotopy $f_s(z) : \partial B^k \to S^k$ by

$$f_s(z) = \frac{d\Gamma_0(z) + sr(z)}{\|d\Gamma_0(z) + sr(z)\|}$$

which is a homotopy between $f_0(z) = \frac{d\Gamma_0 z}{\|d\Gamma_0 z\|}$ and $f_1(z) = f(z)$ as before. By homotopy invariance of the degree, we see that the index of f at 0 is the same as the index of f_0 at 0. Now $d\Gamma_0$ is just an invertible matrix, so $d\Gamma_0 \in GL_k(\mathbb{R})$. The two components of $GL_k(\mathbb{R})$, which correspond to $\det > 0$ and $\det < 0$, are path connected. So if the determinant of $d\Gamma_0$ is positive, $d\Gamma_0$ is homotopic to the identity I, and if $\det d\Gamma_0$ is negative, then it is homotopic to the diagonal matrix E with diagonal entries $(-1, 1, \ldots, 1)$, which represents a single reflection through the x_1 axis. Therefore, taking a path $M_t \in GL_n(\mathbb{R})$ such that $M_0 = d\Gamma_0$ and $M_1 = I$ or E according to which component of $GL_k(\mathbb{R})$, we have again by the homotopy invariance of the degree, that $\deg \frac{M_t z}{\|M_t z\|}$ is constant for all $t \in [0,1]$. Consequently,

$$\deg f_0 = \deg \frac{d\Gamma_0 z}{\|d\Gamma_0 z\|} = \deg \frac{z}{\|z\|} = 1$$

if $\det d\Gamma_0 z > 0$, and otherwise, if $\det d\Gamma_0 < 0$,

$$\deg f_0 = \deg \frac{d\Gamma_0 z}{\|d\Gamma_0 z\|} = \deg \frac{Ez}{\|Ez\|} = -1.$$

□

Now let's show how Step 1 can be achieved. We will modify Γ in a neighborhood of x by taking a small compactly supported bump function $\rho : B_x \to [0,1]$, and some element $a \in B_x$, and putting

$$\widetilde{\Gamma}(x) = \Gamma(x) + \rho(x)a.$$

Then $\widetilde{\Gamma}(x) = 0 \iff \Gamma(x) = \rho(x)a$. $\|\Gamma(x)\| = \rho(x)\|a\| \leq \|a\|$, this means that the zeros of $\widetilde{\Gamma}(x)$ are contained in the neighborhood of x where $\|\Gamma(x)\| \leq \|a\|$. If we choose $\|a\|$ to be very small, we can choose the function $\rho(x)$ to be compactly supported in a neighborhood U of x in which Γ has no other zeros, and to satisfy $\rho(x) = 1$ on a tiny neighborhood V containing the zeros of $\widetilde{\Gamma}(x)$. Thus the zeros of $\widetilde{\Gamma}(x)$ correspond to the points where $\Gamma(x) = a$. By definition, the degree of Γ at x is the number of these zeros counted with signs according to whether Γ preserves or changes the orientation. Now by Sard's theorem, the critical values of Γ have measure zero, and therefore, in any neighborhood of x, we can find a regular value of Γ, so we choose a very small a such that a is regular. This means that for every x such that $\Gamma(x) = a$, we have that $d\Gamma_x$ is invertible. Therefore, since at such an x we have $d\rho = 0$, we have $d\widetilde{\Gamma}_x = d\Gamma_x$ and so $d\widetilde{\Gamma}$ is invertible, so each zero has degree 1 or -1, according to whether $d\Gamma_x$ is orientation preserving or orientation reversing. Therefore, the total index of the zeros of $\widetilde{\Gamma}$ in this neighborhood of x is identical to the index of x as a zero of Γ. Moreover, outside of this neighborhood $\rho(x) = 0$ and the zeros of Γ are the same as the zeros of $\widetilde{\Gamma}$. Hence $\operatorname{ind}\Gamma = \operatorname{ind}\widetilde{\Gamma}$. Therefore, if we do this local procedure at every zero of Γ where $d\Gamma$ is not invertible, we can replace Γ with a vector field $\widetilde{\Gamma}$ such that $d\widetilde{\Gamma}$ is invertible at every 0, and such that $\operatorname{ind}\Gamma = \operatorname{ind}\widetilde{\Gamma}$.

Now let us exponentiate $\widetilde{\Gamma}$, to get a function $\Phi_{\widetilde{\Gamma}} : M \to M$ given by $x \mapsto \exp_x(\widetilde{\Gamma}(x))$. We want to relate the intersection number of $\operatorname{Gr}\Phi_{\widetilde{\Gamma}} \cap \triangle$ with the index of $\widetilde{\Gamma}$.

First we want to understand what a point of transverse intersection of $\triangle \cap \operatorname{Gr} f$ looks like. Remember that the intersection $\triangle \cap \operatorname{Gr} f$ consists of points such that $(x, x) = (x, f(x))$, i.e. $f(x) = x$, i.e. fixed points of f.

Lemma. *Let $f : M \to M$ be a smooth function. Let x be an isolated fixed point, $f(x) = x$, and consider $(x, x) \in \triangle \cap \operatorname{Gr}(f)$. Then the intersection $\triangle \cap \operatorname{Gr}(f)$ is* transverse *at (x, x) if and only if $(df - I)_x$ is invertible, and the sign of the intersection at (x, x) is equal to the sign of the determinant of $(df - I)_x$.*

Proof. Suppose that $x \in M$ is a fixed point of f. Let $v_1, \ldots, v_n$ be a positively oriented basis for T_xM. A positively oriented basis for $\triangle$ at (x, x) is therefore

$$\left\{ \begin{bmatrix} v_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} v_2 \\ v_2 \end{bmatrix}, \ldots, \begin{bmatrix} v_n \\ v_n \end{bmatrix} \right\}$$

and a positively oriented basis for $\operatorname{Gr} f$ at (x, x) is

$$\left\{ \begin{bmatrix} v_1 \\ df_x v_1 \end{bmatrix}, \begin{bmatrix} v_2 \\ df_x v_2 \end{bmatrix}, \ldots, \begin{bmatrix} v_n \\ df_x v_n \end{bmatrix} \right\}$$

The intersection is transverse if and only if the collection

$$\left\{ \begin{bmatrix} v_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} v_2 \\ v_2 \end{bmatrix}, \ldots, \begin{bmatrix} v_n \\ v_n \end{bmatrix}, \begin{bmatrix} v_1 \\ df_x v_1 \end{bmatrix}, \begin{bmatrix} v_2 \\ df_x v_2 \end{bmatrix}, \ldots, \begin{bmatrix} v_n \\ df_x v_n \end{bmatrix} \right\} \tag{1}$$

forms a basis for $T_{(x,x)}M \times M$, in other words if and only if

$$\det \begin{pmatrix} v_1 & v_2 & \ldots & v_n & v_1 & v_2 & \ldots & v_n \\ v_1 & v_2 & \ldots & v_n & df_x v_1 & df_x v_2 & \ldots & df_x v_n \end{pmatrix} \neq 0.$$

The sign of the intersection is positive if and only if the basis (1) is positively oriented, in other words if and only if

$$\det \begin{pmatrix} v_1 & v_2 & \ldots & v_n & v_1 & v_2 & \ldots & v_n \\ v_1 & v_2 & \ldots & v_n & df_x v_1 & df_x v_2 & \ldots & df_x v_n \end{pmatrix} > 0,$$

and the sign of the intersection is negative if and only if the basis (1) is negatively oriented, in other words if and only if

$$\det \begin{pmatrix} v_1 & v_2 & \ldots & v_n & v_1 & v_2 & \ldots & v_n \\ v_1 & v_2 & \ldots & v_n & df_x v_1 & df_x v_2 & \ldots & df_x v_n \end{pmatrix} < 0.$$

Let's write $A = (v_1 v_2 \ldots v_n)$, i.e. the $n \times n$ matrix whose columns are the oriented basis for T_xM. Then the matrix above is really just

$$\begin{pmatrix} A & A \\ A & df_x A \end{pmatrix} = \begin{pmatrix} I & 0 \\ I & I \end{pmatrix} \begin{pmatrix} A & A \\ 0 & (df_x A - A) \end{pmatrix}$$

so

$$\begin{aligned} \det \begin{pmatrix} A & A \\ A & df_x A \end{pmatrix} &= \det \begin{pmatrix} I & 0 \\ I & I \end{pmatrix} \det \begin{pmatrix} A & A \\ 0 & (df_x A - A) \end{pmatrix} \\ &= \det(I)\det(I)\det(A)\det(df_x A - A) \\ &= \det(A)\det((df_x - I)A) \\ &= \det(A)^2 \det(df_x - I). \end{aligned}$$

Since $\det A \neq 0$ we see that the invertibility and sign of the determinant are completely determined by the invertibility and sign of the determinant of $df_x - I$. $\square$

Using this we can now finish steps 2 and 3. Having defined the map $\Phi_{\widetilde{\Gamma}} : M \to M$ by $\Phi_{\widetilde{\Gamma}}(x) = \exp_x(\widetilde{\Gamma}(x))$, the fixed points of $\Phi_{\widetilde{\Gamma}}$ are exactly the points at which $\widetilde{\Gamma}(x) = 0$. The previous lemma tells us that the intersection $\triangle \cap \operatorname{Gr}\Phi_{\widetilde{\Gamma}}$ is transverse if and only if $d\Phi_{\widetilde{\Gamma}} - I$ is invertible, with the sign of the intersection determined by the sign of $d\Phi_{\widetilde{\Gamma}} - I$.

The final step is to show that $d\Phi_{\widetilde{\Gamma}} = I + d\widetilde{\Gamma}$. Let $\sigma : (-\epsilon, \epsilon) \to M$ be such that $\sigma(0) = x, \sigma'(0) = \mathbf{v}$. Then, by definition of the differential, we have

$$(d\Phi_{\widetilde{\Gamma}})_x(\mathbf{v}) = \frac{d}{dt}\Phi_{\widetilde{\Gamma}}(\sigma(t)) = \frac{d}{dt}\exp_{\sigma(t)}(\widetilde{\Gamma}(\sigma(t))) = (d\exp)_{(x,0)}(\mathbf{v}, d\widetilde{\Gamma}_x\mathbf{v}).$$

The last equality is because $\exp_p(\gamma)$ is a function of two variables, p and γ. We can write the differential $d\exp_{(x,0)}(\mathbf{v}, d\widetilde{\Gamma}_x\mathbf{v}) = D^1\mathbf{v} + D^2 d\widetilde{\Gamma}_x\mathbf{v}$ where D^1 is the differential of exp with respect to p (i.e. fixing γ) and D^2 is the differential of exp with respect to γ. In our case $D^1\mathbf{v} = \mathbf{v}$ because fixing $\gamma = \widetilde{\Gamma}(x) = 0$, the exponential map is the identity map. And in our case $D^2 d\Gamma_x\mathbf{v} = d\Gamma_x\mathbf{v}$, because the derivative $\frac{d}{dt}|_{t=0}\exp_p(\gamma(t))$ for fixed p is $\gamma'(0)$, by a defining property of the exponential map. Therefore $(d\Phi_{\widetilde{\Gamma}})_x(\mathbf{v}) = \mathbf{v} + d\widetilde{\Gamma}_x\mathbf{v}$, showing that in fact $(d\Phi_{\widetilde{\Gamma}})_x = I + d\widetilde{\Gamma}_x$. Now combining everything, we have all the steps 1, 2 and 3.

3.6. **Gauss-Bonnet.** As an application of the Poincaré-Hopf index theorem let us prove the Gauss-Bonnet theorem. Let $S \subset \mathbb{R}^{k-1}$ be a compact and oriented hypersurface. Being oriented, it has a normal vector field, i.e. a section $N : S \to T\mathbb{R}^k|_S$, defined such that $N(p) \in T_p\mathbb{R}^{k+1}S^{k-1}$ is the unique unit vector such that for a positively oriented ordered basis $\xi_1, \dots, \xi_k$ for T_pS, the $(k+1)$-tuple $\{N(p), \xi_1, \dots, \xi_k\}$ is a positively oriented ordered basis of $T_p\mathbb{R}^k$.

The *Gauss map* for the surface is the map $g : S \to S^k$ given by $g(p) = N(p)$.

The *Gaussian curvature* of the surface is the function $\kappa = \det dg : S \to \mathbb{R}$, which is a measure of how much the normal vector field is changing at a given point. (The more curved the surface the larger the changes in the normal vector field.)

Theorem. *Let $S \subset \mathbb{R}^{k+1}$ be a compact, oriented hypersurface, where k is even. Then*

$$\int_S \kappa \, d\,\mathrm{Vol}_S = \frac{1}{2}\chi(S)\gamma_k$$

where $\gamma_k = \int_{S^k} d\,\mathrm{Vol}_{S^k}$ is the volume of the unit sphere of dimension k with respect to the standard volume form.

Proof. Let $g : S \to S^k$ be the Gauss map. Since S and S^k are both compact and oriented of the same dimension, we have the Degree Formula:

$$\int_S g^*(d\,\mathrm{Vol}_{S^k}) = (\deg g)\int_{S^k} d\,\mathrm{Vol}_{S^k}\,.$$

and in fact, $g^*(d\,\mathrm{Vol}_{S^k}) = \det(dg)d\,\mathrm{Vol}_S = \kappa \; d\,\mathrm{Vol}_S$, and $\gamma_k = \int_{S^k} d\,\mathrm{Vol}_{S^k}$ so we just need to show that $\deg g = \frac{1}{2}\chi(S)$.

We will prove this identity by constructing a vector field Γ such that $\operatorname{ind}\Gamma = 2\deg g$. Then the Poincaré-Hopf index theorem says $\operatorname{ind}\Gamma = \chi(S)$.

Let $a \in S^k$ be a regular value of g, and suppose that $-a$ is also a regular value of g. (By Sard's theorem, the set $\mathrm{Crit}(g) \subset S^k$ has measure 0. A regular value $a \in S^k$ is in the complement of $\mathrm{Crit}(g)$. If $-a$ is also regular it mean that $-a$ is in the complement of $\mathrm{Crit}(g)$, or equivalently a is in the complement of $-\mathrm{Crit}(g)$. So an a satisfying the condition is in the complement of $\mathrm{Crit}(g) \cup (-\mathrm{Crit}(g))$. In any case it means that such an a is in the complement of a set of measure zero, in other words the set $a \in S^k$ for which both a and $-a$ are regular are dense in S^k.)

Now we define a vector field on S as follows: project the vector a onto T_pS at all $p \in S$. In other words, subtract the component of a in the normal direction:

$$\Gamma(p) = a - \langle a, g(p)\rangle g(p).$$

What are the zeros of Γ: $\Gamma = 0 \iff a - \langle a, g(p)\rangle g(p) = 0 \iff \langle a, g(p)\rangle g(p) = a \iff g(p) = \pm a$. Since by hypothesis on a, these are regular values of g, this means that g is a local isomorphism about each zero, and in particular the zeros are isolated. Furthermore

$$d\Gamma_p = -\langle a, dg_p\rangle g_p - \langle a, g_p\rangle dg_p$$

and we claim that the first term on the right is zero. Let $\alpha(t)$ be a cuve for which $\alpha(0) = p, \alpha'(0) = \mathbf{v}$. Then $\langle g(\alpha(t)), g(\alpha(t))\rangle = 1 \forall t$ so that, differentiating, we see that $0 = 2\langle g(\alpha(0)), dg_{\alpha(0)}\mathbf{v}\rangle$ and so $0 = \langle g(p), dg_p\mathbf{v}\rangle = \langle a, dg_p\mathbf{v}\rangle$. So the first term in $d\Gamma_p$ is zero, and we have

$$d\Gamma_p = -\langle a, g_p\rangle dg_p$$

Now since $g_p = \pm a$, and $\langle a, a\rangle = 1$, we see $d\Gamma_p = -dg_p$ if $g(p) = a$, and $d\Gamma_p = dg_p$ if $g(p) = -a$. Since by regularity we know that dg_p is invertible for all $p \in g^{-1}(\pm a)$, we see from this that the differential $d\Gamma_p$ is invertible for all $p \in g^{-1}(\pm a)$, i.e. $d\Gamma_p$ is invertible at all the zeros of Γ_p.

□

4. Singular homology, cohomology, fundamental group

Reference for this part: Allen Hatcher, Algebraic Topology. Available as a free pdf download from his webpage. Chapter 2: Singular homology, Chapter 3: Singular cohomology, Chapter 1: Fundamental group.

4.1. **Simplicial homology.** Definition of n-simplex, definition of a $\triangle$-complex on a topological space X, definition of simplicial chain groups $C_k^{\triangle}(X)$ using coefficients in $\mathbb{Z}$, boundary operator $\partial : C_k^{\triangle}(X) \to C_{k-1}^{\triangle}(X)$, $\partial \circ \partial = 0$, simplicial homology $H_k^{\triangle}(X)$. Small dimensional computations: $\mathbb{R}P^2, T^2, S^2, S^1$, Klein bottle.

4.2. **Singular homology.** Definition of singular n-simplex, definition of singular homology, definition of reduced singular homology. Singular homology of a point. Homotopy invariance of singular homology. Examples: the singular homology of $\mathbb{R}^n$ (or of any open ball) is equal to the homology of a point. The singular homology of the punctured plane (or of any cylinder) is equal to the singular homology of the circle.

Definition 21. *A is a deformation retract of X if $A \subset X$ and there exists a continuous function $r : X \to A$ such that*

- *$r|_A$ is the identity map on A,*
- *r is homotopic to the identity map on X.*

If A is a deformation retract of X then $H_k(A) \cong H_k(X)$ for all k.

4.3. **Relative homology.** Relative chain complex for $A \subset X$, $C_k(X, A) = C_k(X)/C_k(A)$ and relative homology $H_k(X, A)$, long exact sequence for relative homology.

4.4. **Useful properties.** Equivalence of singular homology and simplicial homology, Mayer-Vietoris long exact sequence for singular homology,

$$\ldots \to H_k(U \cap V) \to H_k(U) \oplus H_k(V) \to H_k(U \cup V) \to H_{k-1}(U \cap V) \to \ldots$$

Theorem (Excision theorem)**.** *Given $Z \subset A \subset X$ such that the closure of Z is contained in the interior of A, we have $H_k(X - Z, A - Z) \cong H_k(X, Z)$. Equivalently, given $A \subset X$ and $B \subset X$ such that the interiors of A and B cover all of X, then $H_k(B, A \cap B) \cong H_k(X, A)$.*

Example: Let $n \geq 1$. $H_k(S^n) = \mathbb{Z}$ if $k = 0$ or n, and is 0 otherwise.

Theorem (Brouwer's fixed point theorem)**.** *For $n \geq 1$ let $B^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | \sum x_i^2 \leq 1\}$ be the closed unit ball in n dimensional space. Let $f : B^n \to B^n$ be a continuous function. Then f must have a fixed point. i.e., $\exists x \in B^n$ such that $f(x) = x$.*

Proof. By contradiction: if there were no fixed point we could define a deformation retract $r : B^n \to \partial B^n$ by putting $r(x) = \frac{f(x)-x}{\|f(x)-x\|}$. If $n \geq 2$ then we have $\partial B^n \cong S^{n-1}$, so we should have $H_k(S^{n-1}) \cong H_k(B^n)$ for all k, but we know that $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$ while $H_{n-1}(B^n) = 0$, so this is impossible. If $n = 1$ then ∂B^1 is the boundary of an interval which has two points, and therefore $H_0(\partial B^1) \cong \mathbb{Z} \oplus \mathbb{Z}$, but $H_0(B^1) \cong \mathbb{Z}$, so again this is impossible. □

4.5. **Cohomology.** Definition of cochain groups $C^k(X)$, definition of coboundary operator $\delta : C^k(X) \to C^{k+1}(X)$, cohomology groups $H^k(X)$, multiplication of cohomology groups, $H^*(X) = \bigoplus_{k \geq 0} H^k(X)$ as a graded ring.

4.6. **Poincaré duality.** M compact and orientable $\implies H_k(M) \cong H^{n-k}(M)$.

5. Fundamental group

X topological space, $x_0 \in X$, a loop in X based at x_0 is a continuous function $\gamma : [0,1] \to X$ such that $\gamma(0) = \gamma(1) = x_0$. Let $P(X)$ be the set of loops in X based at x_0.

Definition 22. *Define an equivalence relation on $P(X)$ by $\gamma_1 \sim \gamma_2$ if and only γ_1 and γ_2 are homotopic, i.e. there exists a continuous function $H : [0,1] \times [0,1] \to X$ such that*

$$H(0,t) = \gamma_1(t) \text{ and } H(1,t) = \gamma_2(t) \ \forall t \in [0,1],$$
$$H(s,0) = x_0 \text{ and } H(s,1) = x_0 \ \forall s \in [0,1].$$

Define a *multiplication* operation on $P(X)$ by

$$\begin{aligned} \gamma_1 * \gamma_2(t) &= \gamma_1(2t), \ t \in [0, \tfrac{1}{2}], \\ & \quad \gamma_2(1-2t), \ t \in [\tfrac{1}{2}, 1]. \end{aligned}$$

Properties of *: Let $\gamma \in P(X)$. Write $\overline{\gamma}$ for the loop $\overline{\gamma}(t) = \gamma(1-t)$, i.e., γ parametrized in the reverse direction, and write 1 for the constant loop based at x_0, i.e. $1(t) = x_0 \ \forall t$.

(1) $\gamma_1 * (\gamma_2 * \gamma_3) \sim (\gamma_1 * \gamma_2) * \gamma_3$
(2) $\gamma * \overline{\gamma} \sim 1$
(3) $\gamma * 1 \sim \gamma$

The fundamental group is $\pi_1(X, x_0)$ is the set of equivalence classes of $P(X)$, with the group operation given by $[\gamma_1] \circ [\gamma_2] = [\gamma_1 * \gamma_2]$. The properties (1) (2) (3) show that the axioms of a group are satisfied.

If X is path connected, $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ for any other base point x_1, so we just write $\pi_1(X)$.

The fundamental group is invariant under homotopies. Example: $\pi_1(\mathbb{R}^n) \cong 1$ because $\mathbb{R}^n$ is homotopic to a point.

Proposition. $\pi_1(S^1) \cong \mathbb{Z}$.

Proof. The proof consists of defining a map from $\mathbb{Z}$ to $\pi_1(S^1)$ by means of the covering map $\pi : \mathbb{R} \to S^1$ given by $t \mapsto e^{i2\pi t}$. One has to show that the map is surjective and injective. This uses the fact that π is a local diffeomorphism such that for every $x \in S^1$, there exists an open neighborhood U of x for which $\pi^{-1}(U)$ is a disjoint union of open sets each of which is isomorphic to U. In particular every continuous map $\gamma : [0,1] \to S^1$ has a unique lift $\widetilde{\gamma} : [0,1] \to \mathbb{R}$ for a given initial condition $\widetilde{\gamma}(0) \in \pi^{-1}(0) = \mathbb{Z}$. □

Proposition. $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$

Example: $\pi_1(T^2) = \pi_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z}$.

Definition 23. *A topological space X is called* simply connected *if it is path connected and $\pi_1(X) \cong 1$, i.e. every loop is homotopic to a constant loop.*

Example 20. (1) The punctured plane $\mathbb{R}^2 \setminus \{a\}$ is not simply connected.
(2) The torus T^2 is not simply connected.
(3) For $n \geq 2$ the sphere S^n IS simply connected.

Theorem (Borsuk-Ulam theorem). *Let $f : S^2 \to \mathbb{R}^2$ be a continuous map. Then there exists a pair of antipodal points x and $-x$ on S^2 such that $f(x) = f(-x)$.*

Example: S^2 = earth, $f(x) = (T(x), P(x))$ where $T(x)$ is temperature at x, $P(x)$ is pressure at x. Then there are always two points on opposite sides of the earth that have exactly the same temperature and pressure.

Example: Suppose that A_1, A_2, A_3 are three closed subsets of S^2 that cover S^2, i.e. $S^2 = \cup A_i$. Then, one of the A_i must contain a pair of antipodal points. To show this we define two continuous functions $d_1 : S^2 \to \mathbb{R}$ and $d_2 : S^2 \to \mathbb{R}$ by $d_i(x) = \text{dist}(x, A_i) = \min\limits_{y \in A_i} \|x - y\|$, and define $d : S^2 \to \mathbb{R}^2$ by $d(x) = (d_1(x), d_2(x))$. Then by the Borsuk-Ulam theorem we know that there exists an $x \in S^2$ such that $d(x) = d(-x)$, in other words $(d_1(x), d_2(x)) = (d_1(-x), d_2(-x))$. So if $d_i(x) = 0$, then $d_i(-x) = 0$ which implies x and $-x$ are in A_i for $i = 1$ or 2. If $d_1(x)$ and $d_2(x)$ are not equal to zero, then $d_1(-x)$ and $d_2(-x)$ are also not equal to zero implying that $x, -x \notin A_1$ and $x, -x \notin A_2$. Thus, x and $-x$ must both belong to A_3.

Proof of the Borsuk-Ulam theorem. By contradiction. If $f(x)$ and $f(-x)$ were never equal, then since $f(x) - f(-x) \neq 0$ we could define a continuous map

$$\begin{aligned} g : S^2 &\rightarrow S^1 \\ g(x) &= \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|} \end{aligned}$$

Let $\gamma(t) = (\cos 2\pi t, \sin 2\pi t, 0)$ be the curve in S^2 that winds once around the Equator, and let $\eta(t) = g(\gamma(t))$ be the resulting curve in S^1. Note that since S^2 is simply connected, $\gamma(t)$ is contractible, and therefore $g(\gamma(t))$ is also contractible, so $[\eta(t)]$ is the trivial element in $\pi_1(S^1)$.

On the other hand, for $t \in [0, \frac{1}{2}]$ we have $\gamma(t + \frac{1}{2}) = -\gamma(t)$, and therefore $g(\gamma(t + \frac{1}{2})) = g(-\gamma(t)) = -g(\gamma(t))$. In other words, $\eta(t + \frac{1}{2}) = -\eta(t)$. If we consider the lift $\widetilde{\eta}(t) : [0, 1] \rightarrow \mathbb{R}$ such that $\widetilde{\eta}(0) = 0$, we see that $\widetilde{\eta}(t + \frac{1}{2}) = \widehat{\eta}(t) + \frac{q}{2}$ for some odd integer q. Therefore,

$$\widehat{\eta}(\frac{1}{2}) = \widehat{\eta}(0) + \frac{q}{2}$$
$$\widehat{\eta}(1) = \widehat{\eta}(\frac{1}{2}) + \frac{q}{2}$$

and consequently $\widehat{\eta}(1) - \widehat{\eta}(0) = q$, where q is an odd integer. But then this means that in the isomorphism $\pi_1(S^1) \cong \mathbb{Z}$ the equivalence class of $[\eta(t)]$ corresponds to q, which can't be zero as it is odd. Hence $[\eta(t)]$ can't be the trivial element of $\pi_1(S^1)$ – contradiction. $\square$