# Time-domain analysis of Continuous-Time Systems

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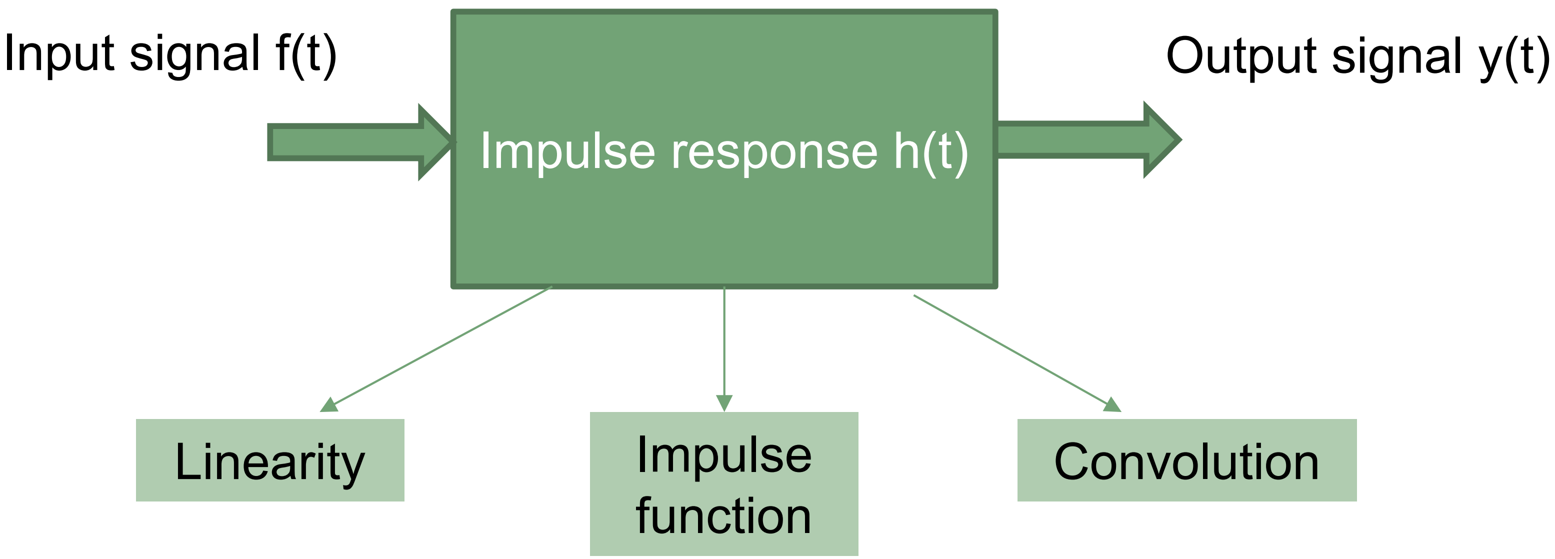
# Classification of systems

1. Linear and non linear
2. Constant parameters and time-varying parameters
3. Instantaneous (memoryless) and dynamic (with memory)
4. Causal and non-causal
5. Lumped-parameters and distributed-parameters
6. Continuous-time and discrete-time
7. Analog and digital

# What do we want to do?

- We want to derive the equations ruling the system, that is those equations that given an input signal allow to calculate the output signal
- The main actor is the *impulse response*, which fully characterizes the behavior of a linear time-invariant system



# What do we need?

- The zero-input response
  - Characteristic modes
  - Characteristic roots
- The zero-state response
  - Unit impulse response
  - Convolution operator

# Linear sytems

- Additivity

$$c_1 \longrightarrow e_1 \qquad \text{and} \qquad c_2 \longrightarrow e_2 \tag{1.37}$$

then for all $c_1$ and $c_2$

$$c_1 + c_2 \longrightarrow e_1 + e_2 \tag{1.38}$$

- Homogeneity

$$c \longrightarrow e$$

then for all real or imaginary $k$

$$kc \longrightarrow ke \tag{1.39}$$

- Linearity implies addditivity and homogeneity

$$k_1c_1 + k_2c_2 \longrightarrow k_1e_1 + k_2e_2$$

**Principle of superposition**

# Response of a linear system

- Total response = zero-input response + zero-state response
- This is possible thanks to linearity that allows to consider the output as the sum of the contributions due to
  - the input signal (the zero-state response) and
  - the initial state of the system (that is the state at t=0, namely the zero-input response)

- Almost all systems become NON linear for sufficiently large signals applied as inputs
  - However, they can be locally linearized and treated as linear for small signal variations

# Zero-input and zero-state response

- The zero-input and zero—state responses are independent of each other.
- The zero-state response only depends on the input signal and the initial conditions are assumed to be zero.
- For each component, the other is totally irrelevant.

# Linear time-invariant systems

- A linear continuous time system is described by a linear differential equation

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y(t) = b_m\frac{d^m f}{dt^m} + b_{m-1}\frac{d^{m-1}f}{dt^{m-1}} + \cdots + b_1\frac{df}{dt} + b_0 f(t) \qquad (2.1a)$$

where all the coefficients $a_i$ and $b_i$ are constants. Using operational notation $D$ to represent $d/dt$, we can express this equation as

$$(D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0)\,y(t) = (b_m D^m + b_{m-1}D^{m-1} + \cdots + b_1 D + b_0)\,f(t) \qquad (2.1b)$$

or

$$Q(D)y(t) = P(D)f(t) \qquad (2.1c)$$

# LTIS

where the polynomials $Q(D)$ and $P(D)$ are

$$Q(D) = D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0$$
$$P(D) = b_m D^m + b_{m-1}D^{m-1} + \cdots + b_1 D + b_0$$

$$\textbf{Total response} = \textbf{zero-input response} + \textbf{zero-state response} \qquad (2.3)$$

The zero-input component is the system response when the input $f(t) = 0$ so that it is the result of internal system conditions (such as energy storages, initial conditions) alone. It is independent of the external input $f(t)$. In contrast, the zero-state component is the system response to the external input $f(t)$ when the system is in zero state, meaning the absence of all internal energy storages; that is, all initial conditions are zero.

# Zero-input response

- System response to internal conditions

The zero-input response $y_0(t)$ is the solution of Eq. (2.1) when the input $f(t) = 0$ so that

$$Q(D)y_0(t) = 0 \tag{2.4a}$$

or

$$\left(D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0\right) y_0(t) = 0 \tag{2.4b}$$

A solution to this equation can be obtained systematically.[1] However, we will take a short cut by using heuristic reasoning. Equation (2.4b) shows that a linear combination of $y_0(t)$ and its $n$ successive derivatives is zero, not at *some* values of $t$, but for all $t$. Such a result is possible *if and only if* $y_0(t)$ and all its $n$ successive derivatives are of the same form. Otherwise their sum can never add to zero for all values of $t$. We know that only an exponential function $e^{\lambda t}$ has this property. So let us assume that

$$y_0(t) = ce^{\lambda t}$$

is a solution to Eq. (2.4b). Then

# Zero-input response

Then

$$Dy_0(t) = \frac{dy_0}{dt} = c\lambda e^{\lambda t}$$

$$D^2 y_0(t) = \frac{d^2 y_0}{dt^2} = c\lambda^2 e^{\lambda t}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$D^n y_0(t) = \frac{d^n y_0}{dt^n} = c\lambda^n e^{\lambda t}$$

# Zero-input response

Substituting these results in Eq. (2.4b), we obtain

$$c\left(\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0\right)e^{\lambda t} = 0$$

For a nontrivial solution of this equation,

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0 \tag{2.5a}$$

This result means that $ce^{\lambda t}$ is indeed a solution of Eq. (2.4), provided that $\lambda$ satisfies Eq. (2.5a). Note that the polynomial in Eq. (2.5a) is identical to the polynomial $Q(D)$ in Eq. (2.4b), with $\lambda$ replacing $D$. Therefore, Eq. (2.5a) can be expressed as

$$Q(\lambda) = 0 \tag{2.5b}$$

When $Q(\lambda)$ is expressed in factorized form, Eq. (2.5b) can be represented as

$$Q(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n) = 0 \tag{2.5c}$$

# Zero-input response

Clearly, $\lambda$ has $n$ solutions: $\lambda_1, \lambda_2, \ldots, \lambda_n$. Consequently, Eq. (2.4) has $n$ possible solutions: $c_1e^{\lambda_1 t}, c_2e^{\lambda_2 t}, \ldots, c_ne^{\lambda_n t}$, with $c_1, c_2, \ldots, c_n$ as arbitrary constants. We can readily show that a general solution is given by the sum of these $n$ solutions,† so that

$$y_0(t) = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t} + \cdots + c_ne^{\lambda_n t} \tag{2.6}$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants determined by $n$ constraints (the auxiliary conditions) on the solution.

# Zero-input response

Observe that the polynomial $Q(\lambda)$, which is characteristic of the system, has nothing to do with the input. For this reason the polynomial $Q(\lambda)$ is called the **characteristic polynomial** of the system. The equation

$$Q(\lambda) = 0 \tag{2.7}$$

is called the **characteristic equation** of the system. Equation (2.5c) clearly indicates that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the roots of the characteristic equation; consequently, they are called the **characteristic roots** of the system. The terms **characteristic values**, **eigenvalues**, and **natural frequencies** are also used for characteristic roots.‡ The exponentials $e^{\lambda_i t} (i = 1, 2, \ldots, n)$ in the zero-input response are the **characteristic modes** (also known as **modes** or **natural modes**) of the system. There is a characteristic mode for each characteristic root of the system, and the *zero-input response is a linear combination of the characteristic modes of the system.*

# Zero-input response

- Repeated roots
  - The considered solution assumes that the n roots of the characteristic polynomial are distinct. If this is not the case then the form of the solution is slightly different
  - For a double root ($\lambda$) the differential equation becomes

$$(D-\lambda)^2 y_0(t) = 0$$

$$y_0(t) = (c_1 + c_2 t)e^{\lambda t}$$

In this case the root $\lambda$ repeats twice. Observe that the characteristic modes in this case are $e^{\lambda t}$ and $te^{\lambda t}$. Continuing this pattern, we can show that for the differential equation

$$(D-\lambda)^r y_0(t) = 0 \tag{2.8}$$

the characteristic modes are $e^{\lambda t}$, $te^{\lambda t}$, $t^2 e^{\lambda t}$, ..., $t^{r-1}e^{\lambda t}$, and that the solution is

$$y_0(t) = \left(c_1 + c_2 t + \cdots + c_r t^{r-1}\right) e^{\lambda t} \tag{2.9}$$

# Zero-input response with simple and repeated roots

Consequently, for a system with the characteristic polynomial

$$Q(\lambda) = (\lambda - \lambda_1)^r (\lambda - \lambda_{r+1}) \cdots (\lambda - \lambda_n)$$

the characteristic modes are $e^{\lambda_1 t}$, $te^{\lambda_1 t}$, ..., $t^{r-1}e^{\lambda_1 t}$, $e^{\lambda_{r+1} t}$, ..., $e^{\lambda_n t}$ and the solution is

$$y_0(t) = (c_1 + c_2 t + \cdots + c_r t^{r-1})e^{\lambda_1 t} + c_{r+1}e^{\lambda_{r+1} t} + \cdots + c_n e^{\lambda_n t}$$

In order to obtain the zero-response, n auxiliary conditions must be given to calculate the $c_i$ coefficients.
If the auxiliary conditions are at t=0 they are called initial conditions.

# Characteristic equation

- System equation in zero-input conditions

$$Q(\lambda) = 0$$

- The polynomial Q is called characteristic polynomial because it has nothing to do with the input signal and only depends on the system itself
- Its solutions are the characteristic roots, or eigenvalues or natural frequencies of the system
- Similarly, the corresponding functions

$$y_i(t) = e^{\lambda_i t}$$

Are the characteristic modes, or natural modes or modes of the system.

There is a characteristic mode for each characteristic root

# Zero-input response

- The zero-input response is a linear combination of the characteristic modes of the system
- Some insights into the zero-input behavior
  - Characteristic modes also determine the zero-state response and thus are the most important attribute of a LTIS
  - The zero-input response describes the way the system recovers the rest position after an instantaneous perturbation occurs
    - The system uses a proper combination of characteristic modes to come back to the rest position while satisfying appropriate boundary (or initial) conditions.

  - Resonance: the external input is a characteristic mode
    - The system "sustains" such input and the output tends to diverge
      - "This would be as pouring gasoline on a fire in a dry forest"

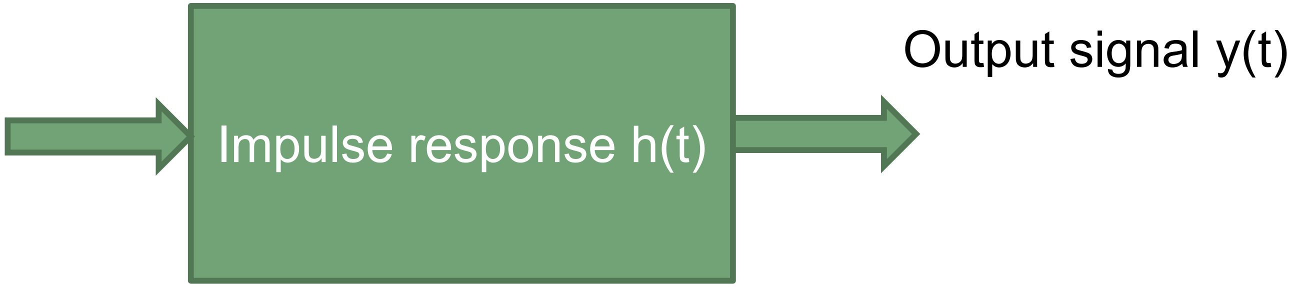
# The unit impulse response h(t)

- To calculate the response of the system to the generic input signal f(t) we first need to calculate the response to the unit delta function


- How to get there
  - Starting from the linearity implications, the response to a complex signal will be derived as a linear combination (sum) of the responses to its instantaneous components

# LTIS: the impulse-based view

- Due to additivity and homogeneity




$f(t)=a_1f_1(t)+a_2f_2(t)+\ldots+a_nf_n(t)$

$y(t)=a_1y_1(t)+a_2y_2(t)+\ldots+a_ny_n(t)$

$y_i(t)$=zero-state response to the input $f_i(t)$

# Impulse-based view

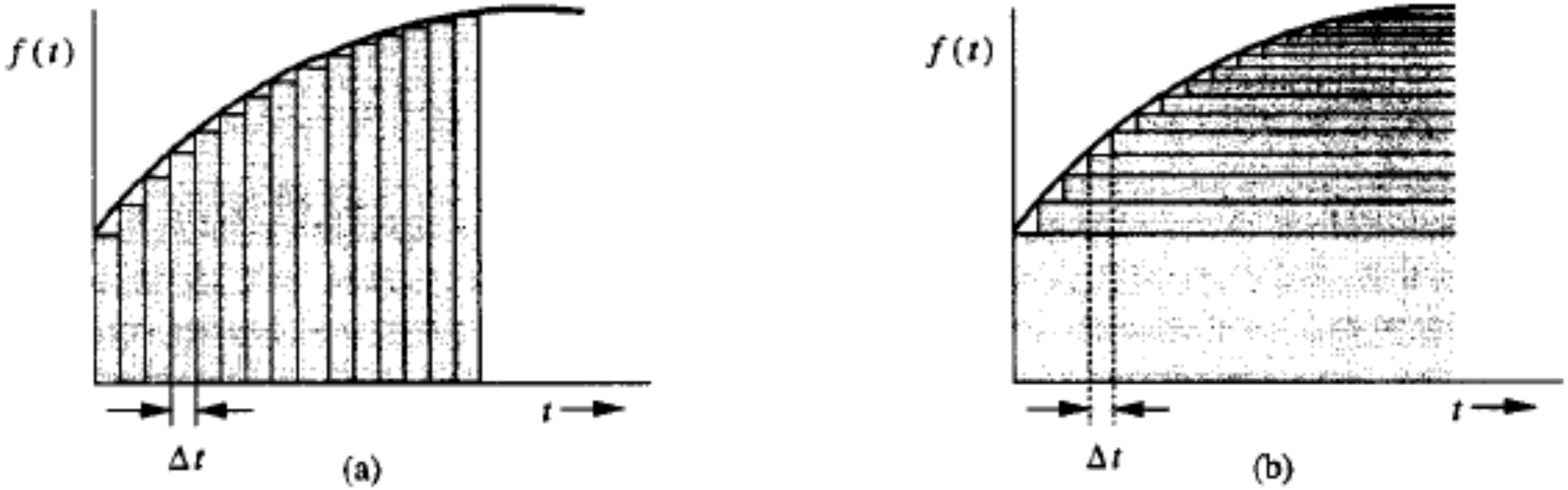



Fig. 1.27 Signal representation in terms of impulse and step components.

Input signal=sum of rectangular pulses of width Delta(t) of different amplitude and delayed in time
Output signal=sum of the responses to such pulses
Importantly: the system's response is the same because of time-invariance

# Time invariance

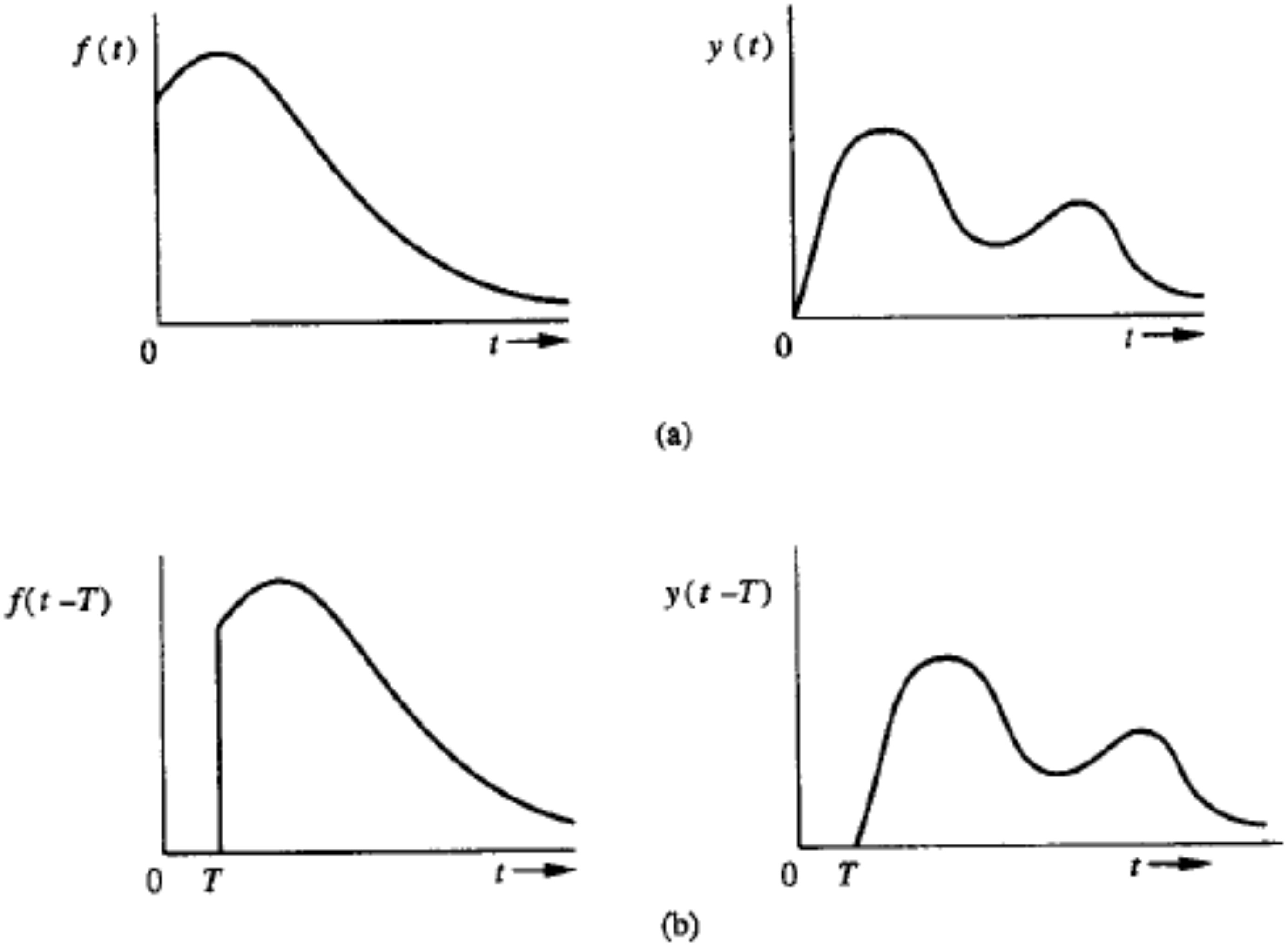



**Fig. 1.28** Time-invariance property.

# Back to the Unit impulse response

- If we know the system response to an input impulse we can determine the system response to any input f(t)
- Such a response is the Unit impulse response h(t) that is the response of the system to a unit delta function delta(t) applied at $t=0^-$ and with all the initial conditions zero at $t=0^-$.
- Hint: the delta creates initial conditions at $t=0^+$ and then disappears, but it ”generates” energy storage in the system. From then on, even though no input is present, the system evolves because of these newly created initial conditions.
- Therefore, the system’s response will be of zero-input kind and thus consist of a linear combination of characteristic modes for $t>0^+$.

# Unit impulse response

- Overall, including t=0, the system response is

$$h(t)=A_0\delta(t)+\text{characteristic modes terms}$$

$$h(t)=h_n\delta(t)+[P(D)y_n(t)]\mathrm{u}(t)$$

where $b_n$ is the coefficient of the $n$th-order term in $P(D)$ [see Eq. (2.17b)], and $y_n(t)$ is a linear combination of the characteristic modes of the system subject to the following initial conditions:

$$y_n^{(n-1)}(0)=1, \quad \text{and} \quad y_n(0)=\dot{y}_n(0)=\ddot{y}_n(0)=\cdots=y_n^{(n-2)}(0)=0 \tag{2.20}$$

where $y_n^{(k)}(0)$ is the value of the $k$th derivative of $y_n(t)$ at $t=0$. We can express this condition for various values of $n$ (the system order) as follows:

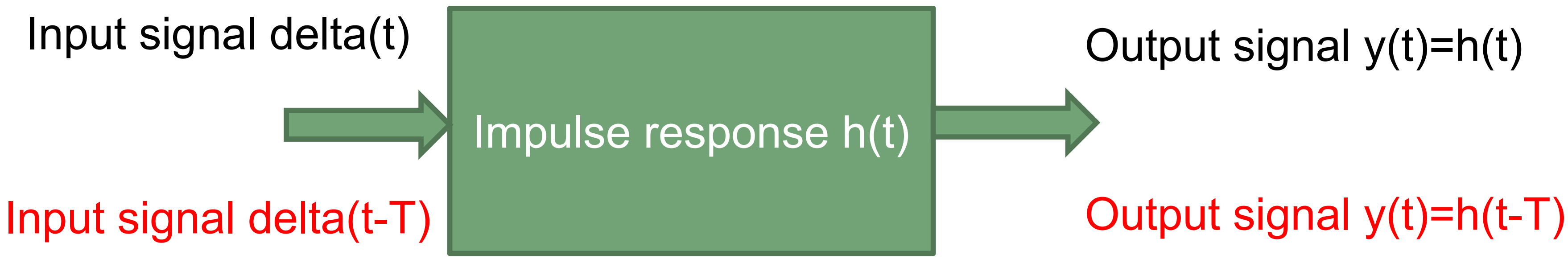
$$\begin{aligned} n=1 &: \; y_n(0)=1 \\ n=2 &: \; y_n(0)=0 \text{ and } \dot{y}_n(0)=1 \\ n=3 &: \; y_n(0)=\dot{y}_n(0)=0 \text{ and } \ddot{y}_n(0)=1 \\ n=4 &: \; y_n(0)=\dot{y}_n(0)=\ddot{y}_n(0)=0 \text{ and } \dddot{y}_n(0)=1 \end{aligned} \tag{2.21}$$

and so on.

If the order of $P(D)$ is less than the order of $Q(D)$, $b_n=0$, and the impulse term $b_n\delta(t)$ in $h(t)$ is zero.

# System response to delayed input

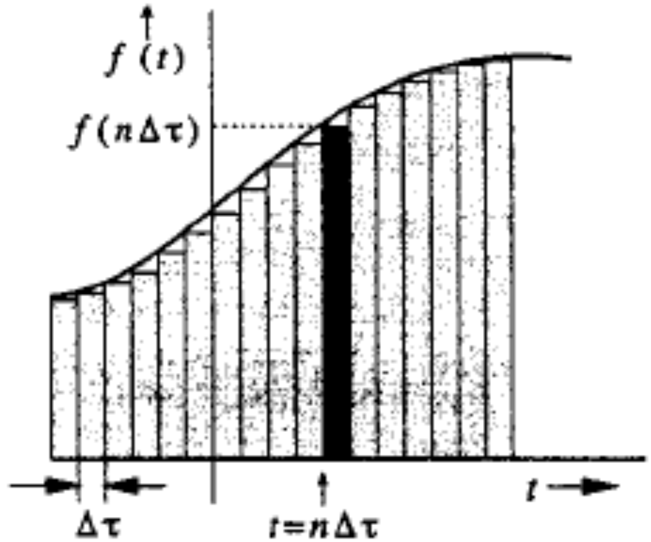
Thanks to the time-invariance property



# Zero-state response

- System response to an external input assuming that the system is in zero-state
- Method:
  - model the input signal f(t) as a superposition of delayed rectangular pulses of different amplitude
  - Express the global response as the sum of the responses to these pulses
  - In the limit where the pulse duration Delta_t tends to zero, each pulse tends to a delta having a strength equal to the area under the pulse. Then the pulse at the same location will have the amplitude $f(n\Delta t)\Delta t$ and can therefore be represented as $f(n\Delta t)\Delta t\delta(t - n\Delta t)$.

# Zero-state response



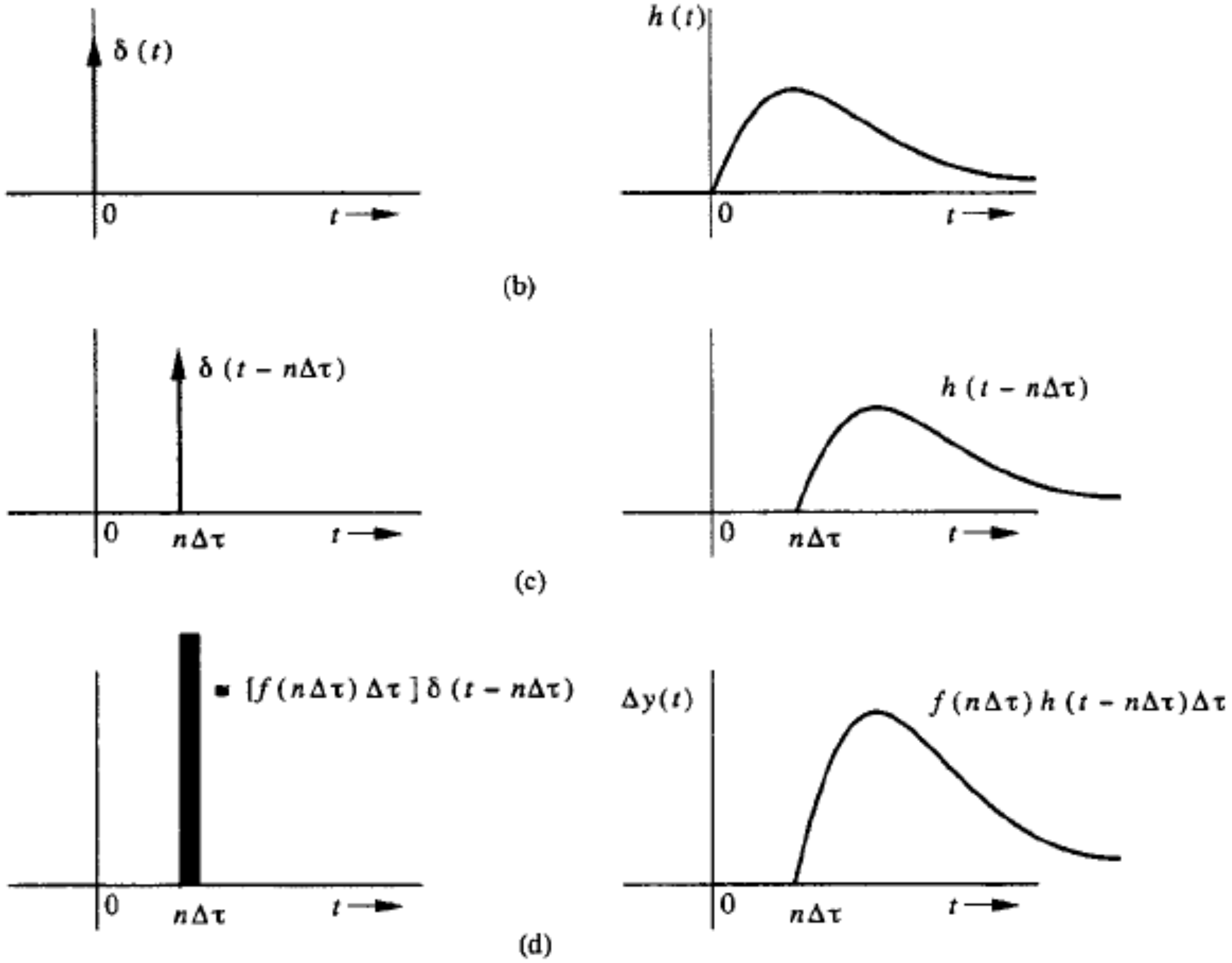

(a)

$$\delta(t) \Longrightarrow h(t)$$

$$\delta(t - n\Delta\tau) \Longrightarrow h(t - n\Delta\tau)$$

$$\underbrace{[f(n\Delta\tau)\Delta\tau]\delta(t - n\Delta\tau)}_{\text{input}} \Longrightarrow \underbrace{[f(n\Delta\tau)\Delta\tau]h(t - n\Delta\tau)}_{\text{output}} \qquad (2.27)$$

# Zero-state response



# Zero-state response

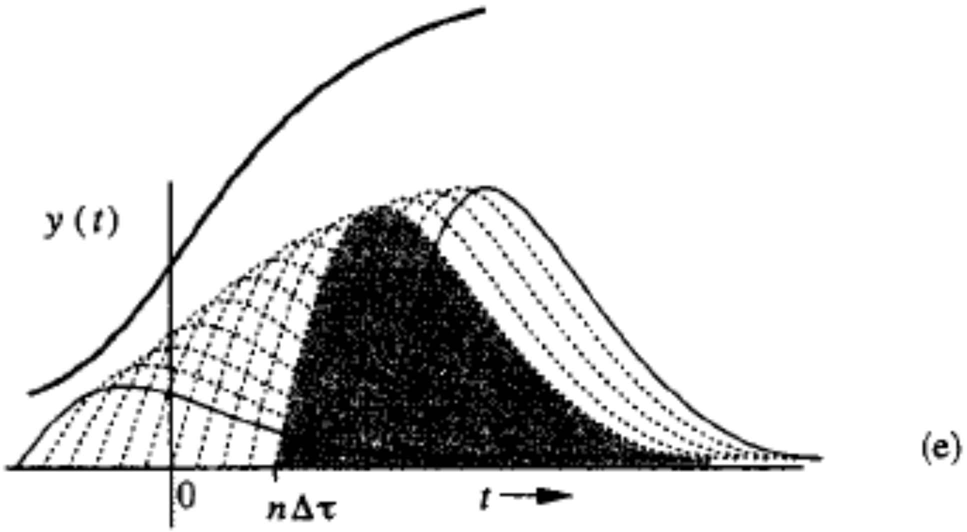



Fig. 2.3 Finding the system response to an arbitrary input $f(t)$.

# Zero-state response

$$\underbrace{\lim_{\Delta\tau\to 0}\sum_{n=-\infty}^{\infty} f(n\Delta\tau)\delta(t-n\Delta\tau)\Delta\tau}_{\text{The input } f(t)} \Longrightarrow \underbrace{\lim_{\Delta\tau\to 0}\sum_{n=-\infty}^{\infty} f(n\Delta\tau)h(t-n\Delta\tau)\Delta\tau}_{\text{The output } y(t)}$$

The left-hand side is the input $f(t)$ represented as a sum of all the impulse components in a manner illustrated in Fig. 2.3a. The right-hand side is the output $y(t)$ represented as a sum of the output components as shown in Fig. 2.3e. Both the left-hand side and the right-hand side, by definition, are integrals given by†

$$\underbrace{\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)\,d\tau}_{f(t)} \Longrightarrow \underbrace{\int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau}_{y(t)} \tag{2.28}$$

The left-hand side expresses the input $f(t)$ as made up of the impulse components in a manner depicted in Fig. 2.3a. The right-hand expresses the output as made up of the sum of the system responses to all the impulse components of the input as illustrated in Fig. 2.3e. To summarize, the (zero-state) response $y(t)$ to the input $f(t)$ is given by

Convolution integral

$$y(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau \tag{2.29}$$

# Zero-state response and characteristic modes

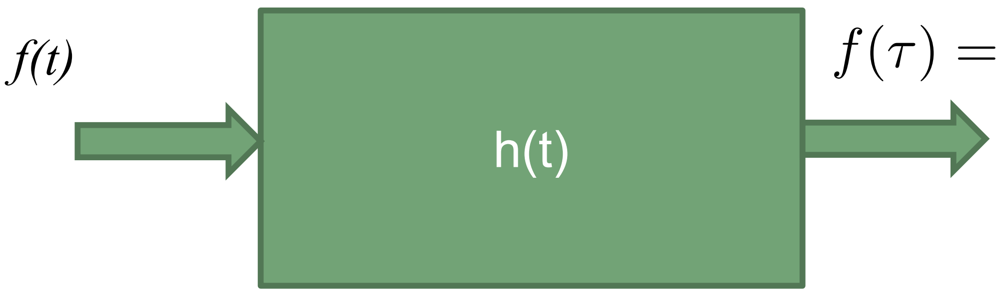
- The characteristic modes shape the zero-state response through the unit impulse response

$$y(t) = \int_{-\infty}^{+\infty} f(\tau)h(t-\tau)\, d\tau$$

$h(t)$=$A_0\delta(t)$+characteristic modes terms

$$y(t) = \int_{-\infty}^{+\infty} f(\tau)h(t-\tau)d\tau$$

$$f(\tau) = 0 \text{ for } \tau < 0$$



# The convolution integral: properties

$$f_1(t) * f_2(t) \equiv \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, d\tau$$

1. **The Commutative Property**: Convolution operation is commutative; that is, $f_1(t) * f_2(t) = f_2(t) * f_1(t)$. This property can be proved by a change of variable. In Eq. (2.30), if we let $x = t - \tau$ so that $\tau = t - x$ and $d\tau = -dx$, we obtain

$$\begin{aligned} f_1(t) * f_2(t) &= -\int_{\infty}^{-\infty} f_2(x) f_1(t-x)\, dx \\ &= \int_{-\infty}^{\infty} f_2(x) f_1(t-x)\, dx \\ &= f_2(t) * f_1(t) \end{aligned} \quad (2.31)$$

2. **The Distributive Property**: According to this property:

$$f_1(t) * [f_2(t) + f_3(t)] = f_1(t) * f_2(t) + f_1(t) * f_3(t) \quad (2.32)$$

3. **The Associative Property**: According to this property:

$$f_1(t) * [f_2(t) * f_3(t)] = [f_1(t) * f_2(t)] * f_3(t) \quad (2.33)$$

The proofs of (2.32) and (2.33) follow directly from the definition of the convolution integral. They are left as an exercise for the reader.

# The convolution integral: properties

4. **The Shift Property**: If

$$f_1(t) * f_2(t) = c(t)$$

then

$$f_1(t) * f_2(t-T) = c(t-T) \tag{2.34a}$$

$$f_1(t-T) * f_2(t) = c(t-T) \tag{2.34b}$$

and

$$f_1(t-T_1) * f_2(t-T_2) = c(t-T_1-T_2) \tag{2.34c}$$

Proof: We are given

$$f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, d\tau = c(t)$$

Therefore

$$\begin{aligned} f_1(t) * f_2(t-T) &= \int_{-\infty}^{\infty} f_1(\tau) f_2(t-T-\tau)\, d\tau \\ &= c(t-T) \end{aligned}$$

Equation (2.34b) follows from (2.34a) and the commutative property of convolution; Eq. (2.34c) follows directly from (2.34a) and (2.34b).

# The convolution integral: properties

5. **Convolution with an Impulse**: Convolution of a function $f(t)$ with a unit impulse results in the function $f(t)$ itself. By definition of convolution

$$f(t) * \delta(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)\,d\tau \tag{2.35}$$

Because $\delta(t-\tau)$ is an impulse located at $\tau = t$, according to the sampling property of the impulse [Eq. (1.24)], the integral in the above equation is the value of $f(\tau)$ at $\tau = t$, that is, $f(t)$. Therefore

$$f(t) * \delta(t) = f(t) \tag{2.36}$$

Actually this result has been derived earlier in Eq. (2.28).

# The convolution integral: properties

6. **The Width Property**: If the durations (widths) of $f_1(t)$ and $f_2(t)$ are $T_1$ and $T_2$ respectively, then the duration (width) of $f_1(t) * f_2(t)$ is $T_1 + T_2$ (Fig. 2.4). The proof of this property follows readily from the graphical considerations discussed later in Sec. 2.4-2. This rule may superficially appear to be violated in some special cases discussed later.

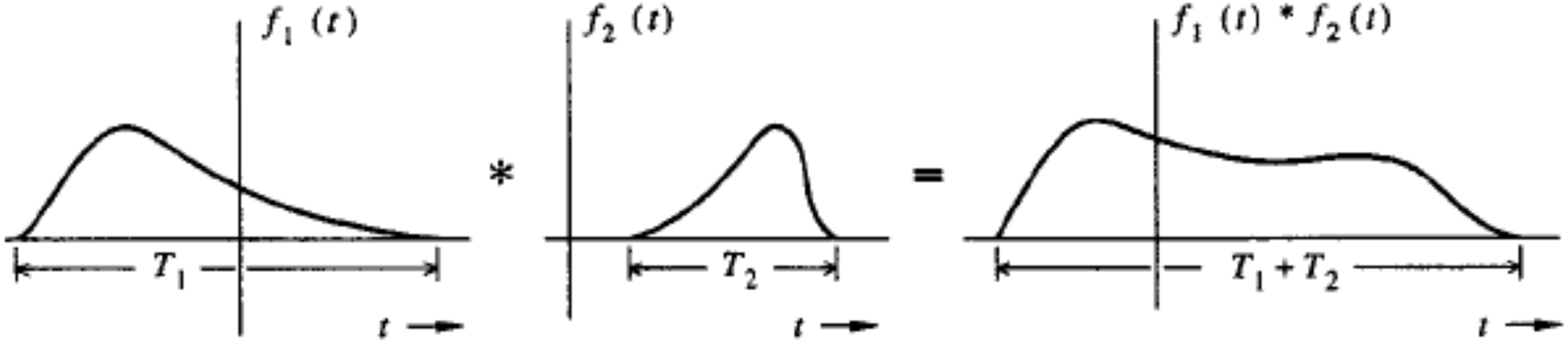



**Fig. 2.4** Width property of convolution.

# Zero-state response and causality

- So far we have hypothesized linearity and time-invariance
- However, most signals are also causal, so they are zero before t=0. This simplifies the convolution integral limits
  - A causal system impulse response function is a causal signal because it cannot start before the t=0
- In formulas

$$y(t) = \int_{-\infty}^{+\infty} f(\tau)h(t-\tau)d\tau$$

$$f(\tau) = 0 \text{ for } \tau < 0$$

- Similarly

$$h(t) = 0 \text{ for } t < 0 \rightarrow h(t-\tau) = 0 \text{ for } t-\tau < 0 \rightarrow \text{ for } t < \tau$$

$$\rightarrow f(\tau) \times h(t-\tau) \neq 0 \text{ only for } 0 \leq \tau \leq t$$

  - If t<0 there is no interval where the product is different from zero

# Limits of the convolution integral

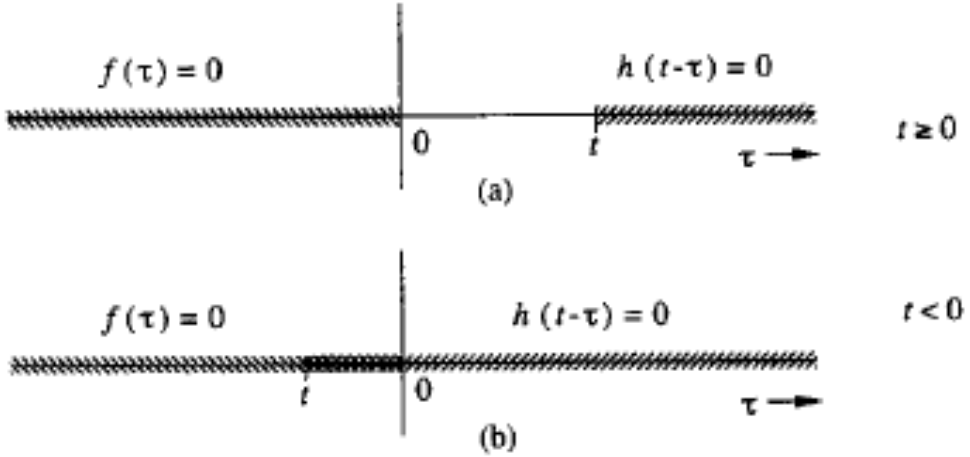



**Fig. 2.5** Limits of convolution integral.

- The convolution integral thus becomes

$$y(t) = \int_0^t f(\tau)h(t-\tau)d\tau \text{ for } t > 0$$

$$0 \text{ for } t \leq 0$$

y is causal

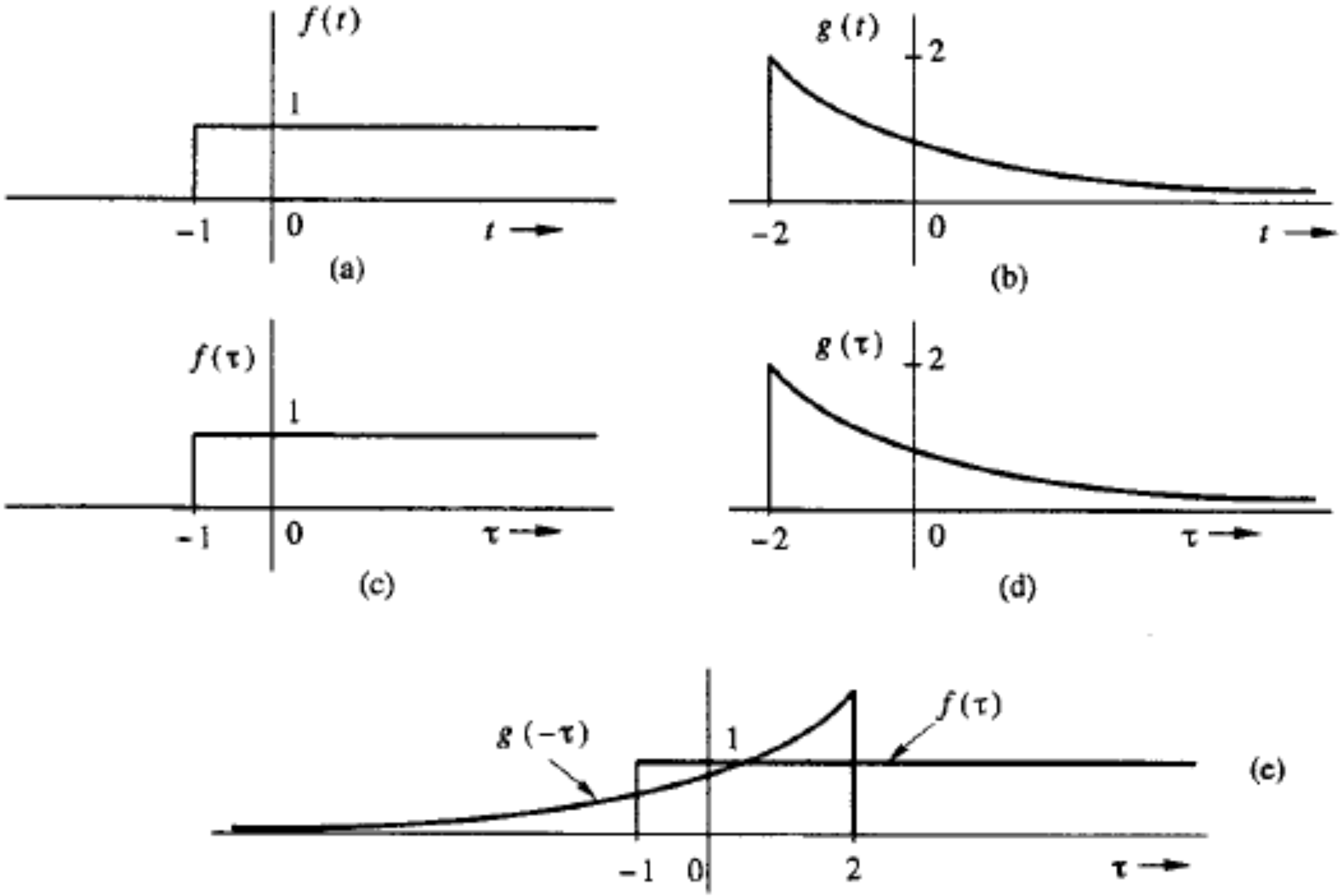
This result shows that if f(t) and h(t) are both causal, the response y(t) is also causal

# The convolution table

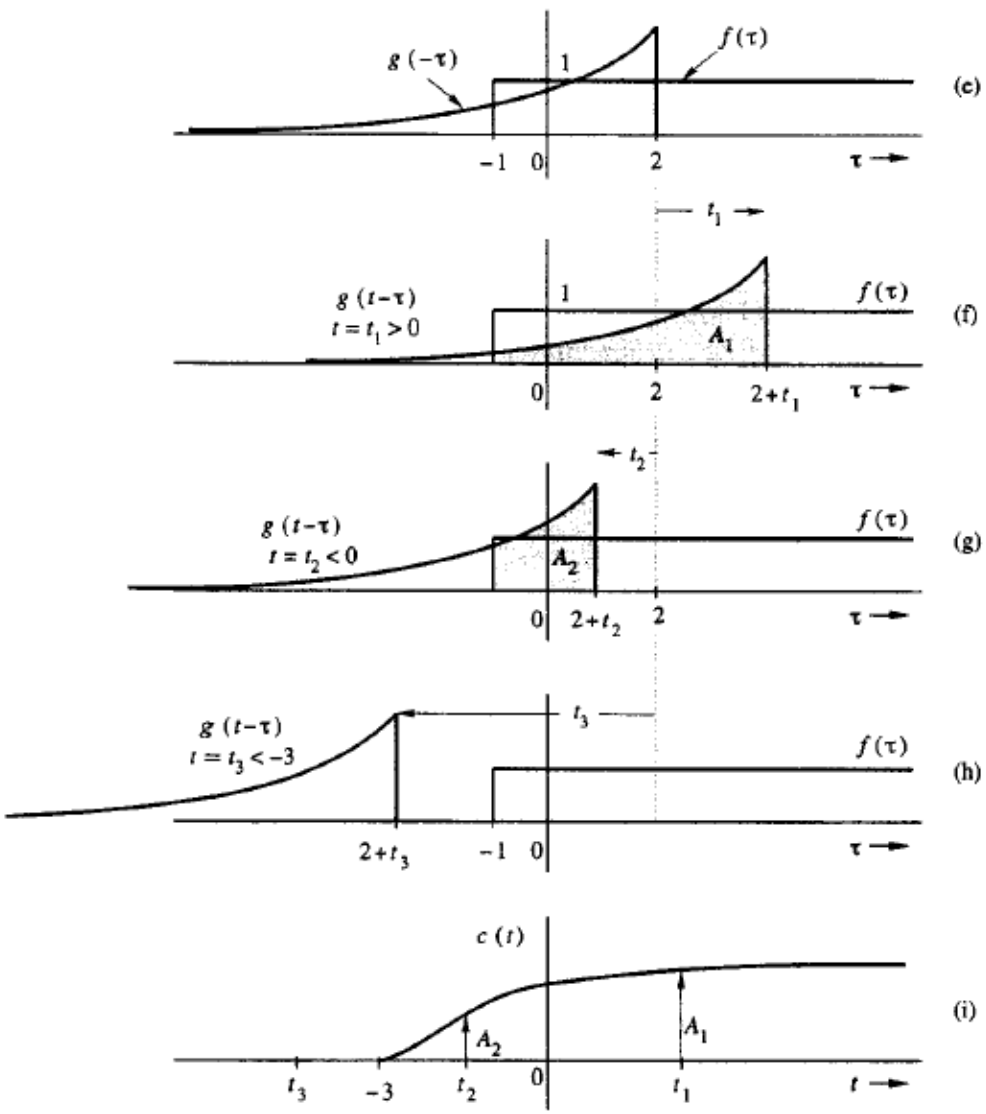
TABLE 2.1: Convolution Table

| No | $f_1(t)$ | $f_2(t)$ | $f_1(t)*f_2(t) = f_2(t)*f_1(t)$ |
|---|---|---|---|
| 1 | $f(t)$ | $\delta(t-T)$ | $f(t-T)$ |
| 2 | $e^{\lambda t}u(t)$ | $u(t)$ | $\dfrac{1-e^{\lambda t}}{-\lambda}u(t)$ |
| 3 | $u(t)$ | $u(t)$ | $tu(t)$ |
| 4 | $e^{\lambda_1 t}u(t)$ | $e^{\lambda_2 t}u(t)$ | $\dfrac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}u(t) \quad \lambda_1 \neq \lambda_2$ |
| 5 | $e^{\lambda t}u(t)$ | $e^{\lambda t}u(t)$ | $te^{\lambda t}u(t)$ |
| 6 | $te^{\lambda t}u(t)$ | $e^{\lambda t}u(t)$ | $\frac{1}{2}t^2 e^{\lambda t}u(t)$ |
| 7 | $t^n u(t)$ | $e^{\lambda t}u(t)$ | $\dfrac{n!\,e^{\lambda t}}{\lambda^{n+1}}u(t) - \sum_{j=o}^{n}\dfrac{n!\,t^{n-j}}{\lambda^{j+1}(n-j)!}u(t)$ |
| 8 | $t^m u(t)$ | $t^n u(t)$ | $\dfrac{m!n!}{(m+n+1)!}t^{m+n+1}u(t)$ |
| 9 | $te^{\lambda_1 t}u(t)$ | $e^{\lambda_2 t}u(t)$ | $\dfrac{e^{\lambda_2 t}-e^{\lambda_1 t}+(\lambda_1-\lambda_2)te^{\lambda_1 t}}{(\lambda_1-\lambda_2)^2}u(t)$ |
| 10 | $t^m e^{\lambda t}u(t)$ | $t^n e^{\lambda t}u(t)$ | $\dfrac{m!\,n!}{(n+m+1)!}t^{m+n+1}e^{\lambda t}u(t)$ |
| 11 | $t^m e^{\lambda_1 t}u(t)$ | $t^n e^{\lambda_2 t}u(t)$ | $\sum_{j=0}^{m}\dfrac{(-1)^j m!(n+j)!\,t^{m-j}e^{\lambda_1 t}}{j!(m-j)!(\lambda_1-\lambda_2)^{n+j+1}}u(t)$ |
| | $\lambda_1 \neq \lambda_2$ | | $+\sum_{k=0}^{n}\dfrac{(-1)^k n!(m+k)!\,t^{n-k}e^{\lambda_2 t}}{k!(n-k)!(\lambda_2-\lambda_1)^{m+k+1}}u(t)$ |
| 12 | $e^{-\alpha t}\cos(\beta t+\theta)u(t)$ | $e^{\lambda t}u(t)$ | $\dfrac{\cos(\theta-\phi)e^{\lambda t}-e^{-\alpha t}\cos(\beta t+\theta-\phi)}{\sqrt{(\alpha+\lambda)^2+\beta^2}}u(t)$ |
| | | | $\phi = \tan^{-1}[-\beta/(\alpha+\lambda)]$ |
| 13 | $e^{\lambda_1 t}u(t)$ | $e^{\lambda_2 t}u(-t)$ | $\dfrac{e^{\lambda_1 t}u(t)+e^{\lambda_2 t}u(-t)}{\lambda_2-\lambda_1} \quad \mathrm{Re}\,\lambda_2 > \mathrm{Re}\,\lambda_1$ |
| 14 | $e^{\lambda_1 t}u(-t)$ | $e^{\lambda_2 t}u(-t)$ | $\dfrac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_2-\lambda_1}u(-t)$ |

# Graphical understanding of convolution



# Graphical understanding of convolution



# The convolution integral: summary

**Summary of the Graphical Procedure**

The procedure for graphical convolution can be summarized as follows:

1. Keep the function $f(\tau)$ fixed.
2. Visualize the function $g(\tau)$ as a rigid wire frame, and rotate (or invert) this frame about the vertical axis ($\tau = 0$) to obtain $g(-\tau)$.
3. Shift the inverted frame along the $\tau$ axis by $t_0$ seconds. The shifted frame now represents $g(t_0 - \tau)$.
4. The area under the product of $f(\tau)$ and $g(t_0 - \tau)$ (the shifted frame) is $c(t_0)$, the value of the convolution at $t = t_0$.
5. Repeat this procedure, shifting the frame by different values (positive and negative) to obtain $c(t)$ for all values of $t$.

# Zero-state response: the everlasting exponential

- Remember that the everlasting exponential is a characteristic mode of the system
- Response to the everlasting exponential

$$y(t) = h(t) * \exp^{st} = \int_{-\infty}^{+\infty} h(\tau) \exp^{s(t-\tau)} d\tau =$$

$$= \exp^{st} \int_{-\infty}^{+\infty} h(\tau) \exp^{-s\tau} d\tau = H(s) \exp^{st}$$

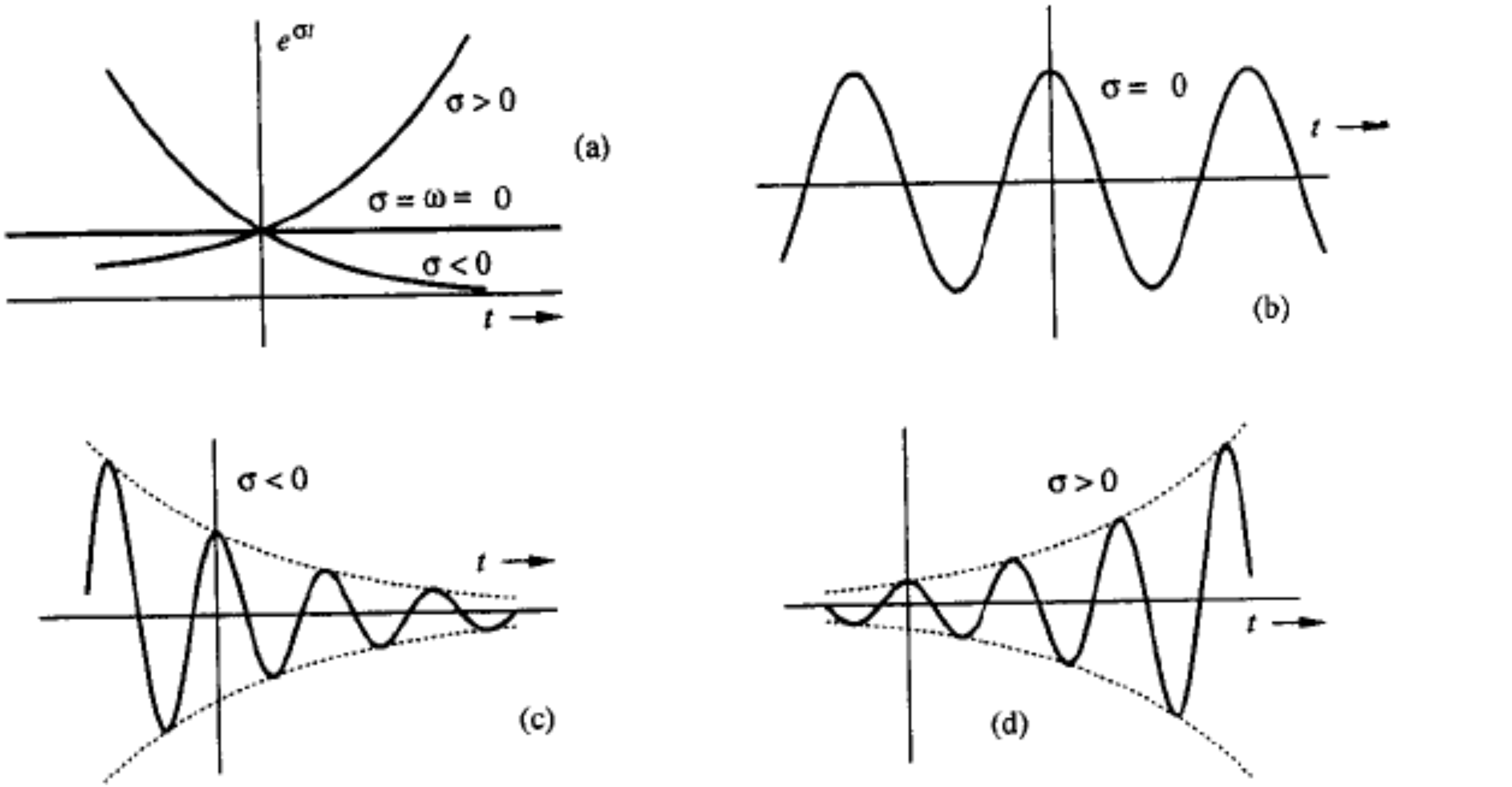
$$H(s) = \int_{-\infty}^{+\infty} h(\tau) \exp^{-s\tau} d\tau$$

**Transfer function of the system**

# Transfer function

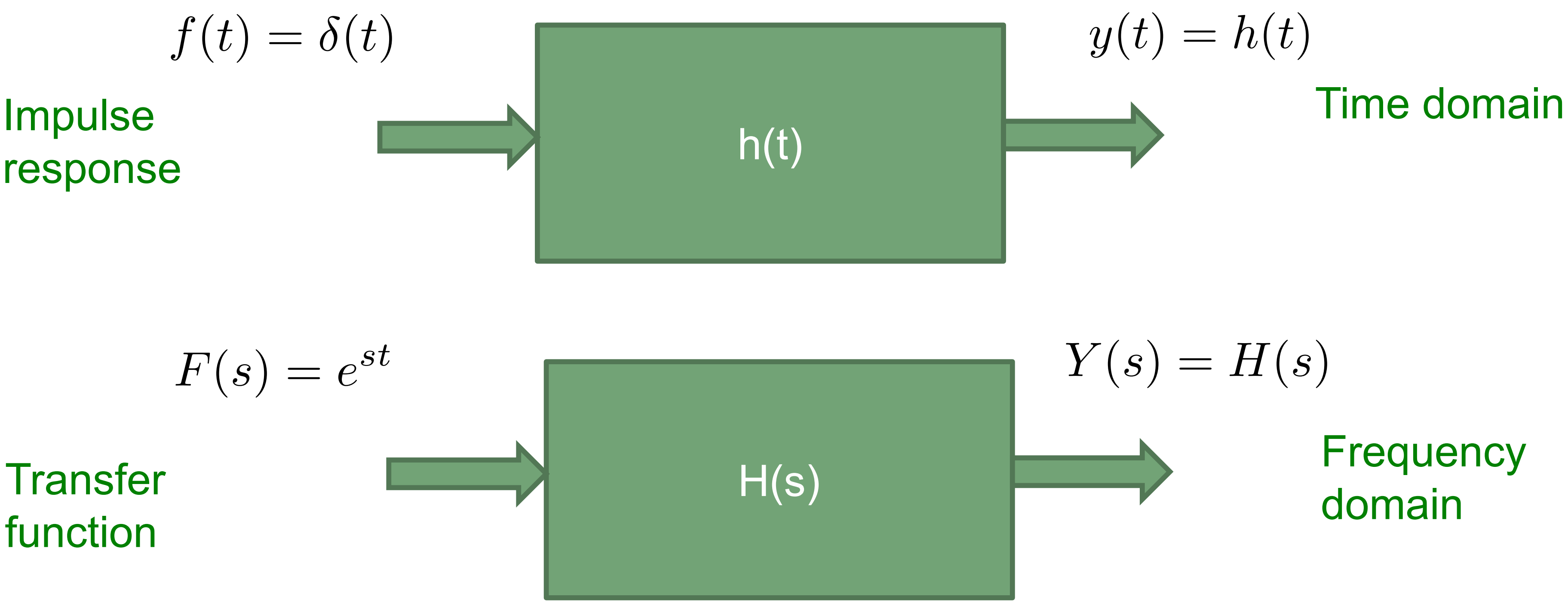
$$H(s) = \frac{output\ signal}{input\ signal}\Big|_{input=everlasting\ exponential}$$

$$H(s) = \frac{P(s)}{Q(s)}$$



# Transfer function

- The transfer function is the response of the system when the input is the everlasting exponential
  - Note: this is not the causal exponential $e^{st}u(t)$



# Total response

$$y(t) = \sum_{j=1}^{n} c_j e^{\lambda_j t} + f(t) * h(t)$$

Zero input → $\sum_{j=1}^{n} c_j e^{\lambda_j t}$

Zero state → $f(t) * h(t)$

- Note that
  - For repeated roots the zero-input response needs to be modified
  - The $c_j$ constants are detetermined by the auxiliary conditions

- There is another way of representing the output that is through the *natural response* and the *forced response*.

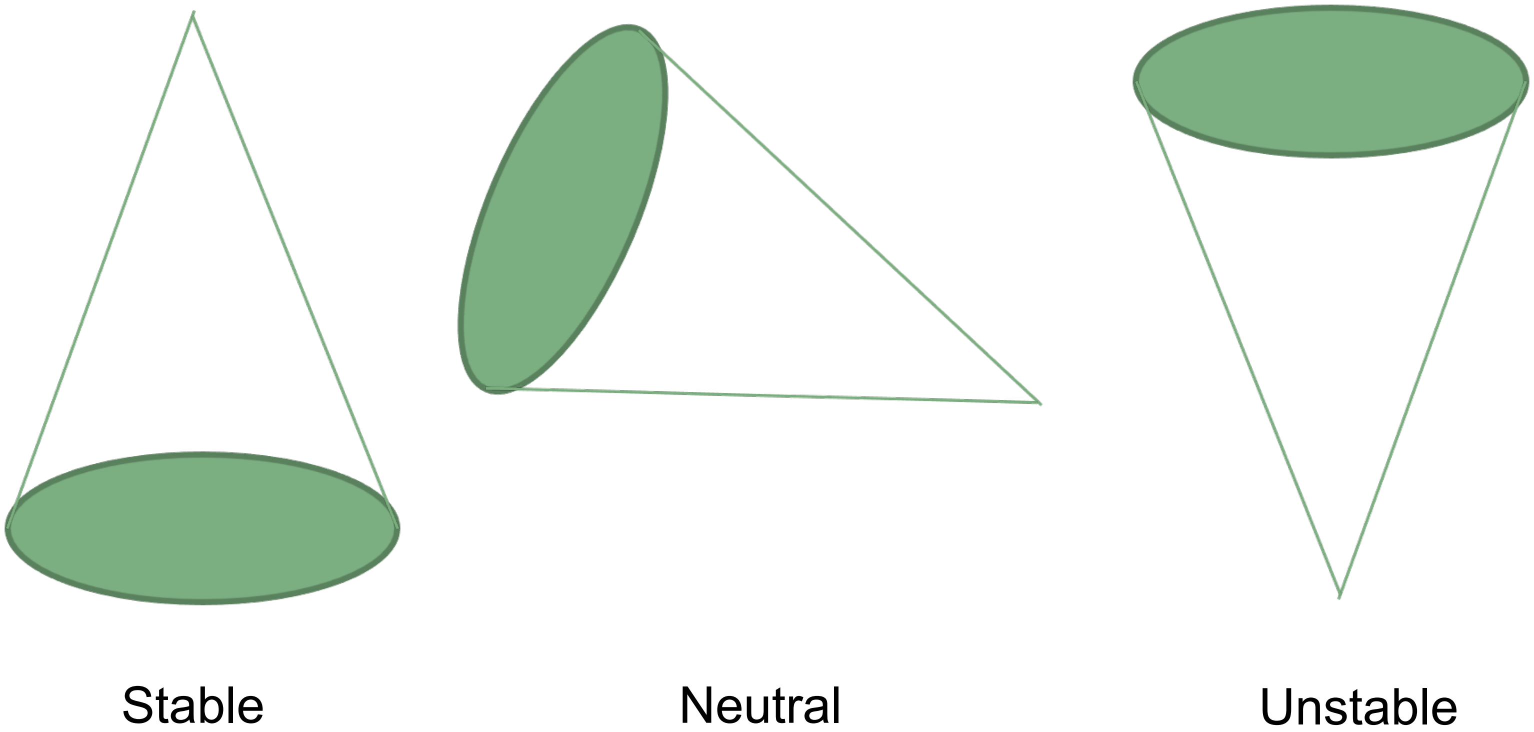
# Natural and forced response: *classical* representation

- Natural response: contains all the characteristic modes of the system
  - It may happen that the zero-state response coincides with one of the characteristic modes.
- Forced response: the part of the total response that cannot be expressed by characteristic modes
- In this case the initial conditions must be used on the total response to obtain the constants $c_j$ that appear in the natural response

- We will rely on the zero-input/zero-state formulation

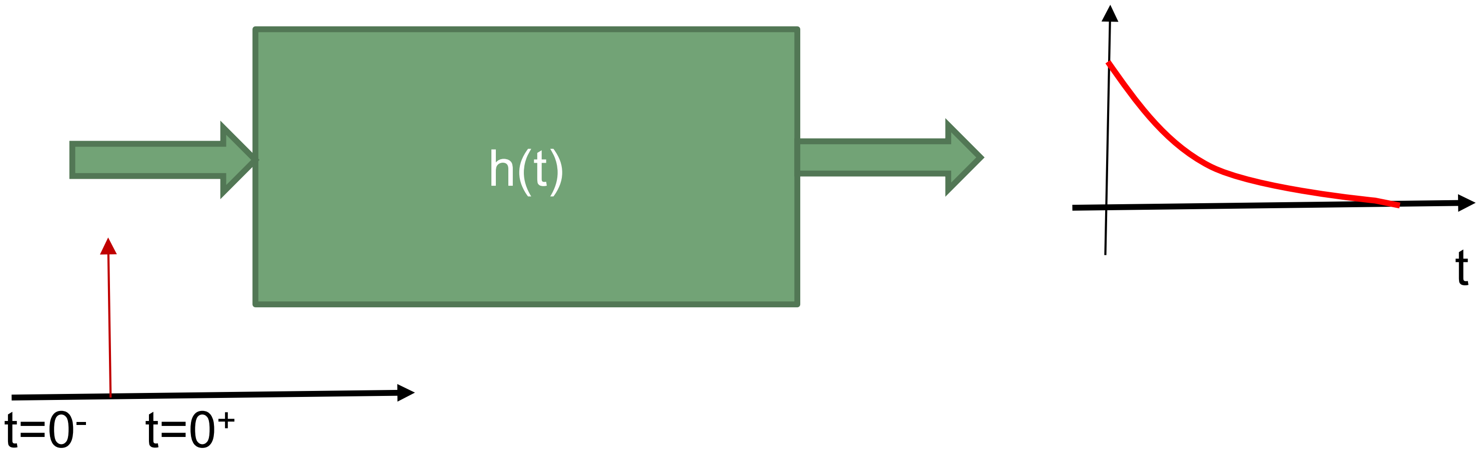
# System stability

- We consider causal, linear, time-invariant systems
- Key concept: equilibrium state



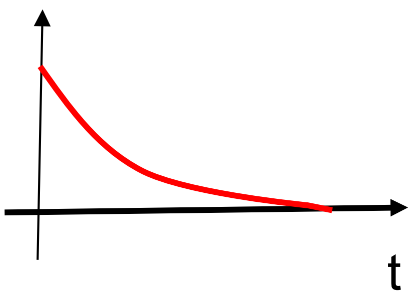
# System stability

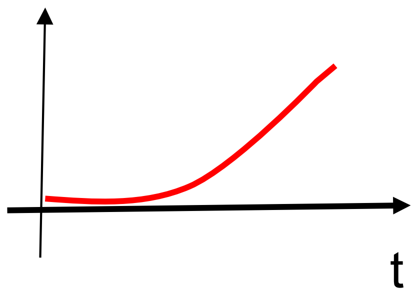
- Equilibrium : zero-state
- Let's assume that the system is in zero-state and we perturb it by creating non-zero initial conditions. Then the system is stable if it naturally evolves to the zero—state initial condition
- Since the zero-input response of the system is shaped by its characteristic modes, the output of a stable system in zero-input condition must tend to zero as t goes to infinity

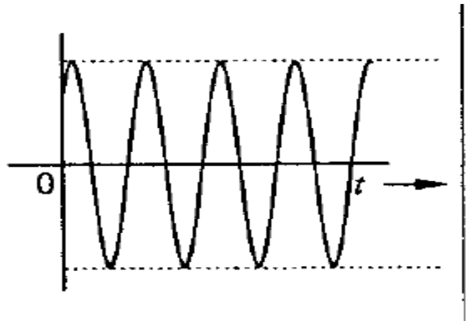


# System stability

- The system is (asymptotically) stable if and only if its characteristic modes tend to zero as t goes to infinity
- If any of the modes grows without bound the system is unstable
- If the zero-input response remains bounded approaching a constant or oscillating with a constant amplitude as t → infinity then the system is marginally stable



# System stability: dependence on characteristic roots

- Assuming n distinct characteristic modes

$$y(t) = \sum_{j=1}^{n} c_j e^{\lambda_j t}$$

- Then

$$\lim_{t \to \infty} e^{\lambda_j t} = 0 \text{ if } Re\{\lambda_j < 0\}$$

$$\lim_{t \to \infty} e^{\lambda_j t} = \infty \text{ if } Re\{\lambda_j > 0\}$$

- A system is stable if and only if its characteristic roots lie in the left half (LHP) of the complex plane
  - If any of the roots lie in the right half (RHP) then the system is unstable
  - If any of the simple (unrepeated) roots lie on the imaginary axis then the system is marginally stable
  - Repeated roots do not cause instability unless thay are on the imaginary axis ( $t^k e^{j\omega t}$ )

# System stability

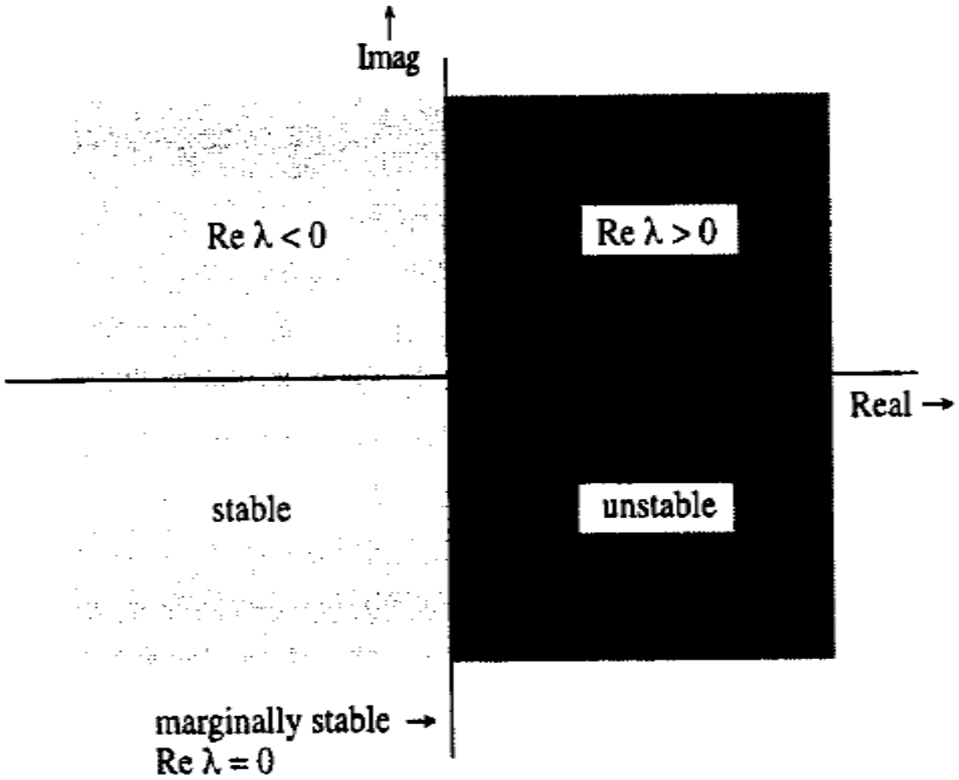
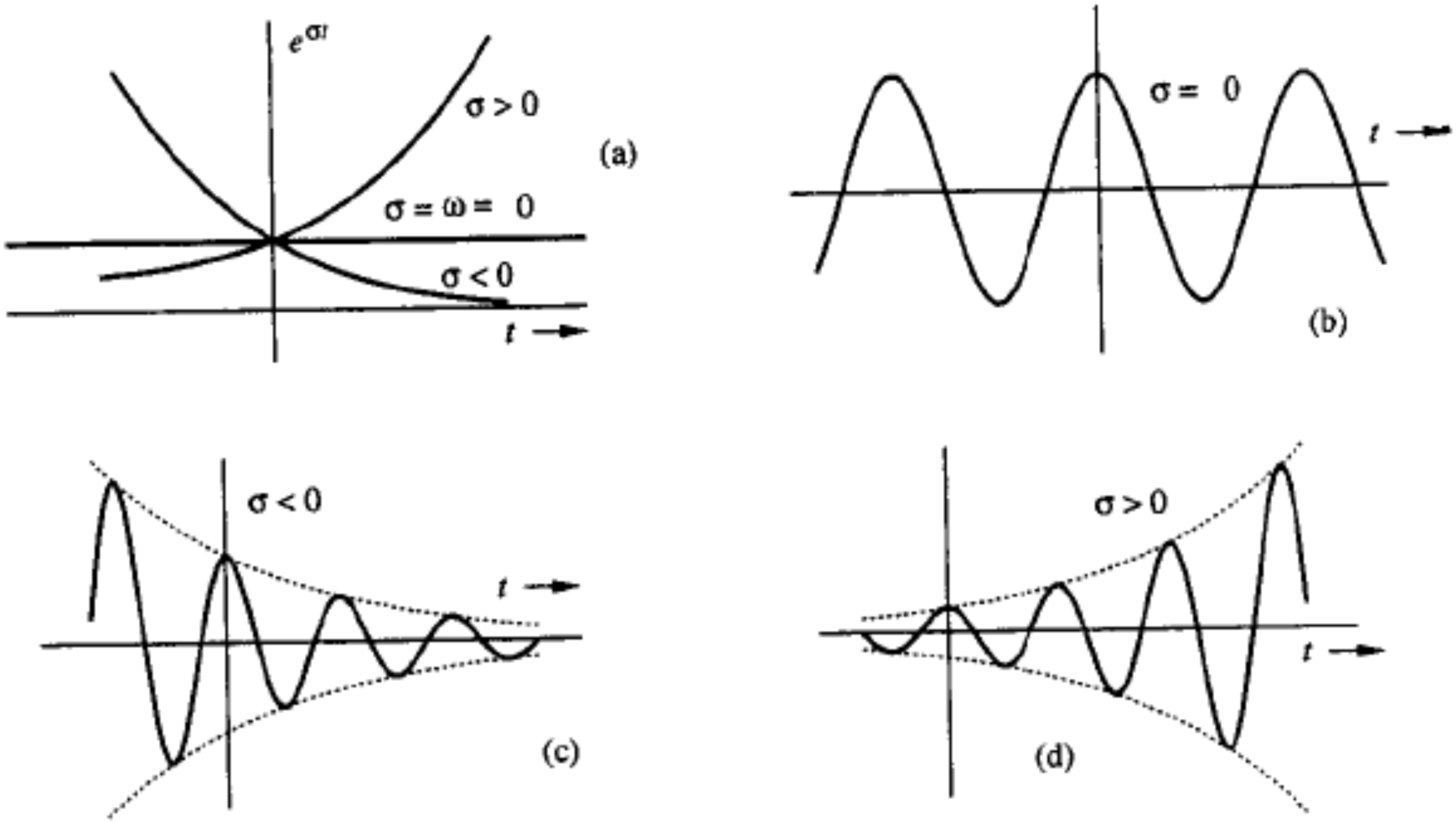


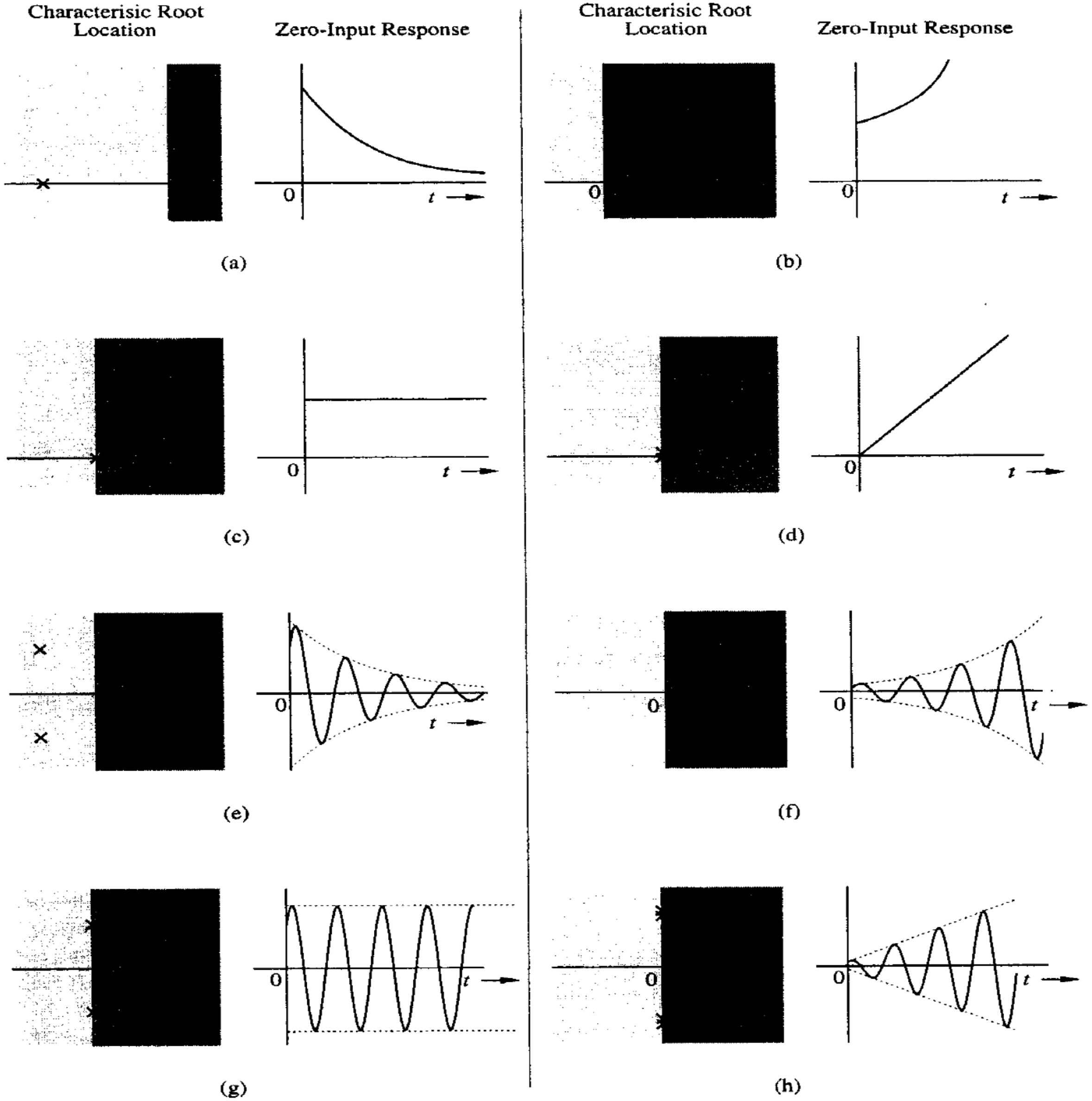

**Fig. 2.15** Characteristic roots location and system stability.

# System stability

$$y(t) = Ae^{\lambda t} = e^{\sigma t} \times e^{j\omega t}$$



# System stability



# BIBO systems

- In a stable system, a bounded input generates a bounded output

Recall that for an LTIC system

$$y(t) = h(t) * f(t)$$
$$= \int_{-\infty}^{\infty} h(\tau) f(t-\tau)\, d\tau$$

Therefore

$$|y(t)| \le \int_{-\infty}^{\infty} |h(\tau)||f(t-\tau)|\, d\tau$$

Moreover, if $f(t)$ is bounded, then $|f(t-\tau)| < K_1 < \infty$, and

$$|y(t)| \le K_1 \int_{-\infty}^{\infty} |h(\tau)|\, d\tau$$

Because $h(t)$ contains terms of the form $e^{\lambda_j t}$ or $t^k e^{\lambda_j t}$, $h(t)$ decays exponentially with time if $\mathrm{Re}\, \lambda_j < 0$. Consequently, for an asymptotically stable system†

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < K_2 < \infty \qquad (2.65)$$

and

$$|y(t)| \le K_1 K_2 < \infty$$

# Response time: the system time constant

- Since y(t)=h(t)*f(t), the width of the output is the sum of the widths of the input and the impulse response
- If the input is a delta, the output has a finite width which means that the system requires time to fully respond (Th, the width of h(t)
  - If the input has width $T_f$, the output has width $T_f+T_h$

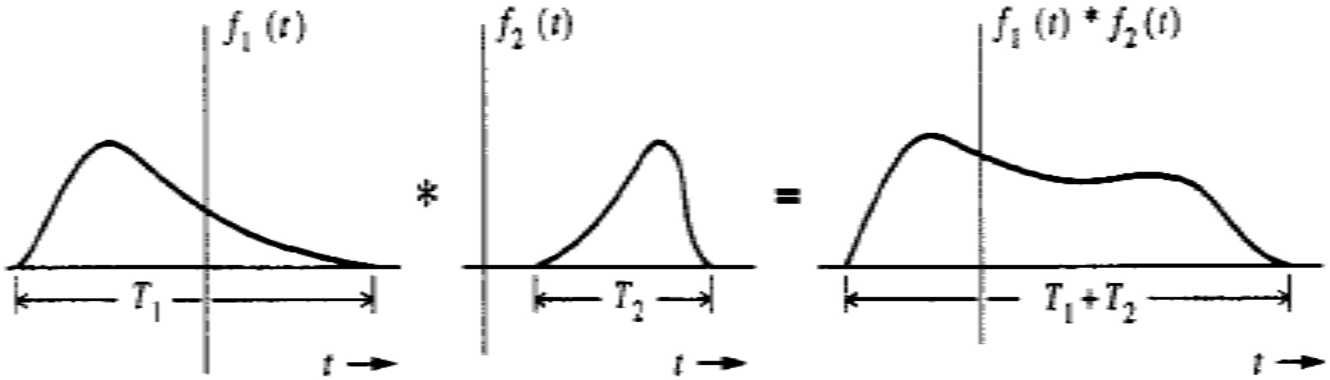



Fig. 2.4 Width property of convolution.

# System time constant

- The system time constant describes how fast the system is
  - The smaller the time constant, the fastest the system
- How to define it?

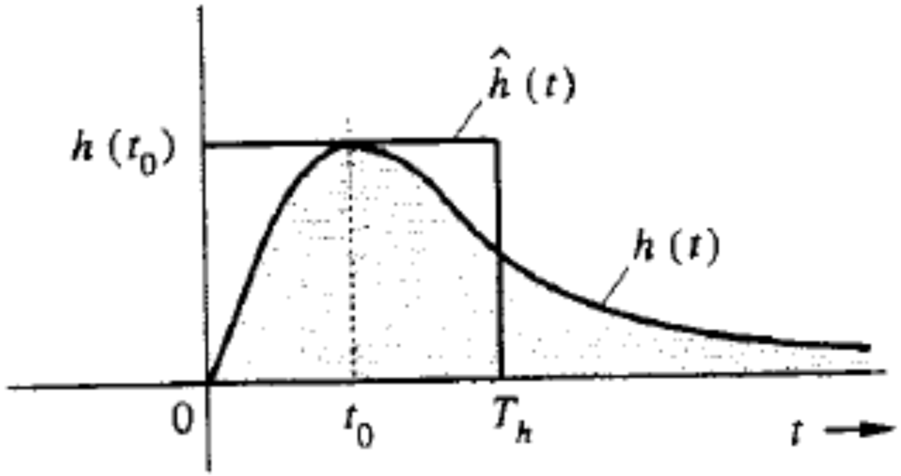



**Fig. 2.18** Effective duration of an impulse response.

# System time constant

- One possible way is to define it as the height of the rectangular pulse with same area than h(t)

$$T_h h(t_0) = \int_{-\infty}^{+\infty} h(t)dt \rightarrow T_h = \frac{\int_{-\infty}^{+\infty} h(t)dt}{h(t_0)}$$

- For a system with a single characteristic root

$$h(t) = Ae^{\lambda t}u(t)$$
$$h(0) = A$$
$$T_h = \frac{1}{A}\int_{0}^{+\infty} e^{\lambda t}dt = \frac{-1}{\lambda}$$

- For a multimodal system, h(t) is a weighted sum of the characteristic modes thus $T_h$ is a weighted sum of the time constants of the characteristic modes of the system

# Time constant and rise time

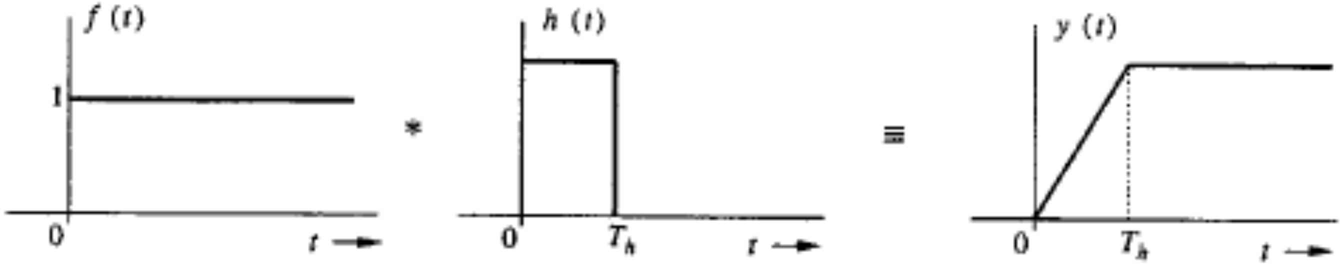



**Fig. 2.19** Rise time of a system.

$T_r = T_h$

# Time constant, filtering and rate of information

- There is a strict connection between the system time constant and its filtering properties
- A short time constant means that the system is able to react quickly to input variations

- A system with time constant $T_h$ behaves as a low-pass filter with cut-off frequency $F_c = 1/T_h$ Hertz
- The system takes $T_h$ seconds to fully react to a single impulse → in order to avoid interference among impulse responses these must be spaced by at least $T_h$ seconds
- → the maximum rate attainable by the system cannot exceed $1/T_h$ pulses/sec.

# Time constant and filtering

$$F_c = \frac{k}{T_r}$$

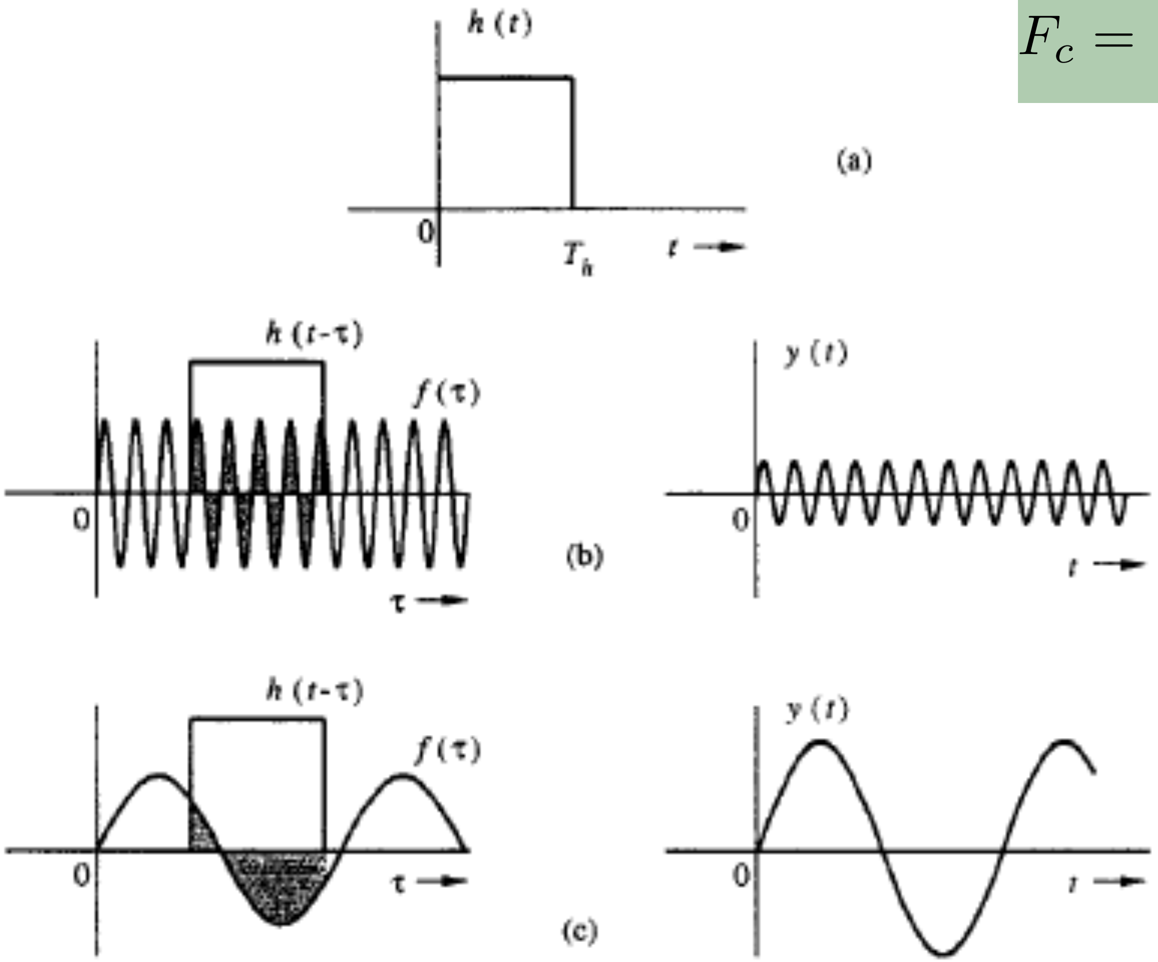



**Fig. 2.20** Time constant and filtering.