2.8 Natural Deduction

We extend the system of section 1.5 to predicate logic. For reasons similar to the ones mentioned in section 1.5 we consider a language with connectives $\wedge, \rightarrow, \perp$ and $\forall$. The existential quantifier is left out, but will be considered later.

We adopt all the rules of propositional logic and we add

$$\forall I \, \frac{\varphi(x)}{\forall x \varphi(x)} \quad \forall E \, \frac{\forall x \varphi(x)}{\varphi(t)}$$

where in $\forall I$ the variable x may not occur free in any hypothesis on which $\varphi(x)$ depends, i.e. an uncancelled hypothesis in the derivation of $\varphi(x)$. In $\forall E$ we, of course, require t to be free for x.

$\forall I$ has the following intuive explanation: if an arbitrary object x has the property φ, then every object has the property φ. The problem is that none of the objects we know in mathematics can be considered "arbitrary". So instead of looking for the "arbitrary object" in the real world (as far as mathematics is concerned), let us try to find a syntactic criteria. Consider a variable x (or a constant) in a derivation, are there reasonable grouns for calling x "arbitrary" ? Here is a plausible suggestion: in the context of the derivations we shall call x arbitrary if nothing has been assumed concerning x. In more technical terms, x is arbitrary at its particular occurrence in a derivation if the part of the derivation above it contains no hypotheses containing x free.

We will demonstrate the necessity of the above restrictions, keeping in mind that the system at least has to be *sound*, i.e. that derivable statements should be true.

Restriction on $\forall I$:

$$\dfrac{\dfrac{\dfrac{\dfrac{[x=0]}{\forall x(x=0)}}{x=0\to\forall x(x=0)}}{\forall x(x=0\to\forall x(x=0))}}{0=0\to\forall x(x=0)}$$

The $\forall$ introduction at the first step was illegal.

So $\vdash 0=0\to\forall x(x=0)$, but clearly $\not\models 0=0\to\forall x(x=0)$ (take any structure containing more than just 0).

Restriction on $\forall E$:

$$\dfrac{\dfrac{[\forall x\neg\forall y(x=y)]}{\neg\forall y(y=y)}}{\forall x\neg\forall y(x=y)\to\neg\forall y(y=y)}$$

The $\forall$ elimination at the first step was illegal.
Note that y is not free for x in $\neg\forall y(x=y)$. The derived sentence is clearly not true in structures with at least two elements.

We now give some examples of derivations. We assume that the reader has by now enough experience in cancelling hypotheses, so that we will not longer indicate the cancellations by encircled numbers.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall x\forall y\varphi(x,y)]}{\forall y\varphi(x,y)}\,\forall E}{\varphi(x,y)}\,\forall E}{\forall x\varphi(x,y)}\,\forall I}{\forall y\forall x(\varphi(x,y)}\,\forall I}{\forall x\forall y\varphi(x,y)\to\forall y\forall x\varphi(x,y)}\to I
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall x(\varphi(x)\wedge\psi(x))]}{\varphi(x)\wedge\psi(x)}}{\varphi(x)}}{\forall x\varphi(x)}\quad\dfrac{\dfrac{\dfrac{[\forall x(\varphi(x)\wedge\psi(x))]}{\varphi(x)\wedge\psi(x)}}{\psi(x)}}{\forall x\psi(x)}}{\forall x\varphi(x)\wedge\forall x\psi(x)}}{\forall x(\varphi\wedge\psi)\to\forall x\varphi\wedge\forall x\psi}$$

Let $x \notin FV(\varphi)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall x(\varphi \to \psi(x))]}{\varphi \to \psi(x)}\forall E \qquad [\varphi]}{\psi(x)} \to E}{\forall x \psi(x)} \forall I}{\varphi \to \forall x \psi(x)} \to I}{\forall x(\varphi \to \psi(x)) \to (\varphi \to \forall x(\psi(x))}$$

$$\dfrac{\dfrac{[\varphi]}{\forall x \varphi}\forall I \qquad \dfrac{[\forall x \varphi]}{\varphi}\forall E}{\varphi \leftrightarrow \forall x \varphi}$$

In the righthand derivation $\forall I$ is allowed, since $x \notin FV(\varphi)$, and $\forall E$ is applicable.

Note that $\forall I$ in the bottom left derivation is allowed because $x \notin FV(\varphi)$, for at that stage φ is still (part of) a hypothesis.

The reader will have grasped the technique behind the quantifier rules: reduce a $\forall x \varphi$ to φ and reintroduce $\forall$ later, if necessary. Intuitively, one makes the following step: to show " for all $x \ldots x \ldots$" it suffices to show "$\ldots x \ldots$" for an arbitrary x. The latter statement is easier to handle. Without going into fine philosophical distinctions, we note that the distinction "for all x $\ldots x \ldots$" – "for an arbitrary x $\ldots x \ldots$" is embodied in our system by means of the distinction. "quantified statement" – " free variable statement".

2.9 Adding the Existential Quantifier

Let us introduce $\exists x \varphi$ as an abbreviation for $\neg \forall x \neg \varphi$ (Theorem 2.5.1 tells us that there is a good reason for doing so). We can prove the following:

Lemma 2.9.1 *(i)* $\varphi(t) \vdash \exists x \varphi(x)$ *(t free for x in φ)*
(ii) $\Gamma, \varphi(x) \vdash \psi \Rightarrow \Gamma, \exists x \varphi(x) \vdash \psi$
if x is not free in ψ or any formula of Γ.

Proof. (i)

$$\dfrac{\dfrac{\dfrac{[\forall x \neg \varphi(x)]}{\neg \varphi(t)}\forall E \qquad \varphi(t)}{\bot} \to E}{\neg \forall x \neg \varphi(x)} \to I$$

so $\varphi(t) \vdash \exists x \varphi(x)$

(ii)

$$\dfrac{\dfrac{\neg\forall x\neg\varphi(x) \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}[\varphi(x)]\\ \mathcal{D}\\ \psi\end{matrix} \qquad [\neg\psi]}{\bot}\,{\to} E}{\neg\varphi(x)}\,{\to} I}{\forall x\neg\varphi(x)}\,\forall I}{\bot}\,{\to} E}{\psi}\,\text{RAA} \qquad \square$$

Explanation. The subderivation top left is the given one; its hypotheses are in $\Gamma \cup \{\varphi(x)\}$ (only $\varphi(x)$ is shown). Since $\varphi(x)$ (that is, all occurrences of it) is cancelled and x does not occur free in Γ or ψ, we may apply $\forall I$. From the derivation we conclude that $\Gamma, \exists x\varphi(x) \vdash \psi$.

We can compress the last derivation into an elimination rule for $\exists$:

$$\dfrac{\exists x\varphi(x) \qquad \begin{matrix}[\varphi]\\ \vdots\\ \psi\end{matrix}}{\psi}\,\exists E$$

with the conditions: x is not free in ψ, or in a hypothesis of the subderivation of ψ, other than $\varphi(x)$.

This is easily seen to be correct since we can always fill in the missing details, as shown in the preceding derivation.

By (i) we also have an introduction rule: $\dfrac{\varphi(t)}{\exists x\ \varphi(x)}\,\exists I$ for t free for x in φ.

Examples of derivations.

$$\dfrac{\dfrac{[\exists x\varphi(x)]^2 \qquad \dfrac{\dfrac{[\forall x(\varphi(x)\to\psi)]^3}{\varphi(x)\to\psi}\,\forall E \qquad [\varphi(x)]^1}{\psi}\,{\to} E}{\dfrac{\psi}{\exists x\varphi(x)\to\psi}\,{\to} I_2}\,\exists E_1}{\forall x(\varphi(x)\to\psi)\to(\exists x\varphi(x)\to\psi)}\,{\to} I_3 \qquad x\notin FV(\psi)$$

$$
\dfrac{\dfrac{[\exists x(\varphi(x) \vee \psi(x))]^3 \qquad \dfrac{[\varphi(x) \vee \psi(x)]^2 \qquad \dfrac{\dfrac{[\varphi(x)]^1}{\exists x \varphi(x)}}{\exists x \varphi(x) \vee \exists x \psi(x)} \qquad \dfrac{\dfrac{[\psi(x)]^1}{\exists x \psi(x)}}{\exists x \varphi(x) \vee \exists x \psi(x)}}{\exists x \varphi(x) \vee \exists x \psi(x)} \vee E_1}{\exists x \varphi(x) \vee \exists x \psi(x)} \exists E_2}{\exists x(\varphi(x) \vee \psi(x)) \rightarrow \exists x \varphi(x) \vee \exists x \psi(x)} \rightarrow I_3
$$

We will also sketch the alternative approach, that of enriching the language.

Theorem 2.9.2 *Consider predicate logic with the full language and rules for all connectives, then* $\vdash \exists x \varphi(x) \leftrightarrow \neg \forall x \neg \varphi(x)$.

Proof. Compare 1.6.3. □

It is time now to state the rules for $\forall$ and $\exists$ with more precision. We want to allow substitution of terms for some occurrences of the quantified variable in $(\forall E)$ and $(\exists E)$. The following example motivates this.

$$
\dfrac{\dfrac{\forall x(x = x)}{x = x} \forall E}{\exists y(x = y))} \exists I
$$

The result would not be derivable if we could only make substitutions for *all* occurrences at the same time. Yet, the result is evidently true.

The proper formulation of the rules now is:

$$
\forall I \ \dfrac{\varphi}{\forall x \varphi} \qquad \forall E \ \dfrac{\forall x \varphi}{\varphi[t/x]}
$$

$$
\exists I \ \dfrac{\varphi[t/x]}{\exists x \varphi} \qquad \exists E \ \dfrac{\exists x \varphi \qquad \begin{matrix} [\varphi] \\ \vdots \\ \psi \end{matrix}}{\psi}
$$

with the appropriate restrictions.