o input and zero-state responses

initial conditions are assumed to be zero

i.e. in zero state)

input signal has no influence

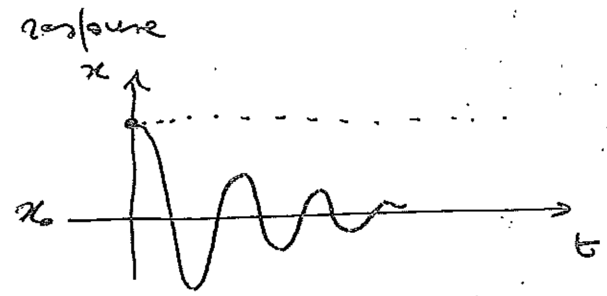
zero input responses are independent.

see

motion is irreversible unless some pieces of

needed. More precisely, N auxiliary condition

conditions at $t=0 \Leftrightarrow$ initial conditions



a proper combination of characteristic modes

The Unit impulse response

Goal: determine the impulse response $h(t)$ of a LTIC system, described by an n^{th} order differential equation:

$$Q(D)\,y(t) = P(D)\cdot f(t)$$

Due to considerations on noise: $m \leq n$ (to be shown)

Let's consider the most general case:

$$m = n$$

$$\left(D^n + a_{n-1}D^{n-1} + \dots + a_1 D + a_0\right)y(t) = \left(b_n D^n + b_{n-1}D^{n-1} + \dots + b_1 D + b_0\right)$$

When solicited by an impulse, the response of the system for $t>0$ is due to the newly created initial conditions.

$\Rightarrow$ $h(t) \Leftrightarrow$ characteristic modes $t \geq 0^+$

and:

$$h(t) = A_0\delta(t) + \text{characteristic modes} \quad t \geq 0$$

Formally:

$$h(t) = b_n\delta(t) + \left[P(D)y_n(t)\right]u(t)$$

$u(t)$ ↳ unit step function

$\left[P(D)y_n(t)\right]$ ↓ L.C. of characteristic modes subject to the following initial conditions:

$y^{(n-1)}(0) = 1$

$y^{(k)}(0) = 0 \quad \forall\ 0 \leq k \leq n-2$

At a single moment $t=0$ there can be at most an impulse

The everlasting exponential e^{st}

* The LTI system zero-state response to everlasting exponential input e^{st} is the same everlasting exponential *

$$e^{st} \longrightarrow \boxed{h(t)} \longrightarrow e^{st}$$

e^{st} is the eigenfunction or characteristic function of the system.

⇒ Sinusoids are also eigenfunctions of LTI systems.

Formally:

$$y(t) = h(t) * e^{st} = \int_{-\infty}^{+\infty} h(\tau) e^{s(t-\tau)} d\tau =$$

$$= \int_{-\infty}^{+\infty} e^{st} h(\tau) e^{-s\tau} d\tau = e^{st} \underbrace{\int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau}_{H(s)}$$

$$\Rightarrow \left\{ \begin{array}{l} y(t) = H(s)\, e^{st} \\ H(s) = \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau \end{array} \right.$$

where, ⇒ TRANSFER FUNCTION

So:

$$H(s) = \left. \frac{\text{output}}{\text{input}} \right|_{\text{input} = e^{st}}$$

NOTE: The transfer function is meaningful only for LTI systems and in general does not exist for non-linear and time-varying systems.

Total response

$$y(t) = \underbrace{\sum_1^n c_j e^{\lambda_j t}}_{\text{zero input}} + \underbrace{f(t) * h(t)}_{\text{zero-state}}$$

NOTE: in case the zero-state response contains characteristic modes, these can be lumped together in the total response, giving the NATURAL RESPONSE $y_n(t)$. Then, the remainder is known as FORCED RESPONSE $y_\phi(t)$.

$$\Rightarrow \quad y(t) = \underset{\substack{\downarrow \\ \text{natural} \\ \downarrow}}{y_n(t)} + \underset{\substack{\downarrow \\ \text{forced}}}{y_\phi(t)}$$

$\left\{ \begin{array}{l} \text{homogeneous solution} \\ \text{complementary} \end{array} \right.$

Rewriting the relations:

$$Q(D)\, y(t) = P(D)\, f(t)$$

where:

$$Q(D) = D^n + a_{n-1} D^{n-1} + \dots + a_1 D + a_0$$

$$P(D) = b_m D^m + b_{m-1} D^{m-1} + \dots + b_1 D + b_0$$

$$\Rightarrow \quad Q(D)\left[y_n(t) + y_\phi(t) \right] = P(D) f(t)$$

$$Q(D)\, y_n(t) + Q(D)\, y_\phi(t) = P(D) f(t)$$

but: $Q(D)\, y_n(t) = 0$ because $y_n(t)$ consists of characteristic modes

$$\Rightarrow \quad \boxed{Q(D)\, y_\phi(t) = P(D) f(t)}$$ Forced response (to the input $f(t)$)

Forced response: The method of undetermined coefficients

CI → forced

anything causal → natural

Consider a LTIC system.

Suppose $f(t)$ yields only a finite number of independent derivatives

Then: The $y_\varphi(t)$ for $f(t)$ can be expressed as a linear combination of the input and its independent derivatives.

$f(t) \Leftrightarrow t^r, e^{\xi t}$ satisfy this constraint.

Ex: $f(t) = at^2 + bt + c$

Then: $y_\varphi(t) = \beta_2 t^2 + \beta_1 t + \beta_0$

Replacing $y_\varphi(t)$ in $Q(D)y_\varphi(t) = P(D)f(t)$ and equating the coefficients of corresponding terms the values of the parameters β_i can be determined.

Note: initial conditions are used to determine the coefficients in the natural modes $y_n(t)$.

(see example 2.8).

Summary (Table 2.2 pp. 161)

$f(t)$	$y_\varphi(t)$
1. $e^{\xi t}$	$\beta e^{\xi t}$
2. $e^{\xi t}$, $\xi = \lambda_i$	$\beta t e^{\xi t}$
3. k	β
4. $\cos(\omega t + \varphi)$	$\beta \cos(\omega t + \varphi)$
5. $\left(\sum_{0}^{R} {}_r \alpha_r t^r\right) e^{\xi t}$	$\sum_0^R {}_r \left(\beta_r t^r\right) e^{\xi t}$

Exponential input

$$f(t) = e^{\xi t} \quad \rightarrow \quad y_\varphi(t) = \beta e^{\xi t}$$

We show that:

$$\beta = \frac{P(\xi)}{Q(\xi)}$$

Proof:

$$Q(D)\, y_\varphi(t) = P(D) f(t)$$

$$Q(D)\left[\beta e^{\xi t}\right] = P(D)\left[e^{\xi t}\right] \qquad (1)$$

but:

$$D e^{\xi t} = \xi e^{\xi t}$$

$$D^2 e^{\xi t} = \xi^2 e^{\xi t} \qquad \Rightarrow Q(D) e^{\xi t} = Q(\xi) e^{\xi t}$$

$$\vdots$$

$$D^n e^{\xi t} = \xi^n e^{\xi t}$$

similarly: $P(D) e^{\xi t} = P(\xi) e^{\xi t}$

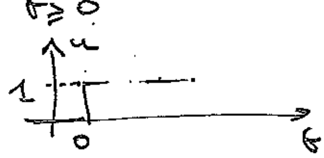
$\Rightarrow$ replacing in (1):

$$\beta Q(\xi) e^{\xi t} = P(\xi) e^{\xi t}$$

$$\rightarrow \beta = \frac{P(\xi)}{Q(\xi)} \qquad \text{CVD}$$

Limiting the response/stimulation to $t \geq 0$

$$f(t) = e^{\xi t} u(t)$$



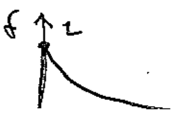
Then:

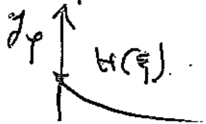
$$y_\varphi(t) = H(\xi) e^{\xi t} \qquad t > 0$$

$$H(\xi) = P(\xi) / Q(\xi)$$

Summary:

$$f(t) = e^{\xi t} u(t) \quad \Rightarrow \quad y_\varphi(t) = H(\xi) e^{\xi t}$$





$(\xi < 0)$

Total system response to an exponential input

$$y(t) = \sum_{1}^{n} j\, k_j e^{\lambda_j t} + H(\xi) e^{\xi t}$$

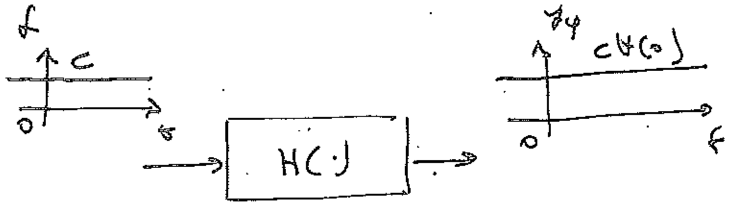
Special cases:

1. $f(t) = c$

   $f(t) = c \cdot e^{\xi t} \qquad \xi = 0$

   $\to \beta = H(0)$

   $y_\varphi(t) = c \cdot H(0)$



2. $f(t) = e^{j\omega t} \qquad \xi = j\omega \qquad \Rightarrow y_\varphi(t) = H(j\omega) e^{j\omega t}$

3. $f(t) = \cos(\omega t)$

   $\cos(\omega t) = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)$

$$\Rightarrow y_\varphi(t) = \frac{1}{2}\left\{ H(j\omega) e^{j\omega t} + H(-j\omega) e^{-j\omega t} \right\} =$$

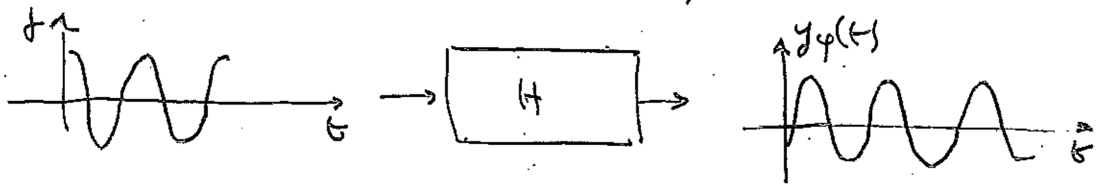
$$= \mathrm{Re}\left\{ H(j\omega) e^{j\omega t} \right\} = \mathrm{Re}\left\{ |H(j\omega)| \cdot e^{j[\omega t + \measuredangle H(j\omega)]} \right\} =$$

$$= |H(j\omega)| \cdot \mathrm{Re}\left\{ e^{j[\omega t + \measuredangle H(j\omega)]} \right\} =$$

$$= |H(j\omega)| \cos(\omega t + \measuredangle H(j\omega))$$

generalization:

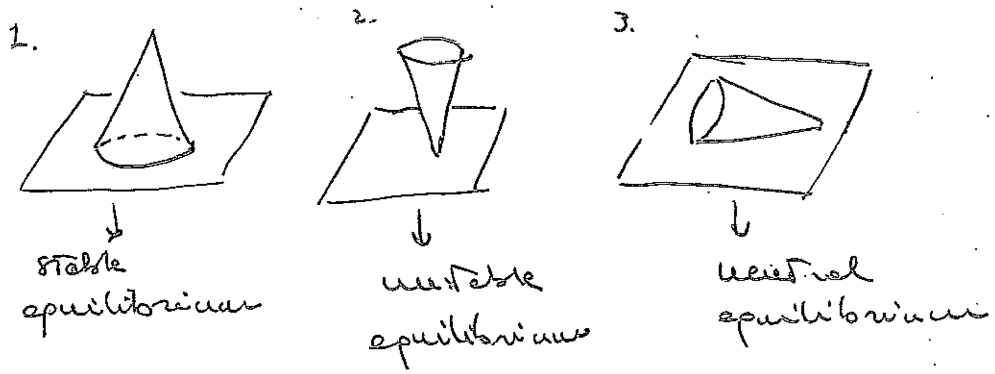
$$f(t) = \cos(\omega t + \theta) \quad \to \quad y_\varphi(t) = |H(j\omega)| \cos(\omega t + \theta + \measuredangle H(j\omega))$$



$(\theta = 0)$

System stability

We restrict to causal, linear, time-invariant (LTI) systems



equilibrium states

Definition:

* If in absence of an external input a system remains in a particular state, or condition, indefinitely, then the state is said to be an equilibrium state of the system.

If this state is perturbed (in $t=0$) and some new initial cond. are created, then if the system is stable it should return to the zero state ⇒ when left to itself. The system's output due to initial conditions should approach zero as $t \to +\infty$.

But the zero-input response (i.e. the response due to the initial conditions) is due to the characteristic modes ⇒

* A system is (asymptotically) stable iff its characteristic modes $\to 0$ as $t \to +\infty$.

⇒ If any of the characteristic modes grows without bound for $t \to \infty$ the system is unstable.

* If the zero-input response remains constant or oscillates with a constant amplitude as $t \to \infty$ the system is said to be marginally stable or simply stable.

Stability condition.

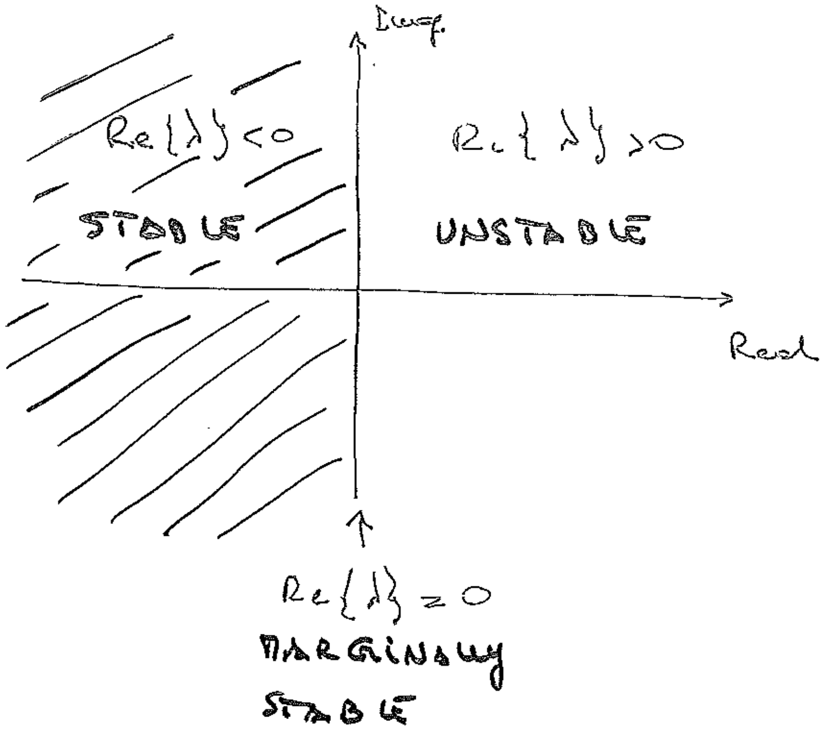
characteristic roots: $\lambda_1, \lambda_2, \dots, \lambda_m$

$$\Rightarrow \quad y_o(t) = \sum_{1}^{n}{}_j e^{\lambda_j t} \cdot c_j$$

$$\lambda_j \in \mathbb{C}$$

But: $$\lim_{t \to \infty} e^{\lambda_j t} = \begin{cases} \phi & \text{if } Re\{\lambda_j\} < 0 \\ \infty & \text{if } Re\{\lambda_j\} > 0 \end{cases}$$

$\Rightarrow$ the stability of the system can be studied looking at the position of the characteristic roots in the complex plane



C.R. in the LHP (left-half plane) $\rightarrow$ stable (asymptotically)

CR in the RHP (right-half plane) $\rightarrow$ unstable

CR on the imaginary axis $\longrightarrow$ marginally stable.

A system is asymptotically stable if all its CR are in the LHP.
If any of the roots lies in the RH the syst. unstable $\rightarrow$
it is enough to have one root on the RHP to cause instability.

Repeated roots

CR λ is repeated R times $\Rightarrow$ modes $e^{\lambda t}, te^{\lambda t}, \dots, t^{R-1}e^{\lambda t}$

But: $\lim_{t\to\infty} t^k e^{\lambda t} = 0$ if $Re\{\lambda\} < 0$

$\Rightarrow$ # repeated roots in the LHP do not cause instability.

When the roots are on the imaginary axis:

$\lambda = j\omega \rightarrow \lim_{t\to\infty} t^k e^{j\omega t} = \infty$

$\Rightarrow$ # repeated roots on the imaginary axis cause instability

Summary

1. A LTIC system is asymptotically stable i.i.f. its characteristic roots are in the LHP; the roots may be simple or repeated

2. A LTIC system is unstable iff. either one or both of these conditions hold:

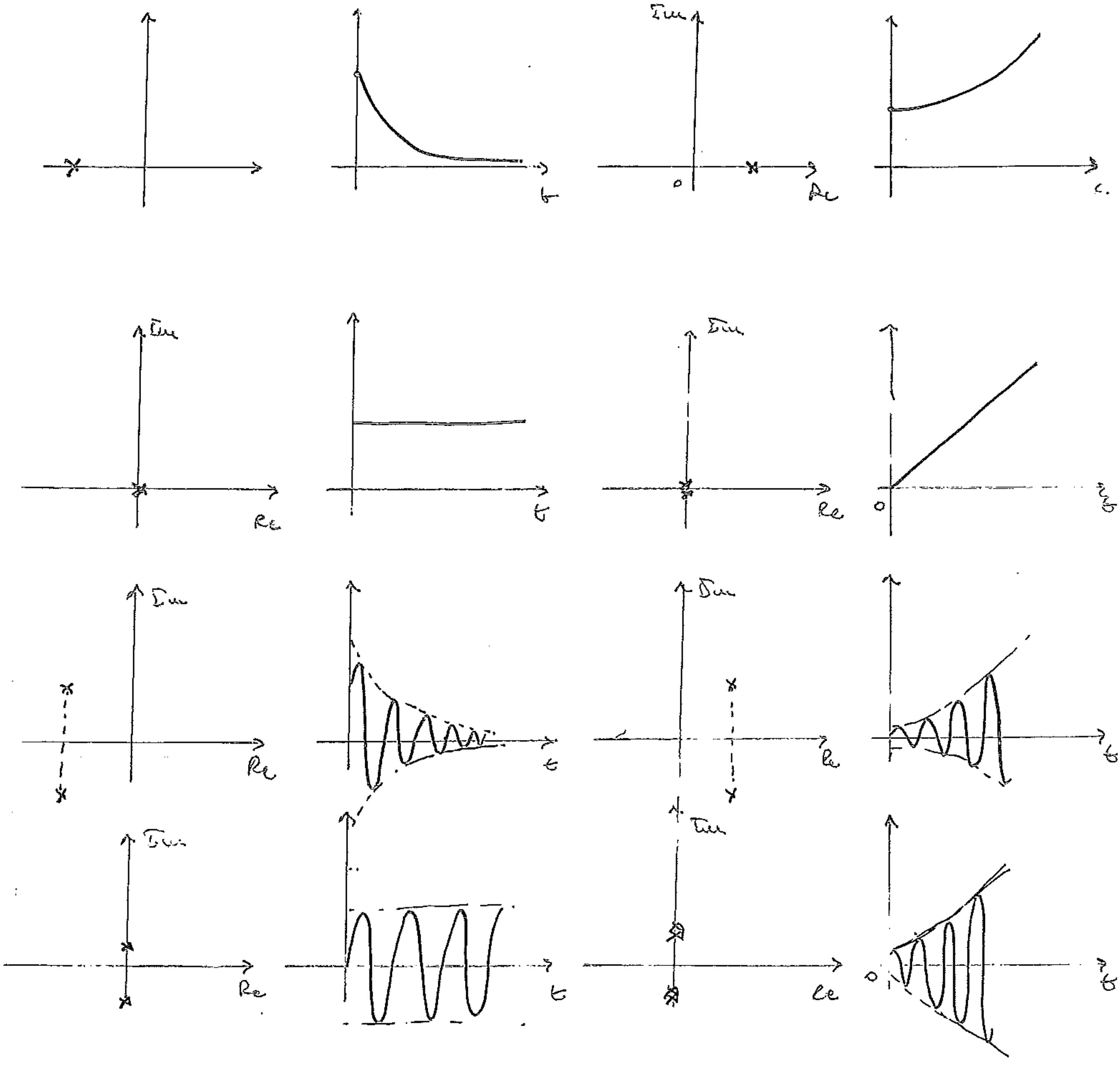
   a. At least one root is in the RHP

   b. There are repeated roots on the imaginary axis

3. A LTIC system is marginally stable if and only if there are no roots in the RHP and there are some unrepeated roots on the imaginary axis.

Location of characteristic roots and the corresponding characteristic modes.

CR zero-input resp. CR zero-input resp.



System Response to a Bounded Input

Intuition : stable system → bounded inputs generate bounded output

Proof:

$$y(t) = h(t) * f(t) = \int_{-\infty}^{+\infty} h(\tau)\, f(t-\tau)\, d\tau$$

$$\Rightarrow \quad |y(t)| \le \int_{-\infty}^{+\infty} |h(\tau)|\,|f(t-\tau)|\, d\tau$$

if $f(t)$ is bounded: $\|f(t)\| < k_1 < +\infty$

Then:

$$|y(t)| \le k_1 \int_{-\infty}^{+\infty} |h(t-\tau)|\, d\tau$$

$$|y(t)| \le k_1 \int_{-\infty}^{+\infty} |h(\tau)|\, d\tau$$

For an asymptotically stable system, $h(t)$ consists of terms of the type: $e^{\lambda_j t}$, $t e^{\lambda_j t}$ with $Re\{\lambda_j\} < 0$, Thus:

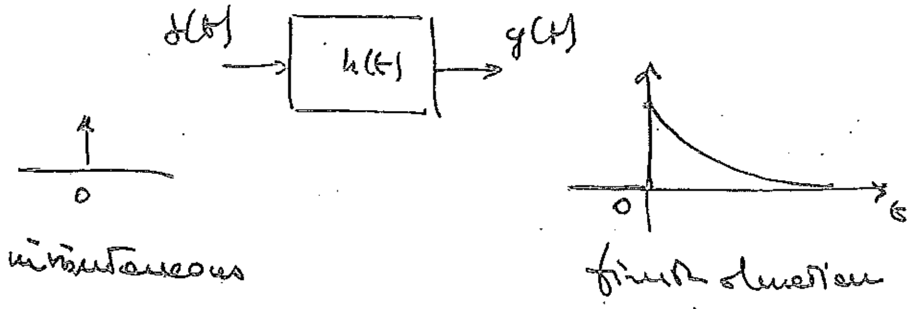
$$\int_{-\infty}^{+\infty} |h(\tau)|\, d\tau \le k_2 < +\infty$$

$\Rightarrow \quad |y(t)| \le k_1 k_2 < +\infty$ CVD

* <u>Asymptotically stable system</u> $\Leftrightarrow$ <u>bounded input $\Rightarrow$ bounded output.</u>

→ BIBO stability (Bounded Input Bounded Output)

Response Time of a system: The System Time constant



Time required to the system to "fully respond" to the input

* The time constant is equal to the width of the impulse response *

Other argument:

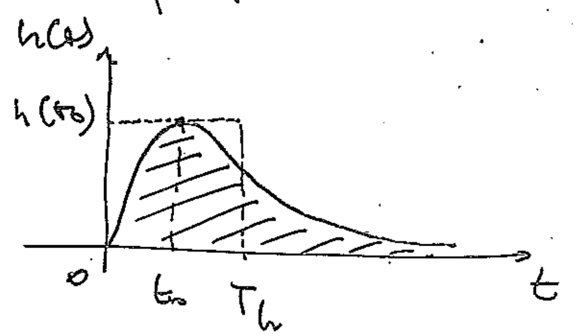
$x(t) \leftrightarrow T_x$

$h(t) \leftrightarrow T_h \quad \Rightarrow x(t) * h(t) \leftrightarrow T_x + T_h \Rightarrow$ width of $y(t)$

* Time constant $\propto \dfrac{1}{\text{speed of response}}$ *

Suitable measures for T_h are to be defined.
These are case-specific.

Ex.



→ infinite duration
Possible criterion: equality of subtended area.

$$h(t_0)\cdot T_h = \int_0^{+\infty} h(t)\,dt \Rightarrow T_h = \frac{\int_0^{+\infty} h(t)\,dt}{h(t_0)}$$

If the system has a single mode: $e^{\lambda t} \to h(t) = Ae^{\lambda t}u(t)$

$h(0) = A$

$$T_h = \frac{1}{A}\int_0^{+\infty} A e^{\lambda t}\, dt$$

If: $\lambda < 0$

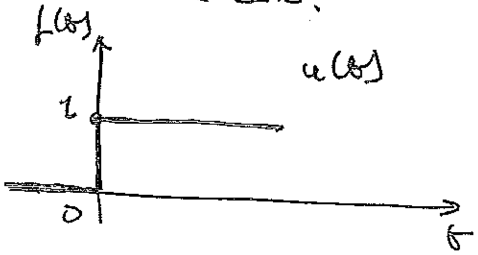
$$T_h = \frac{1}{A}\cdot A \cdot \frac{1}{\lambda}\left(e^{\lambda t}\right)_0^{+\infty} = -\frac{1}{\lambda} \rightarrow \boxed{T_h = -\frac{1}{\lambda}}$$

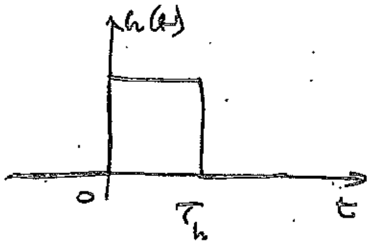
→ negative of 1/characteristic root

Multimode systems: T_h = weighted sum of reciprocal of characteristic roots (with changed signs)

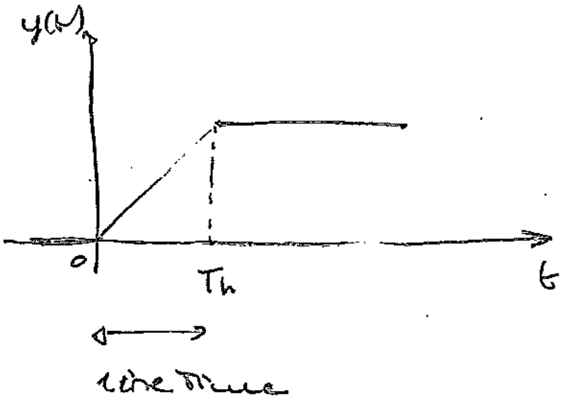
Rise time of a system

Consider the case:





$y(t) = f(t) * h(t)$



Filtering properties

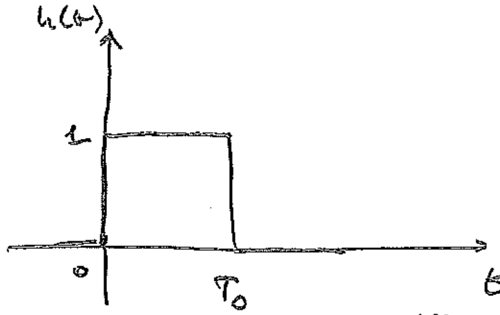
Th small → The system responds to rapid variations of the input → it is able to "follow" rapid input signal's evolution

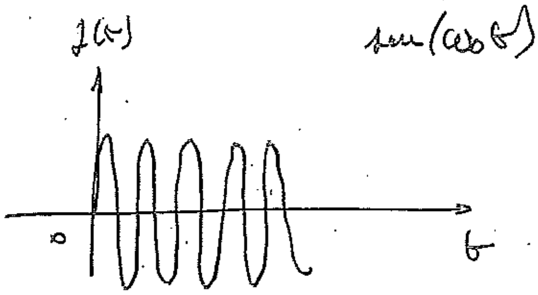
Viceversa: Th large → slow response → rapid input changes are disregarded

$T_h \ll$ → fast response → high frequencies are preserved

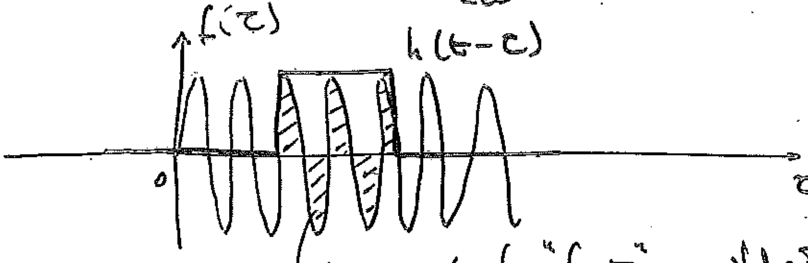
$T_h \gg$ → slow response → " " " "cutted"

Intuition

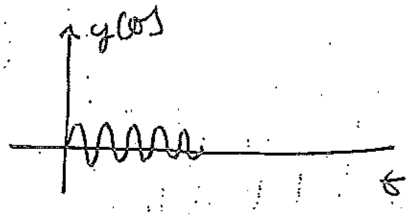




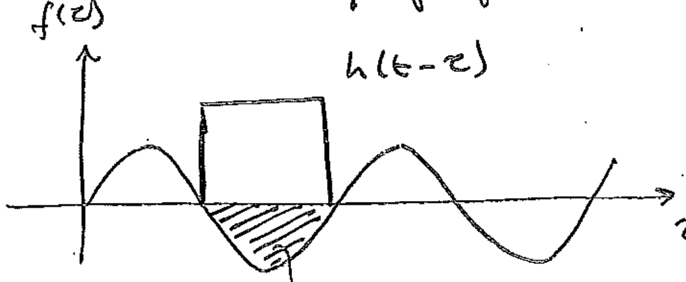
$$y(t) = h(t) * f(t) = \int_{-\infty}^{+\infty} h(\tau) f(t-\tau)\,d\tau = \int_{-\infty}^{+\infty} f(\tau)\, h(t-\tau)\,d\tau$$

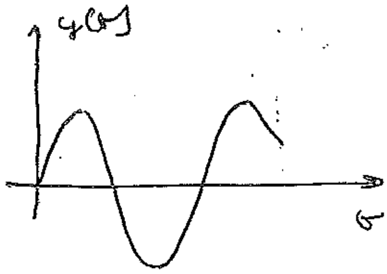


⇒



→ $\emptyset$ for "fast" oscillations





$\neq \emptyset$ for "slow" oscillations

⇒ If the period of the sinusoid is $\ll T_h$ ⇒ The response is "small"

" " " " " " $\gtrsim T_h$ ⇒ " " "large"

⇒ Cutoff frequency : $\boxed{F_c = \frac{1}{T_h}}$ bandwidth of the system

Since: $T_r = T_h$

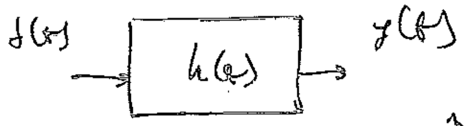
Then: $\boxed{T_r = \frac{1}{F_c}}$ The system's rise time is inversely proportional to the bandwidth

In the general case, it can be proved that:

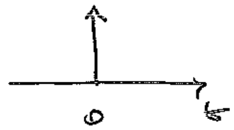
$$F_c = \frac{k}{T_h}$$

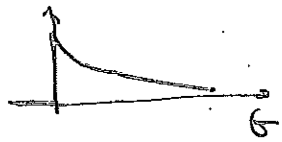
where k depends on $h(t)$.

Pulse dispersion (or spreading)

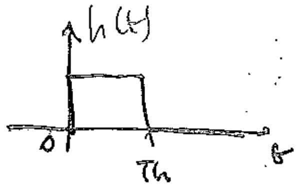


1.



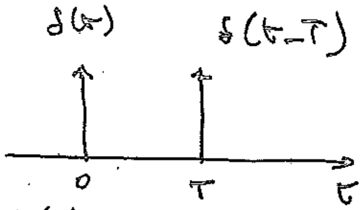


$h(t) = Ae^{\delta t}$, $\delta < 0$

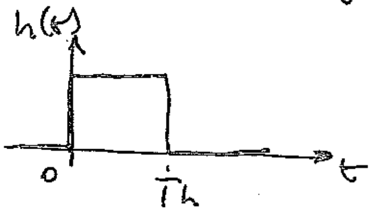


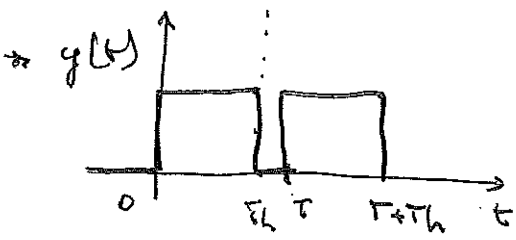
$h(t) \rightarrow$ rectangular pulse

2.

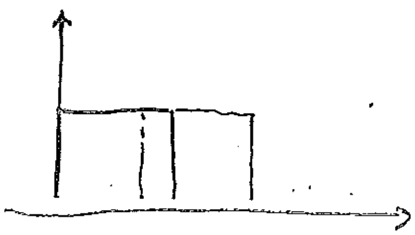


$x(t) = \delta(t) + \delta(t-T)$





$T > T_h$



$T < T_h$

$\rightarrow$ interference $\rightarrow$ The rate of pulse transmission should not exceed $1/T_h$

But :
$$F_c = \frac{1}{T_c}$$

$\Rightarrow$ The frequency of the pulse train cannot not exceed the channel (or system) bandwidth.

NOTE : The real theoretical limit is $2F_c$, but it is very difficult to attain. Usually, F_c pulses/sec is used.

Resonance

Conditions : The input signal is very similar to a characteristic mode of the system.

Let's assume the system has one simple mode $e^{\lambda t}$.
$$h(t) = A e^{\lambda t}$$
and let the input signal be:
$$f(t) = e^{(\lambda - \varepsilon)t}$$
Then:
$$y(t) = A e^{\lambda t} * e^{(\lambda-\varepsilon)t} = \dots = A e^{\lambda t}\left(\frac{1-e^{-\varepsilon t}}{\varepsilon}\right)$$
We are interested in the case $\varepsilon \to \phi$

$$\lim_{\varepsilon\to\phi} y(t) = \lim_{\varepsilon\to 0} A e^{\lambda t} \frac{1-e^{-\varepsilon t}}{\varepsilon} = \underset{d/d\varepsilon}{(\text{l'Hopital})} = A t e^{\lambda t} \qquad \frac{t e^{-\varepsilon t}}{1} \to t$$

1. $\mathrm{Re}\,\lambda < 0$: $e^{\lambda t}$ decays faster than $t \Rightarrow y(t) \to 0$ when $t \to \infty$
2. $\mathrm{Re}\,\lambda = 0 \to \lambda = j\omega \to y(t) = A \cdot t \cdot e^{j\omega t} \to y(t) \to \infty$ linearly with t.
3. Real system $\begin{cases} \lambda = j\omega \\ \lambda^* = -j\omega \end{cases} \Rightarrow h(t) = A\cos(\omega t)$

if : $x(t) = \cos(\omega t) \to y(t) = A\cos(\omega t) * \cos(\omega t) = \dots \approx A t \cos(\omega t)$

LTIC system response to periodic inputs

$f(t)$ periodic $\Rightarrow f(t) = \sum_{-\infty}^{+\infty} D_n e^{jn\omega_0 t}$; $\omega_0 = \frac{2\pi}{T_0}$

LTIC system:

$$e^{st} \longrightarrow \boxed{H(s)} \longrightarrow H(s)e^{st}$$

$$e^{j\omega t} \rightarrow H(j\omega) \rightarrow e^{j\omega t} H(j\omega)$$

$\Rightarrow$ from linearity,

$$f(t) \rightarrow \underbrace{\sum_{-\infty}^{\infty} e^{jn\omega_0 t} H(jn\omega_0) D_n}$$

$\downarrow$

periodic signal with the same period

Signal Transmission Through LTIC systems

$$X(\omega) \rightarrow \boxed{H(\omega)} \rightarrow Y(\omega)$$

in asymptotic marginally stable systems

$Y(\omega) = X(\omega) H(\omega)$

$\downarrow$

frequency response of the system

$|Y(\omega)| = |X(\omega)| \cdot |H(\omega)|$ → amplitude change

$\measuredangle Y(\omega) = \measuredangle X(\omega) \measuredangle H(\omega)$ → $\measuredangle$ change

Distortionless processing: $y(t) = k \cdot f(t - t_d)$

$\rightarrow Y(\omega) = k e^{-j\omega t_d} F(\omega)$

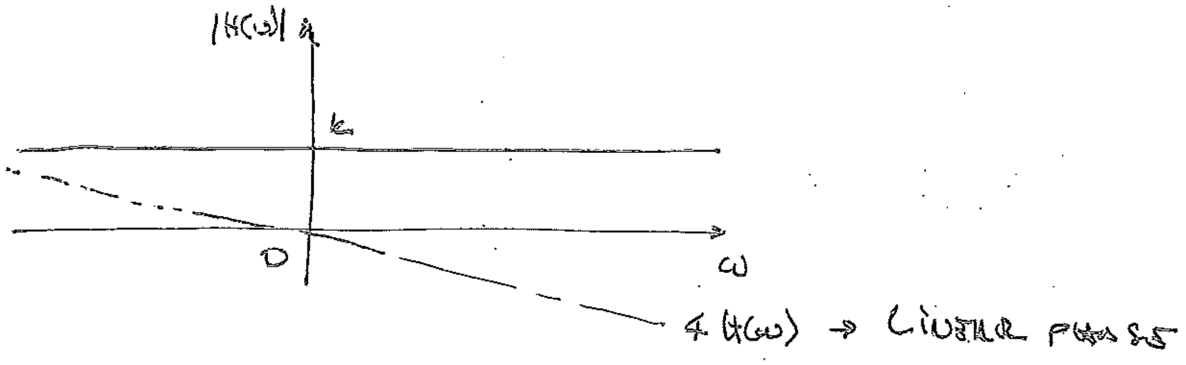
but for a LTIC system:

$Y(\omega) = H(\omega) F(\omega)$ $\Rightarrow$ $\boxed{H(\omega) = k e^{-j\omega t_d}}$

$|H(\omega)| = k$

$\measuredangle H(\omega) = -\omega t_d$

Frequency response of a distortionless system:



Intuition: for the output to be a replica of the input
- all harmonics must be amplified / attenuated by the same factor
- The relative phase between the harmonics remains unchanged if the system is linear phase

Note
- Audio signals: for a ∢ distortion to be perceivable, the slope of ∢ H(ω) should be comparable to the duration of the signal.

spoken syllable ⇒ single signal

$0.01 \div 0.1\,s$ (10 ÷ 100 ms)

In practical audio systems, the slope of ∢ H(ω) is a few ms ⇒ hardly perceivable

⇒ The human hear is relatively insensitive to phase distortion

- Video signals

phase (spatial) delay ⇒ smearing ⇒ highly perceivable

⇒ The human eye is very sensitive to phase changes.

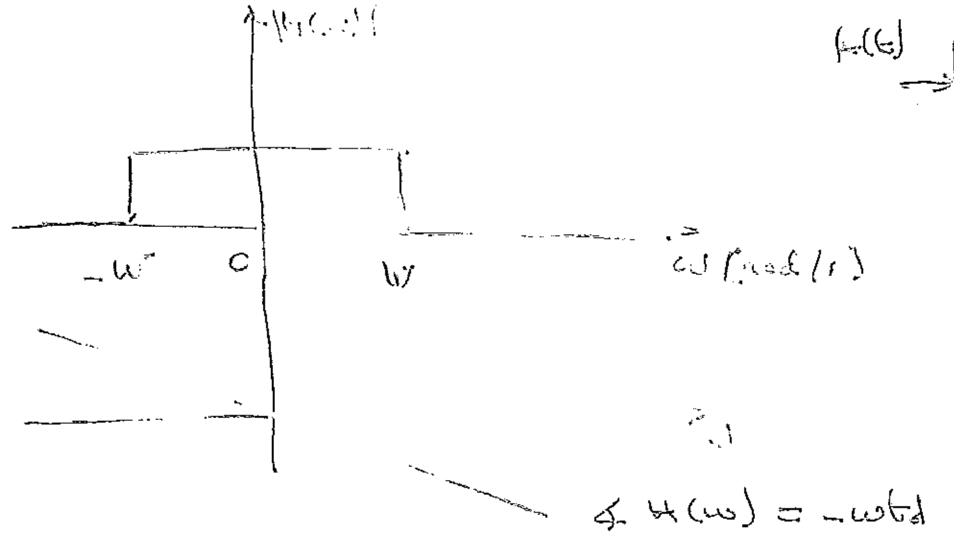
Intuition ⇒ smearing of edges, which are perceptually very important for image interpretation by the HVS

Filters.

Ideal filter: allow distortionless transmission of a given band of frequencies and suppress all the remaining frequencies.

Ex: Ideal low pass filter

$$x(t) \to \boxed{H(\omega)} \to y(t)$$



delay $= t_d$

$y(t) = x(t - t_d)$

Formally:

$$\begin{cases} |H(\omega)| = \text{rect}\left(\dfrac{\omega}{2W}\right) \\ \measuredangle H(\omega) = -\omega t_d \end{cases} \Rightarrow H(\omega) = \text{rect}\left(\dfrac{\omega}{2W}\right) e^{-j\omega t_d}$$

impulse response:

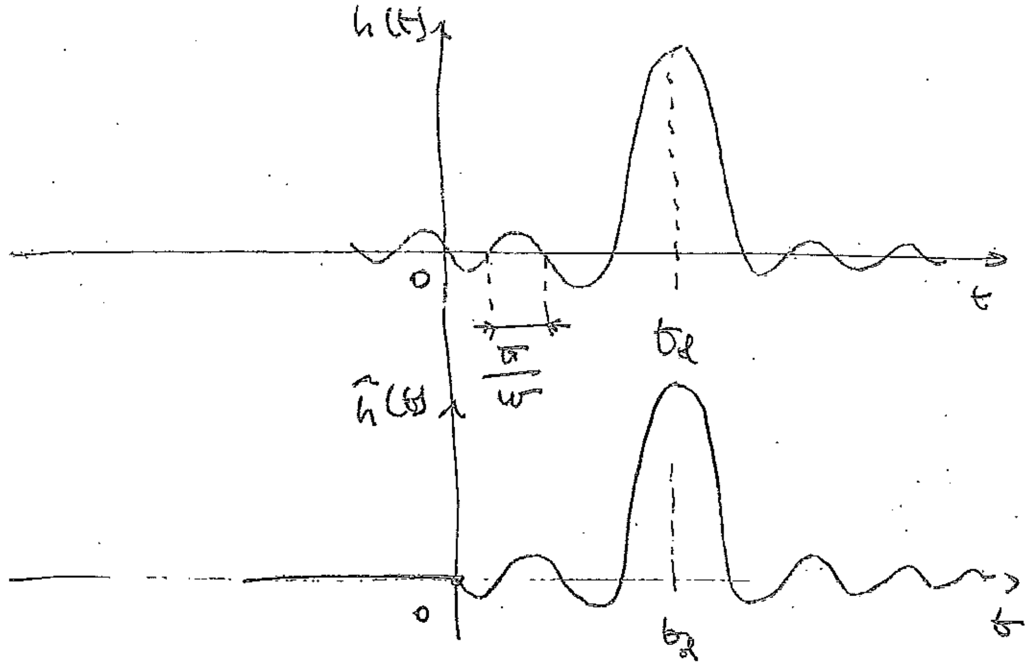
$$h(t) = \frac{W}{\pi} \text{sinc}\left[W(t - t_d)\right]$$ ⇒ not physically realizable

$$\delta(t) \to \boxed{h(t)} \to h(t)$$

For realizability: $h(t) = 0 \quad \forall t < 0$

Solution:

1) $h(t) \to h(t) \cdot u(t)$



→ if t_d is sufficiently large, $\hat{h}(t) \simeq h(t)$ and $\hat{H}(\omega) \simeq H(\omega)$.

→ causality is recovered at the price of a large delay between the input and the output.

Practically: $t_d \simeq 3,4 \frac{\pi}{\omega}$ is suitable

2. PALEY - WIENER criterion: CNS for $|H(\omega)|$ to be realizable is:

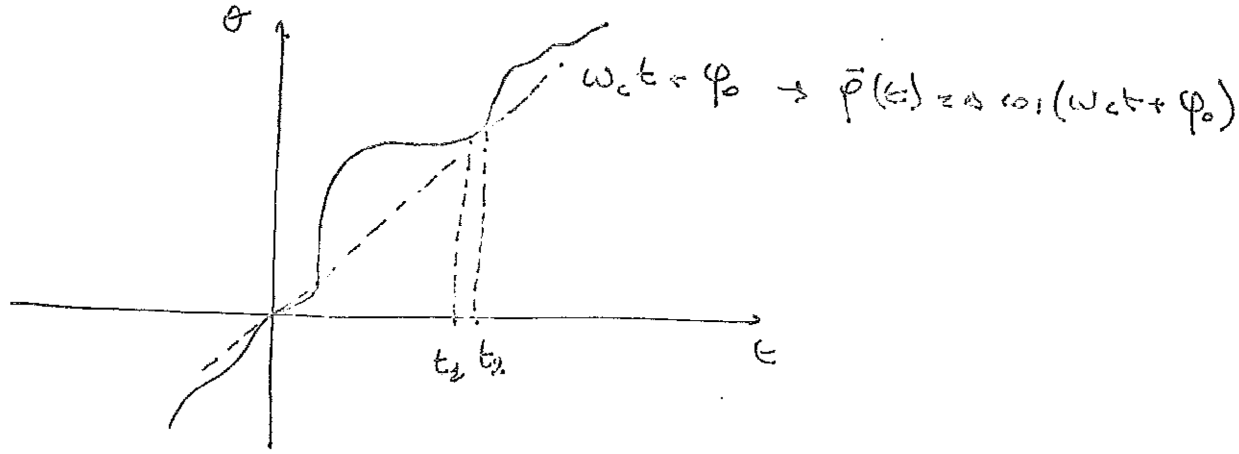
$$\int_{-\infty}^{+\infty} \frac{|\ln |H(\omega)||}{1+\omega^2} d\omega < +\infty$$

→ if $|H(\omega)|$ is zero in a single point or in a set of discrete frequencies it is still realizable. However, it cannot be zero over a finite band.

[illegible]

Generalized sinusoidal signal:

$$\varphi(t) = A\cos\theta(t)$$



for $t_1 \le t \le t_2 \to \varphi(t)$ and $\bar{\varphi}(t)$ have the same frequency.

Generalizing: instantaneous frequency:

$$\omega_i(t) = \frac{d\theta(t)}{dt}$$

$$\to \theta(t) = \int_{-\infty}^{t} \omega_i(t)\, dt$$

Conventional sinusoids: $\bar{\theta}(t) = \omega_c t + \varphi_0 \to \omega_i = \omega_c = \text{const.}$

Data Truncation: The Window function

Data truncation is needed in many situations:

- Numerical computation of the FT of signals of infinite duration
- Approximation of an ideal LP filter (IIR) with a causal one by truncating its IR at some point k,
- Getting rid of aliasing in sampling;
- Signal approximation:

$$f(t) = \sum_{-\infty}^{+\infty} D_n e^{-jn\omega_0 t} \Rightarrow \bar{f}(t) = \sum^{+N} D_n e^{-jn\omega_0 t}$$

Window function

simple truncation → rectangular window $w_R(t)$

→ equal weight is assigned to all samples within the window

$f(t) \rightarrow F(\omega)$

$$f_w(t) = f(t)\cdot w(t) \iff F_w(\omega) = \frac{1}{2\pi} F(\omega) * W(\omega)$$

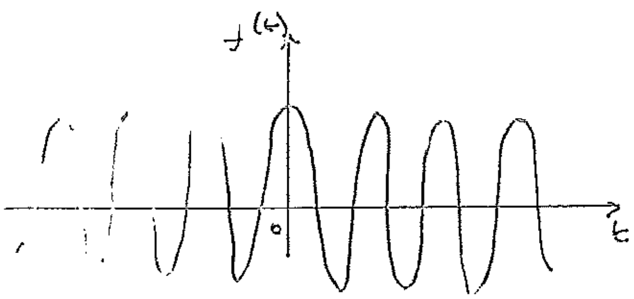
1. $N_F \quad N_w \quad N_F + N_w$ → spectral spreading

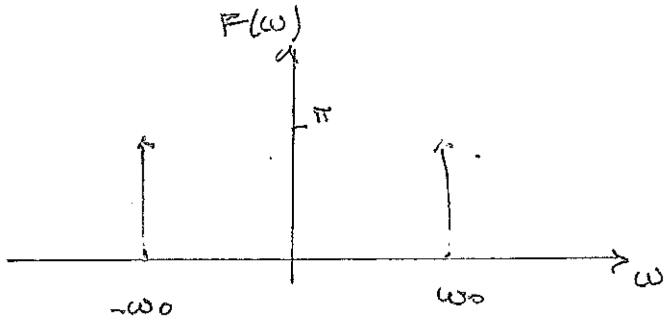
2. If $W(\omega)$ is not strictly band limited ⇒ spectral leakage

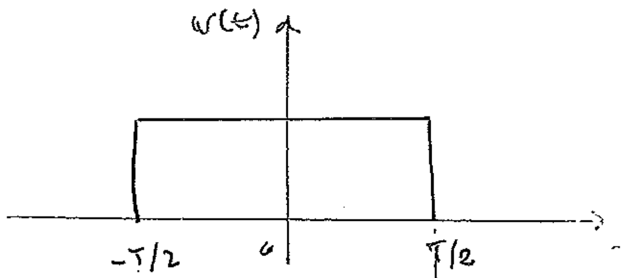
⇒ $F_w(\omega) \to 0$ for $\omega \to \infty$ at the same rate of $W(\omega)$ even though $F(\omega)$ is strictly band limited.

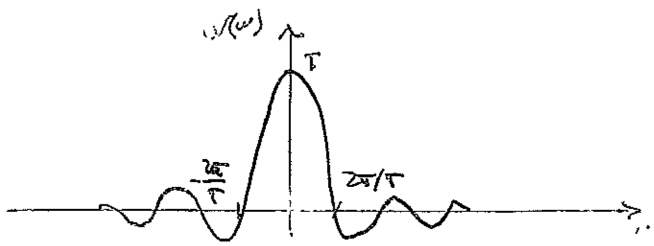
Example

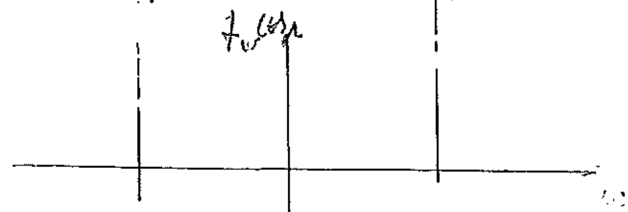
$$\begin{cases} f(t) = \cos(\omega_0 t) \\ w(t) = \mathrm{rect}\left(\frac{t}{T}\right) \end{cases}$$

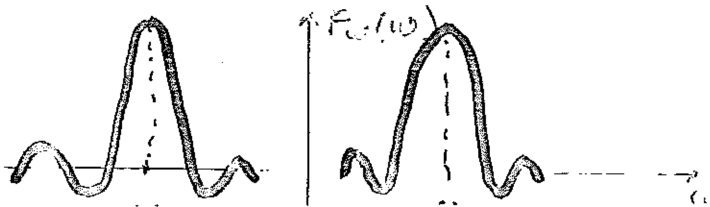


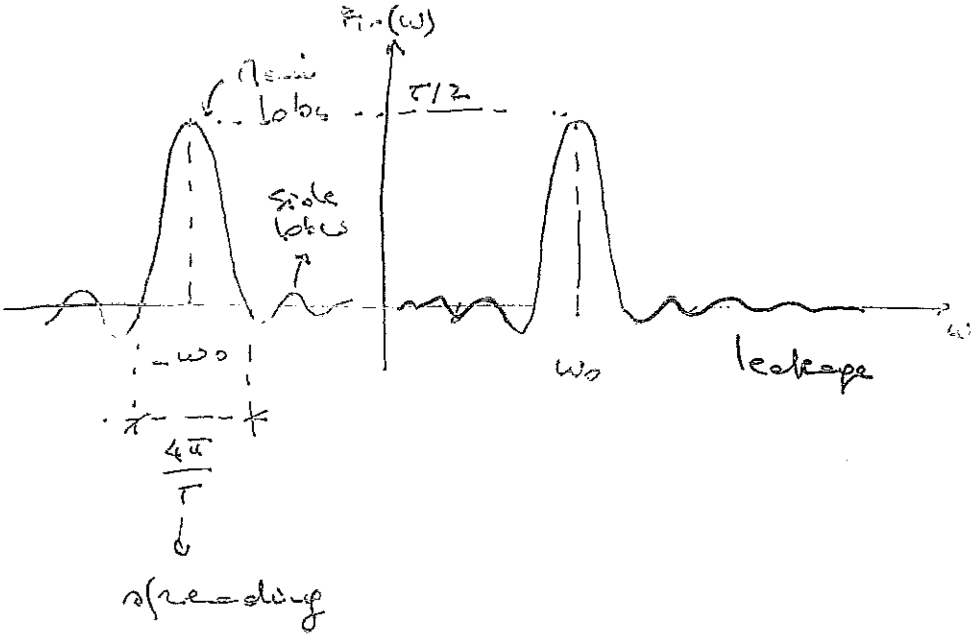




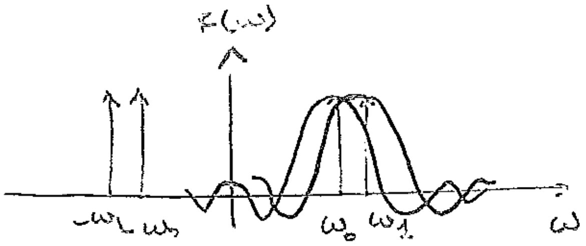








Spectral spreading implies the loss of spectral resolution



If $|\omega_1 - \omega_0| > \frac{4\pi}{T}$ the two sinusoidal components cannot be resolved (distinguished) in the truncated signal $F_T(\omega)$.

Possible remedies.

Signal spread $\propto$ window spectral width $\Delta W(\omega)$
which is $\propto$ to " temporal " $\Delta W(t)$

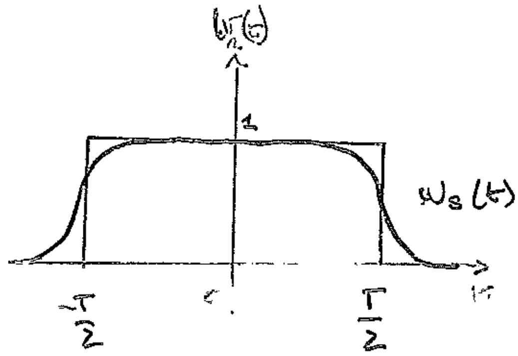
$\Rightarrow$ increase the window function duration $\Delta W(t)$

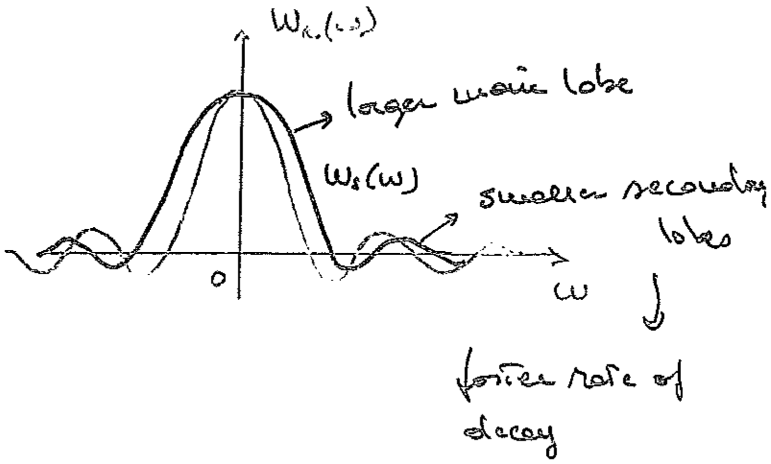
Leakage $\propto$ decay rate in frequency domain
(smoothness of a signal $\propto$ # of derivatives)

$\Rightarrow$ The smoother the signal the faster its rate of decay $\Rightarrow$ choose a smooth window

It is a matter of trade off.

For a given temporal width:





Summary:

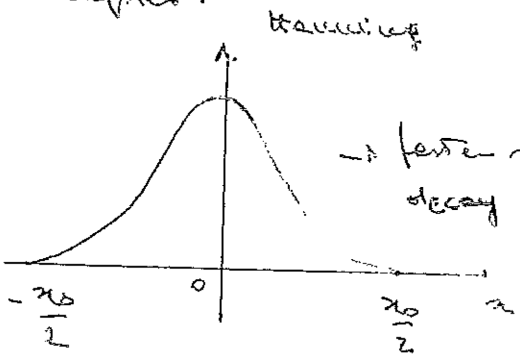
- smoother window →
  - larger main lobe ⇒ larger spectral spreading
    ↓
    could be reduced by increasing temporal duration
  - faster decay of the secondary lobes

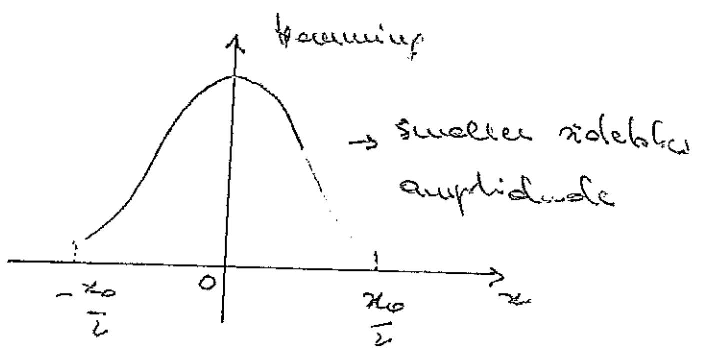
The most famous windows: Hanning, Bartlett, Hamming, Kaiser

Note: All are symmetrical about the origin ⇒ $W(\omega)$ is real.

$\Rightarrow \measuredangle W(\omega) = 0, \pi$ ⇒ minimum phase distortion

Examples:



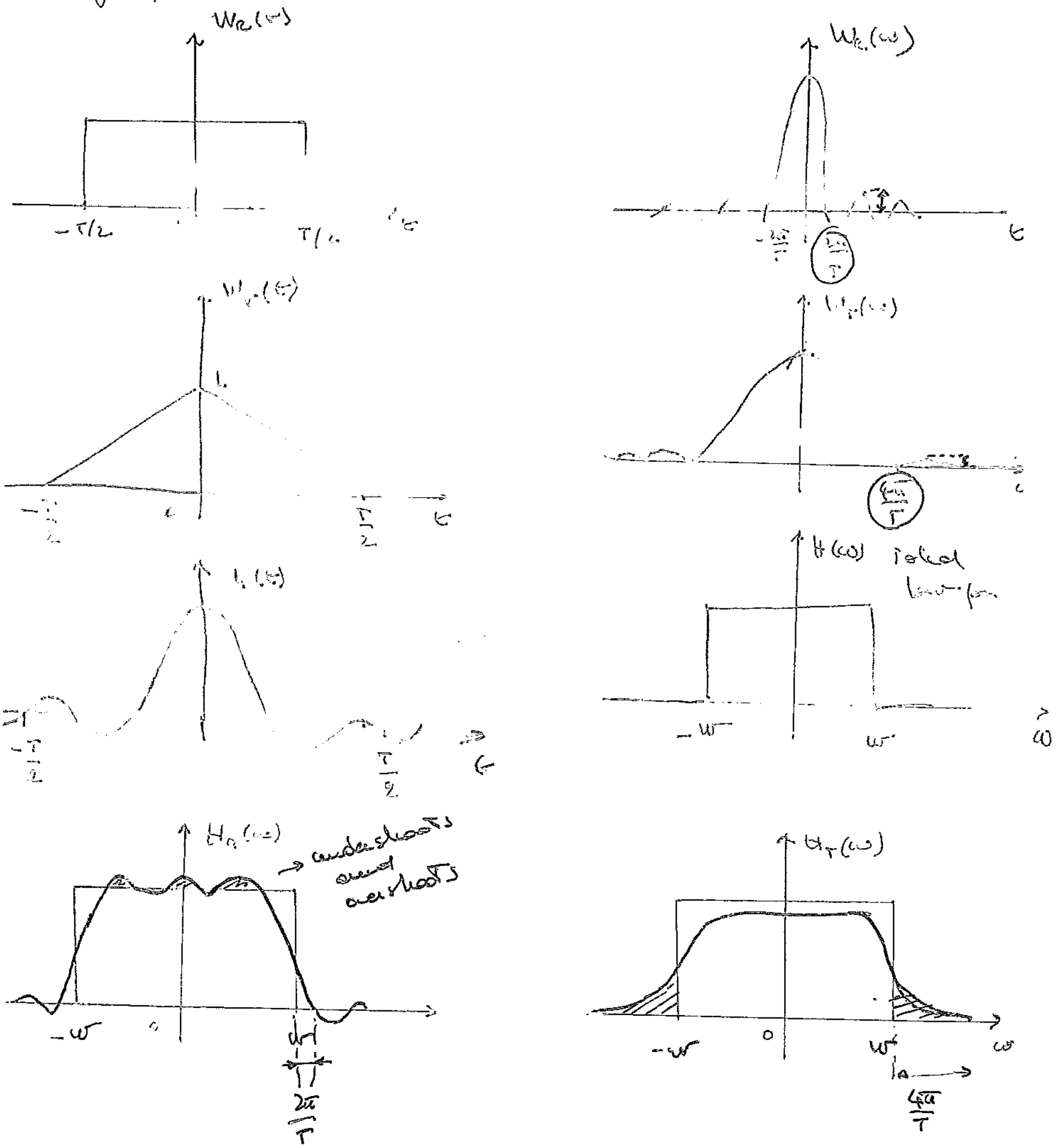


$x = t$, spatial position...

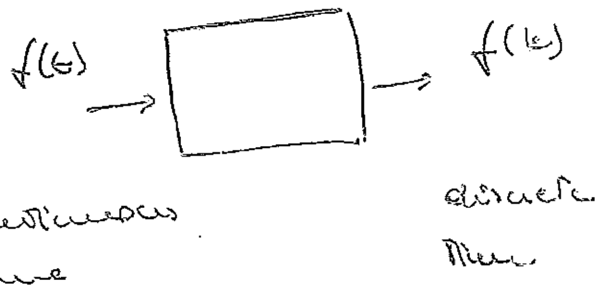
↓
most often used for filtering

(see Table 4.3 page 307 for properties)

Low pass filter design



SAMPLING



Sampling Theorem

States the conditions for which a signal can be reconstructed without error from its samples.

ℓ Bandlimited signals

Let $f(t)$ be a bandlimited signal:

$$f(t) \Rightarrow F(\omega) ; \quad F(\omega) = 0 \text{ for } |\omega| > 2\pi \cdot B$$

A bandlimited signal can be reconstructed without error from its samples if the samples are taken uniformly at a rate

$$F_s > 2B$$

→ minimum sampling frequency:

$$F_s = 2B \text{ Hz}$$

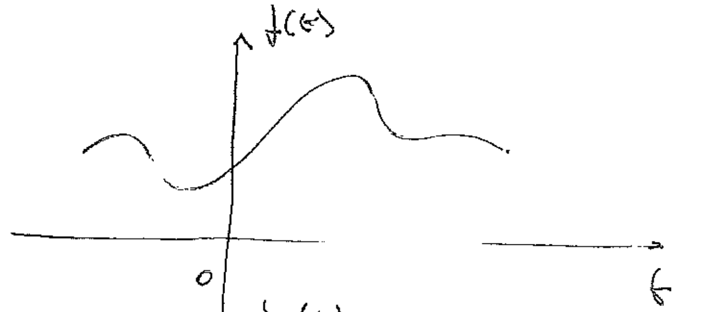
Proof:

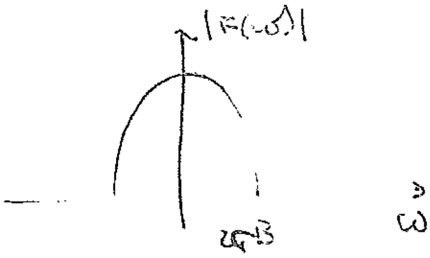
$f(t)$: continuous time signal

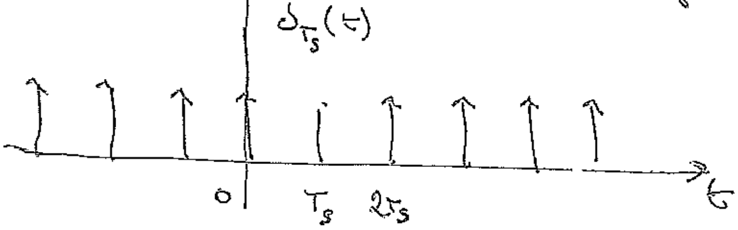
$f_s(t)$: sampled signal

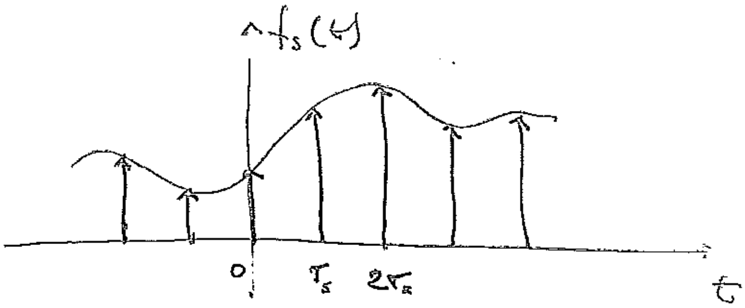
$f[k]$: sample series

$$f_s(t) = f(t) \cdot \delta_T(t) = \sum_n f(nT_s)\,\delta(t - nT_s)$$









Frequency representation of sampling

$$F_s(\omega) = F(\omega) * \delta_{T_0}(\omega)$$

$$\delta_{T_0}(\omega) = \mathcal{F}\left\{ \sum_m \delta(t - mT_s) \right\}$$

but $\delta_{T_0}(t)$ is a periodic function $\Rightarrow$ it can be expressed as a Fourier series:

(1) $$\delta_{T_0}(t) = \sum_m D_m e^{+jm\omega_s t}$$

$\omega_s = \frac{2\pi}{T_s}$ sampling frequency

But:

$$D_m = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \delta_{T_0}(t) e^{-jm\omega_s t} = \frac{1}{T_s}$$ (sampling property of the δ function)

Then:

$$(1) \rightarrow \delta_{T_0}(t) = \frac{1}{T_s} \sum_m e^{jm\omega_s t} \qquad (2)$$

Taking the F-Transform of both sides:

$$\mathcal{F}\{\delta_{T_0}(t)\} = \int_{-\infty}^{+\infty} \left(\sum_n \frac{1}{T_s} e^{jn\omega_s t} \right) \cdot e^{-jn\omega t} dt =$$

$$= \frac{1}{T_s} \int_{-\infty}^{+\infty} \sum_n e^{-jn(\omega-\omega_s)t} dt =$$

$$= \frac{1}{T_s} \sum_n \int_{-\infty}^{+\infty} e^{-jn(\omega-\omega_s)t} dt =$$

$$= \frac{1}{T_s} \sum_n \underbrace{\int_{-\infty}^{+\infty} 1 \cdot e^{-jn(\omega-\omega_s)t} dt}_{\mathcal{F}\{1\}\big|_{\omega=\omega_s} = \delta(\omega)\big|_{\omega=\omega_s} = \delta(\omega-\omega_s)}$$

$$\Rightarrow \quad \boxed{\delta_{T_0}(\omega) = \frac{1}{T_s} \sum_n \delta(\omega - n\omega_s)}$$

→ F-Transform of a δ Train in Time is a δ-Train in frequency, which samples are located at $\omega = \omega_s = \frac{2\pi}{T_s}$ namely corresponding to integer multiples of the sampling frequency.

For the F-Transform of the sampled signal we have:

$$F_s(\omega) = F(\omega) * \delta_{T_0}(\omega) =$$

$$= \frac{1}{T_s} F(\omega) * \sum_n \delta(\omega - n\omega_s) =$$

$$= \frac{1}{T_s} \int_{-\infty}^{+\infty} F(\nu) \sum_n \delta(\omega - n\omega_s - \nu) d\nu =$$

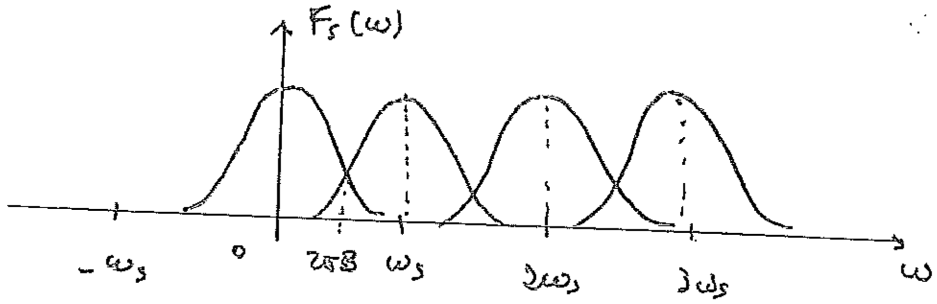
$$= \frac{1}{T_s} \int^{\infty} \sum_n F(\nu) \delta(\omega - \nu - n\omega_s) d\nu =$$

$$= \frac{1}{T_s} \sum_m \int_{-\infty}^{+\infty} F(\nu)\, \delta(\omega - \nu - m\omega_s)\, d\nu = \frac{1}{T_s} \sum_m F(\omega - m\omega_s)$$

$$\Rightarrow \boxed{F_s(\omega) = \frac{1}{T_s} \sum_m F(\omega - m\omega_s)}$$

⇒ The F-Transform of the sampled signal is a sequence of scaled replicas of the spectrum of the original continuous time signal $F(\omega)$. The replicas are centered at integer multiples of the sampling frequency ω_s.

In the previous example:



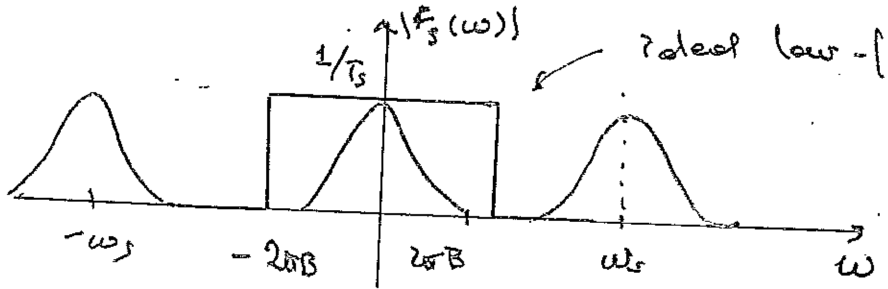
⇒ in order to avoid spectral overlapping: $\omega_s \geq 4B\pi$

$(\omega_s - 2\pi B \geq 2\pi B)$

$\Rightarrow \boxed{f_s \geq 2B} \iff \boxed{T_s \leq \frac{1}{2B}}$

↓ Nyquist rate ↓ Nyquist interval

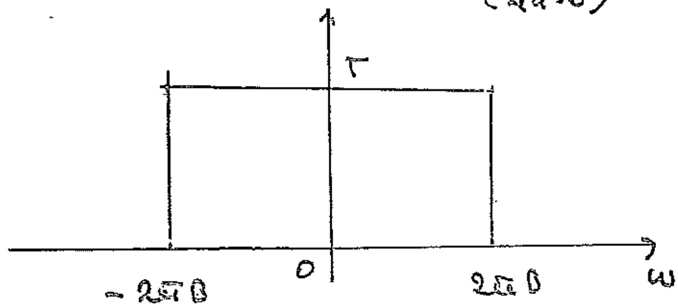
- If the Nyquist condition is satisfied, the signal can be exactly recovered by ideal low pass filtering



Signal reconstruction : Interpolation

The signal can be reconstructed without error by filtering the sampled version with a low pass filter with gain T_s and bandwidth B:

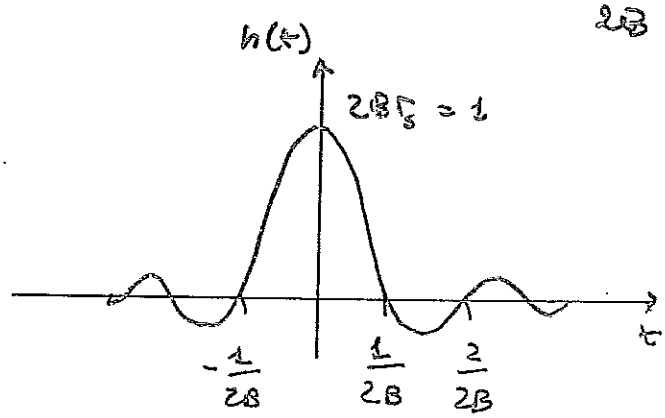
$$H(\omega) = T_s \cdot \mathrm{rect}\left(\frac{\omega}{4\pi \cdot B}\right)$$



Interpretation in time domain

$$h(t) = 2BT\, \mathrm{sinc}\left(2\pi B t\right)$$

zeros: $2\pi B \cdot t = k\pi \rightarrow t = \frac{k}{2B}$



at Nyquist rate: $T_s = \frac{1}{2B} \rightarrow 2BT_s = 1$

Filtering in time domain:

$$\begin{cases} f(t) = f_s(t) * h(t) \\ f_s(t) = \sum_n f(nT_s)\,\delta(t - nT_s) \\ h(t) = \mathrm{sinc}\left(2\pi B t\right) \end{cases}$$

BUT:

$f(mT_s)\,\delta(t - mT_s)$: impulse of amplitude $f(mT_s)$ located in $t = mT_s$

⇒ The corresponding filter output is: $f(mT_s)\,h(t - mT_s)$

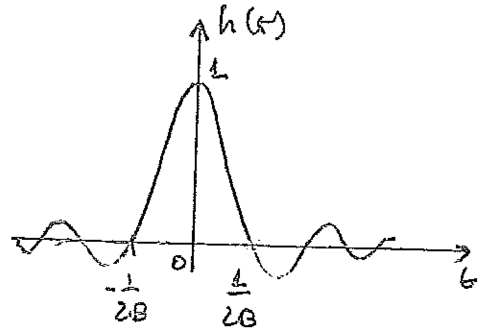
⇒ The filter output generated by $f_s(t)$ is:

$$f_r(t) = \sum_m f(mT_s)\,h(t - mT_s) =$$

$$= \sum_m f(mT_s)\,\mathrm{sinc}\left(2\pi B(t - mT_s)\right) =$$

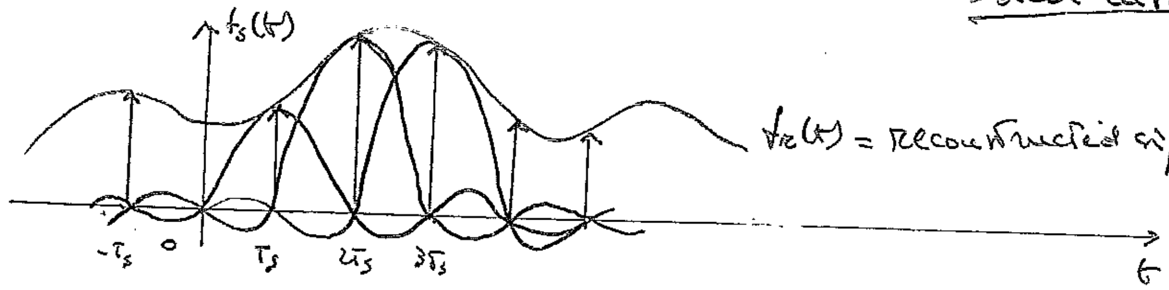
$$= \sum_m f(mT_s)\,\mathrm{sinc}\left(2\pi Bt - \underbrace{2\pi BT_s}_{=1\cdot\pi}\cdot m\right) \Rightarrow$$

$$f_r(t) = \sum_m f(mT_s)\,\mathrm{sinc}\left(2\pi Bt - m\pi\right)$$



$T_s = \frac{1}{2B}$

Ideal interpolation



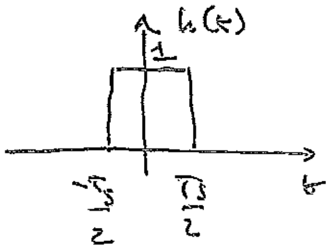
$f_r(t)$ = reconstructed signal $\equiv f(t)$

⇒ The values of $f(t)$ in-between samples are obtained as weighted sums of all sample values.

$$\boxed{\text{Interpolation} \Leftrightarrow \text{low-pass filtering}}$$

Other forms of interpolation

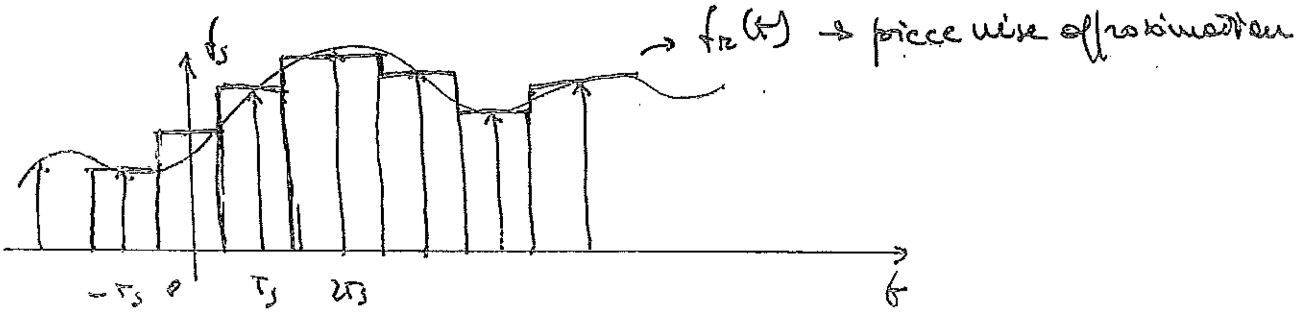
... Zero-order interpolation

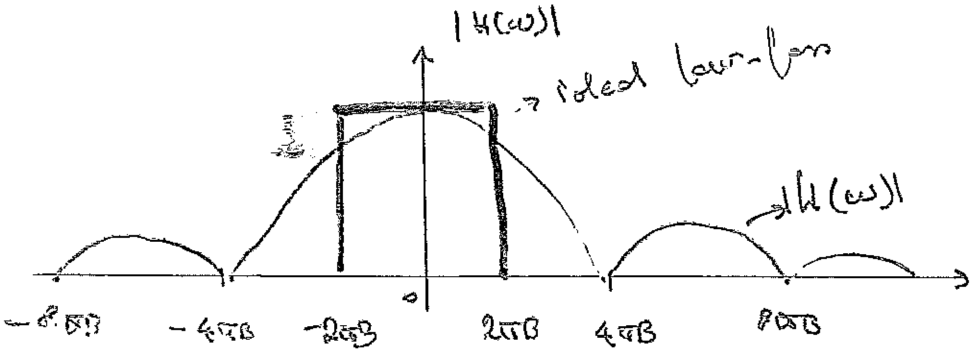


$$h(t) = \mathrm{rect}\left(\frac{t}{T_s}\right) = \mathrm{rect}(2B\cdot t)$$

$$H(\omega) = T_s\,\mathrm{sinc}\left(\frac{\omega T_s}{2}\right) = \frac{1}{2B}\,\mathrm{sinc}\left(\frac{\omega}{4B}\right)$$

$$f_R(t) = \sum_n f(nT_s)\,h(t-nT_s)$$



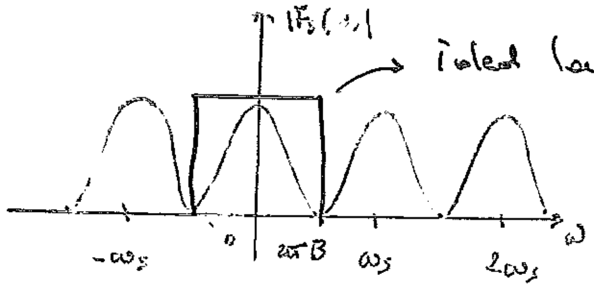


$H(\omega)$ is a bad approximation of the ideal low-pass filter $\Rightarrow$

$f_R(t)$ is a "poor" approximation of $f(t)$

Hint: The tails of the filter $H(\omega)$ make the components of the spectrum repetitions to contribute to the reconstruction, so that the "original" spectrum cannot be recovered.

Signal Reconstruction

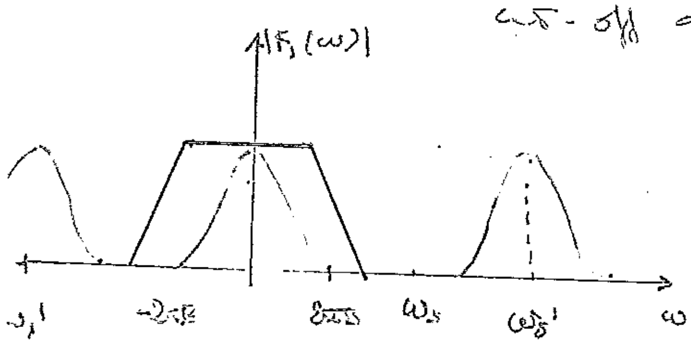


$\omega_s = 4\pi \cdot B$

To recover $F(\omega)$ an ideal low pass filter is needed BUT such a filter is unrealisable (it can be approximated only with an infinite time delay).

Solution: sampling the signal with a higher rate

$\omega_s' > \omega_s$ → The spectral replicas get further from each other → $F(\omega)$ can be recovered using a low pass filter with a gradual cut-off characteristic.



Paley - Wiener → not realisable neither

However, a closer approximation can be obtained with a smaller time delay.

Summary:

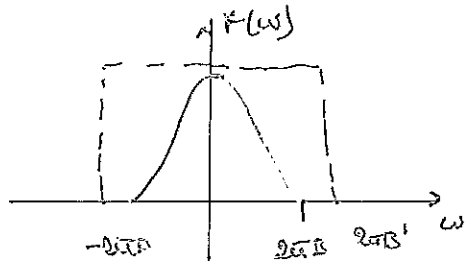
It is impossible in practice to recover a bandlimited signal $f(t)$ exactly from its samples, even if the sampling rate is higher than the Nyquist rate. However, as the sampling rate increases, the recovered signal approaches the original one more closely.

ALIASING

• A signal cannot be time-limited and band-limited simultaneously at the same time *

Intuition

Let $f(t)$ be a signal that is time and band limited, and let B be its bandwidth



$F(\omega) = 0 \quad \forall\, \omega > 2\pi B$

$F(\omega) \equiv F(\omega) \cdot \text{rect}\left(\frac{\omega}{4\pi B'}\right)$

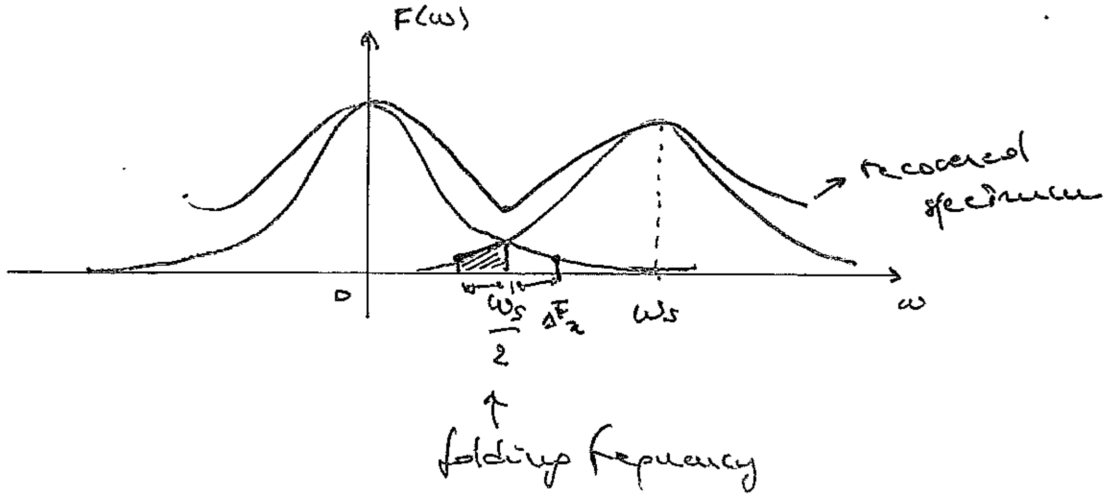
But this would imply: $f(t) = f(t) * 2B' \text{sinc}(2\pi B' t) \Rightarrow$ not time-lim. given the hp.

* All practical signals are time limited *

→ They are not band limited.

Low-pass filtering is not enough because (after sampling)

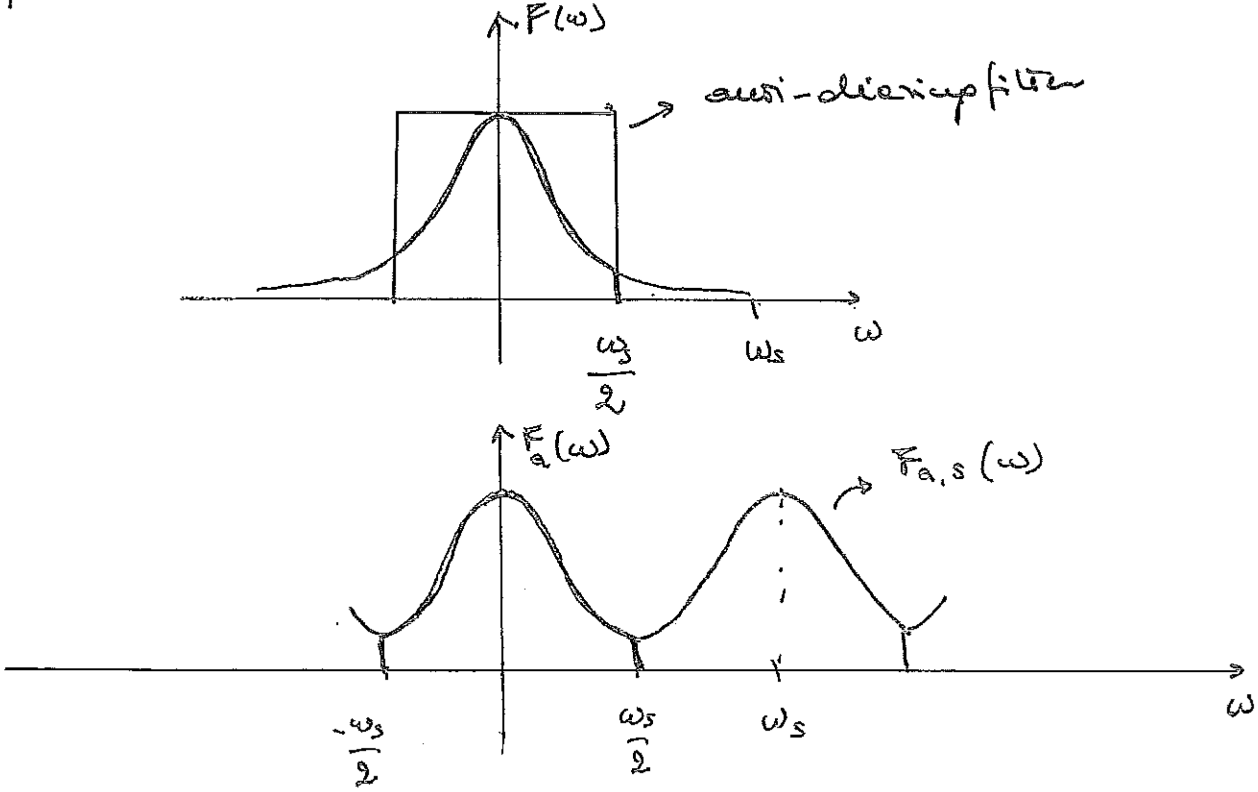
- We lose the tails of the spectrum
- Frequency folding or aliasing



Because of the infinite extent of the spectrum, high frequency tails are "folded back" making the spectrum of the sampled signal differ from that of $f(t)$ even in the base band ($|\omega| \leq 2\pi B$)

Solution: low-pass filtering BEFORE sampling.

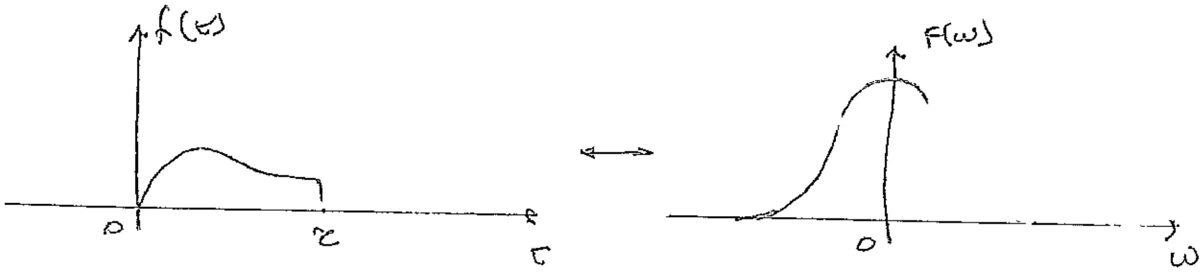
Given ω_s (OR f_s), filter out all the components having higher frequency than $\omega_s/2$

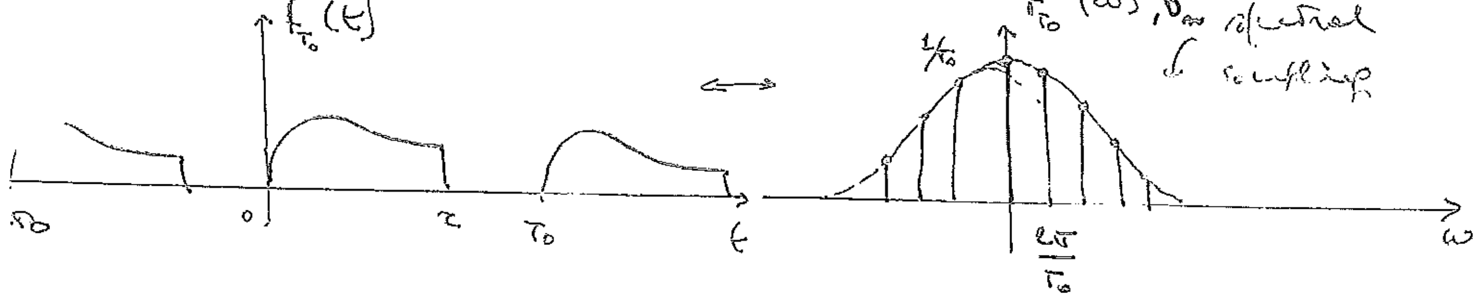


NOTE: $f_a(t)$ is not time-limited

Spectral sampling.

Consider a time-limited signal $f(t)$ and let τ be its ... Now





⇒ as long as $T_0 > \tau$ the time-domain repetitions don't overlap.

The spectrum is sampled with rate $\omega_0 = 2\pi / T_0$

⇒ The further the repetitions the denser the sampling.

Proof

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)\, e^{-j\omega t}\, dt = \int_{0}^{\tau} f(t)\, e^{-j\omega t}\, dt$$

$$f_{T_0}(\omega) = \sum_{-\infty}^{+\infty} {}_n D_n\, e^{-jn\omega_0 t}$$

$$\omega_0 = \frac{2\pi}{T_0}$$

$$D_n = \frac{1}{T_0} \int_{0}^{T_0} f_{T_0}(t)\, e^{-j\omega_0 t n}\, dt \underset{\uparrow}{=} \frac{1}{T_0} \int_{0}^{\tau} f(t)\, e^{-j\omega_0 t n}\, dt = \frac{1}{T_0} F(n\omega_0)$$

by construction

$$\boxed{D_n = \frac{1}{T_0} F(n\omega_0)}$$

⇒ The coeff. of the Fourier series of $f_{T_0}(t)$ are $1/T_0$ times the sampled values of $F(n\omega_0)$ ⇒

The spectrum of the periodic signal is obtained by sampling the spectrum of the original signal. *

Condition for recovery:

$$T_0 \geq \tau \rightarrow F_0 \leq \frac{1}{\tau} \quad \text{or} \quad \omega_0 \leq \frac{2\pi}{\tau}$$

Time - frequency normalization.

$$x_s(t) = \sum_{-\infty}^{+\infty}{}_n\, x_c(nT_s)\,\delta(t - nT_s) \qquad \omega_s = \frac{2\pi}{T_s}$$

$x_s(t)$: sampled

x_c: continuous - time

T_s: sampling period

2) $$X_s(\omega) = \sum_{-\infty}^{+\infty}{}_n\, x(nT_s)\, e^{-j\omega n T_s} = \frac{1}{T_s}\sum_{-\infty}^{+\infty}{}_n X_c(\omega - n\omega_s)$$

Let $x[n]$ be the corresponding sequence:

$$x[n] = x_s(nT_s) \qquad (1)$$

Taking its Fourier series and calling Ω the corresponding freq:

$$X(\Omega) = \sum_{-\infty}^{+\infty}{}_n\, x[n]\, e^{-j\Omega n} = (\text{for } (1))$$

$$= \sum_{-\infty}^{+\infty}{}_n\, x(nT_s)\, e^{-j\Omega n} \qquad (3)$$

Comparing (3) with (2) $\Rightarrow$

$$X_s(\omega) = X(\Omega)\Big|_{\Omega = \omega T_s}$$

$X_s(\omega)$: sampled, $X(\Omega)$: sequence

Equivalently:

$$X(\Omega) = X_s(\omega)\Big|_{\omega = \frac{\Omega}{T_s}} = X_s\left(\frac{\Omega}{T_s}\right) = \frac{1}{T_s}\sum_{-\infty}^{+\infty}{}_n X_c\left(\frac{\Omega}{T_s} - k\frac{2\pi}{T_s}\right)$$

$$\boxed{X(\Omega) = \frac{1}{T_s}\sum_{-\infty}^{+\infty}{}_n X_c\left(\frac{\Omega}{T_s} - k\frac{2\pi}{T_s}\right)}$$

$X(\Omega)$: (sequence), X_c: (continuous)

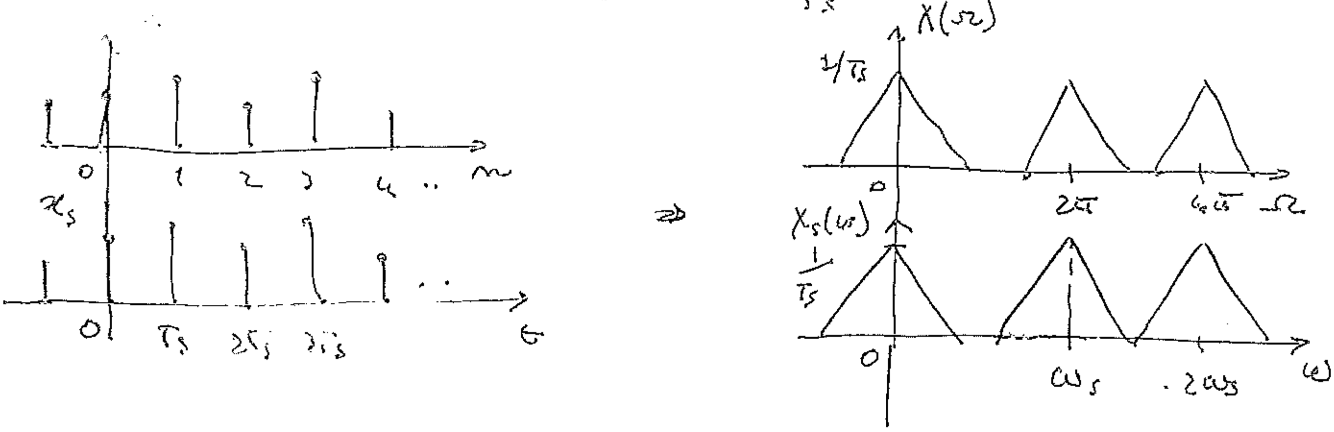
$X(\Omega)$ is simply a frequency scaled version of $X_s(\omega)$ with frequency scaling specified by $\boxed{\Omega = \omega T_s}$.

. This can be considered as a normalization of the frequency axis so that the frequency $\omega = \omega_s$ in $X_s(\omega)$ is normalized to 2π in $X(\Omega)$.

$x[n] \longleftrightarrow X(\Omega) \longleftrightarrow$ period $= 2\pi$

$x(nT_s) \longleftrightarrow X_s(\omega) \longleftrightarrow$ period $= \omega_s$

Time normalization by factor T_s $\Rightarrow$ frequency normalization by factor $\frac{1}{T_s}$



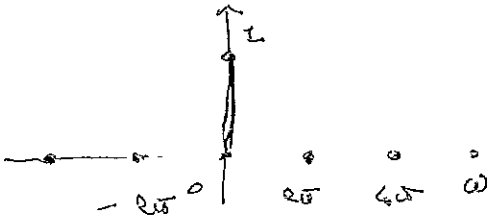
Spectral interpolation

(4) $$F(\omega) = \sum_n F(n\omega_0)\,\mathrm{sinc}\left(\frac{\omega\tau}{2} - n\pi\right) \qquad \omega_0 = \frac{2\pi}{\tau}$$

Example

Let $F(\omega)$ be the F-transform of a unit duration signal, that is sampled at the intervals of 1 Hz (or 2π rad/s) (Nyquist rate).
The samples are:

$$\begin{cases} F(0) = 1 \\ F(\pm 2\pi n) = 0 \quad \forall n \neq 0 \end{cases}$$



Then, from (4):

$$F(\omega)\Big|_{\omega=0} = F(0) = 1 \quad \text{and} \quad \tau = 1$$

$$\rightarrow F(\omega) = \mathrm{sinc}\left(\frac{\omega}{2}\right) \qquad \text{only solution.}$$

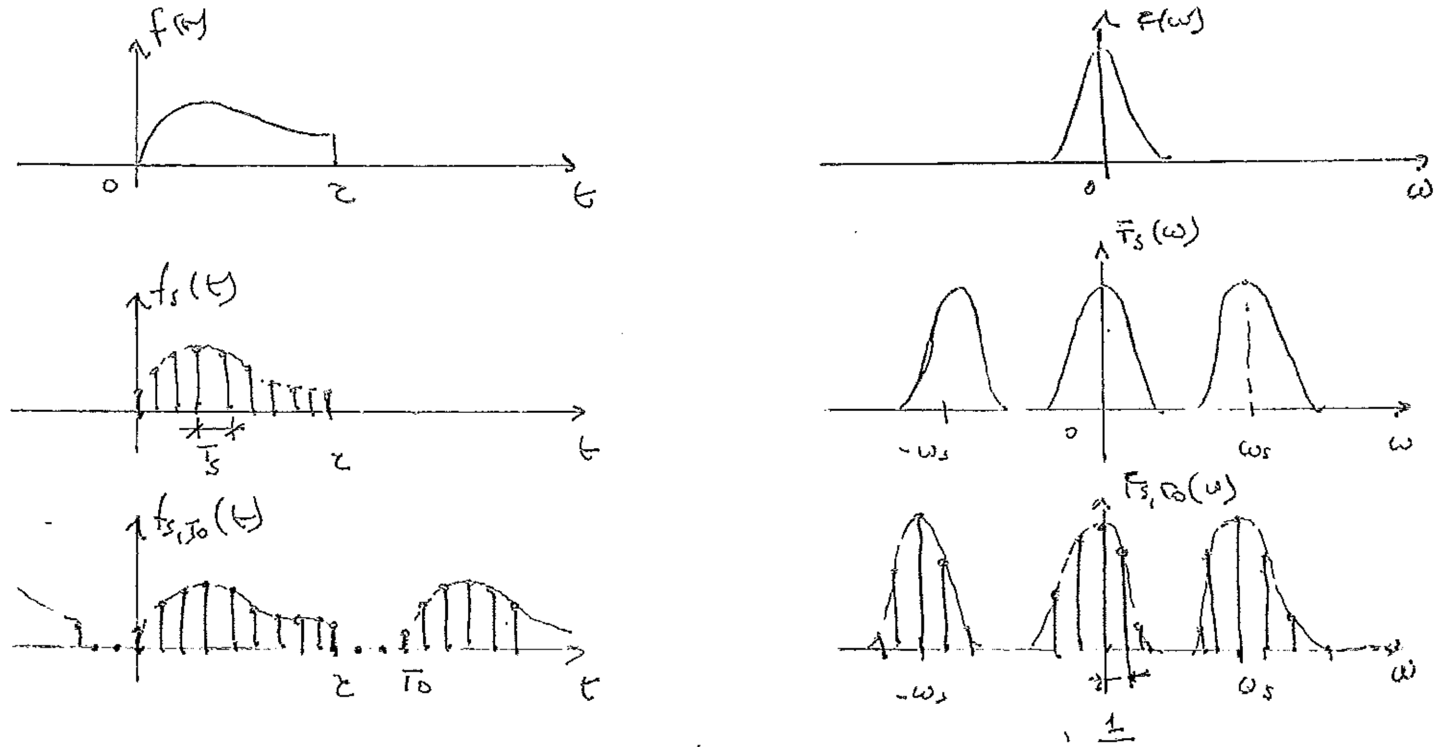
DTFT

Premise:

* when a signal is sampled and then periodically repeated, the corresponding spectrum is also sampled and periodically repeated *

Time domain	Frequency domain
Continuous	Continuous (CTFT)
Periodic, Continuous	Sampled (CTFS)
Sampled	Periodic (DTFT)
Periodic and sampled	Periodic and sample (DTFS)

Time domain | Frequency domain



Goal: relate the samples of $f(t)$ to the samples of $F(\omega)$

<u>Number of samples</u>

signal → $N_0 = \frac{T_0}{T_s}$

Spectrum → $N_0' = \frac{\omega_s}{\omega_0} = \frac{2\pi}{T_s} \cdot \frac{T_0}{2\pi} = \frac{T_0}{T_s} = N_0$

$$\Rightarrow \boxed{N_0' = N_0}$$

The number of samples of the signal in one period T_0 is identical to N_0', namely the number of samples of the spectrum in one period, ω_s.

Time domain		Frequency domain
Finite duration	$\longrightarrow$	Not bandlimited
Truncation	$\longrightarrow$	Leakage and spectral spreading (aliasing)
Temporal spreading (leakage and aliasing in time domain)	$\longleftarrow$	LP before sampling (Band limitation)

* The aliasing can never be eliminated for time-limited signals because the spectrum is NOT bandlimited.

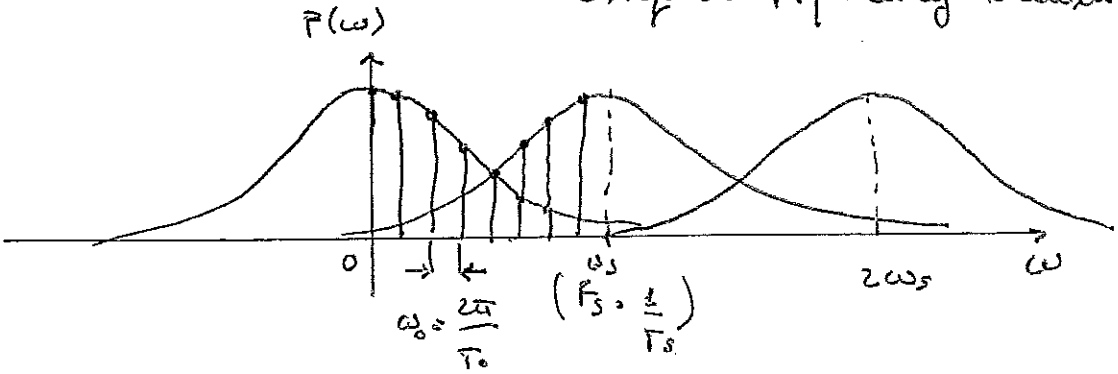
* For bandlimited signals, signals are <u>NOT</u> time limited such that their repetition (spectral sampling) generates aliasing in time domain.

⇒ In computing numerically the F-transform, direct and inverse, the error can be reduced as much as we wish but it can never be eliminated.

To make the signal time limited we can use a window to truncate it. But this introduces spectral spreading and aliasing.
Aliasing can be reduced by increasing the sampling frequency ω_s,
Leakage can be reduced by using a tapered window.
But this increases spectral spreading or smearing. This can be reduced by increasing T_0 and thus reducing F_0 (increase in spectral or frequency resolution).

Aliasing and leakage in numerical computation

$f(t)$ is time-limited → $F(\omega)$ not band-limited
↓
aliasing in frequency domain



T_0 : period in time domain
T_s : sampling interval

To reduce frequency aliasing: $\uparrow \omega_s$ ($\downarrow T_s$) → sample more densely in time domain

However, aliasing can never be eliminated since $f(t)$ is time-lim.

Viceversa :

$F(\omega)$ is band limited → no frequency aliasing BUT temporal aliasing

⇒ In computing the direct & inverse F-transform numerically the error can never be completely eliminated.

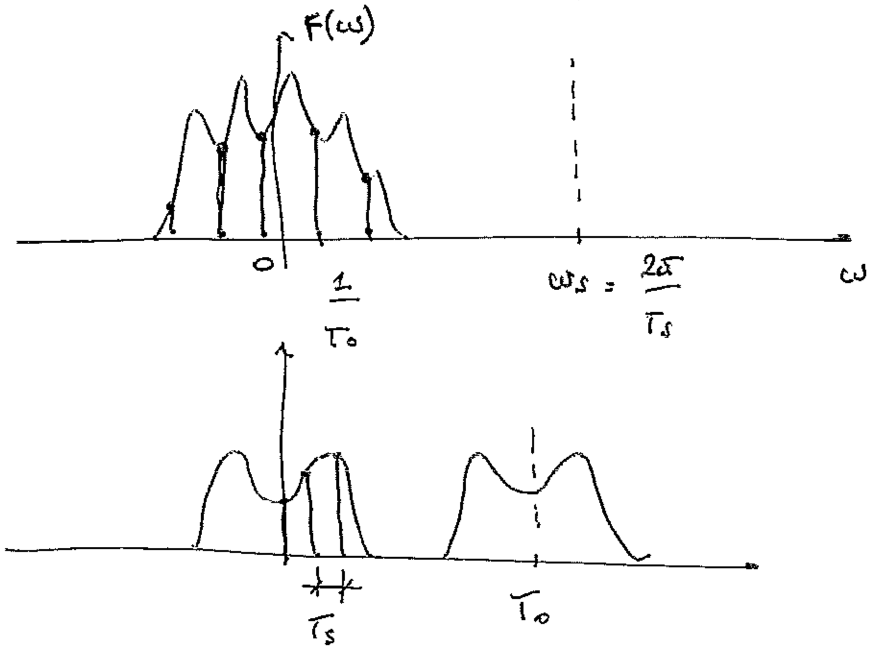
Summary:

Temporal truncation → spectral spreading and leakage → aliasing

This can increase spectral spreading
↓
use a wider window ($\uparrow T_0$)
BUT → this ↑ frequency resolution ($f_0 = \frac{1}{\ldots}$)

can be reduced using a tapered window
BUT

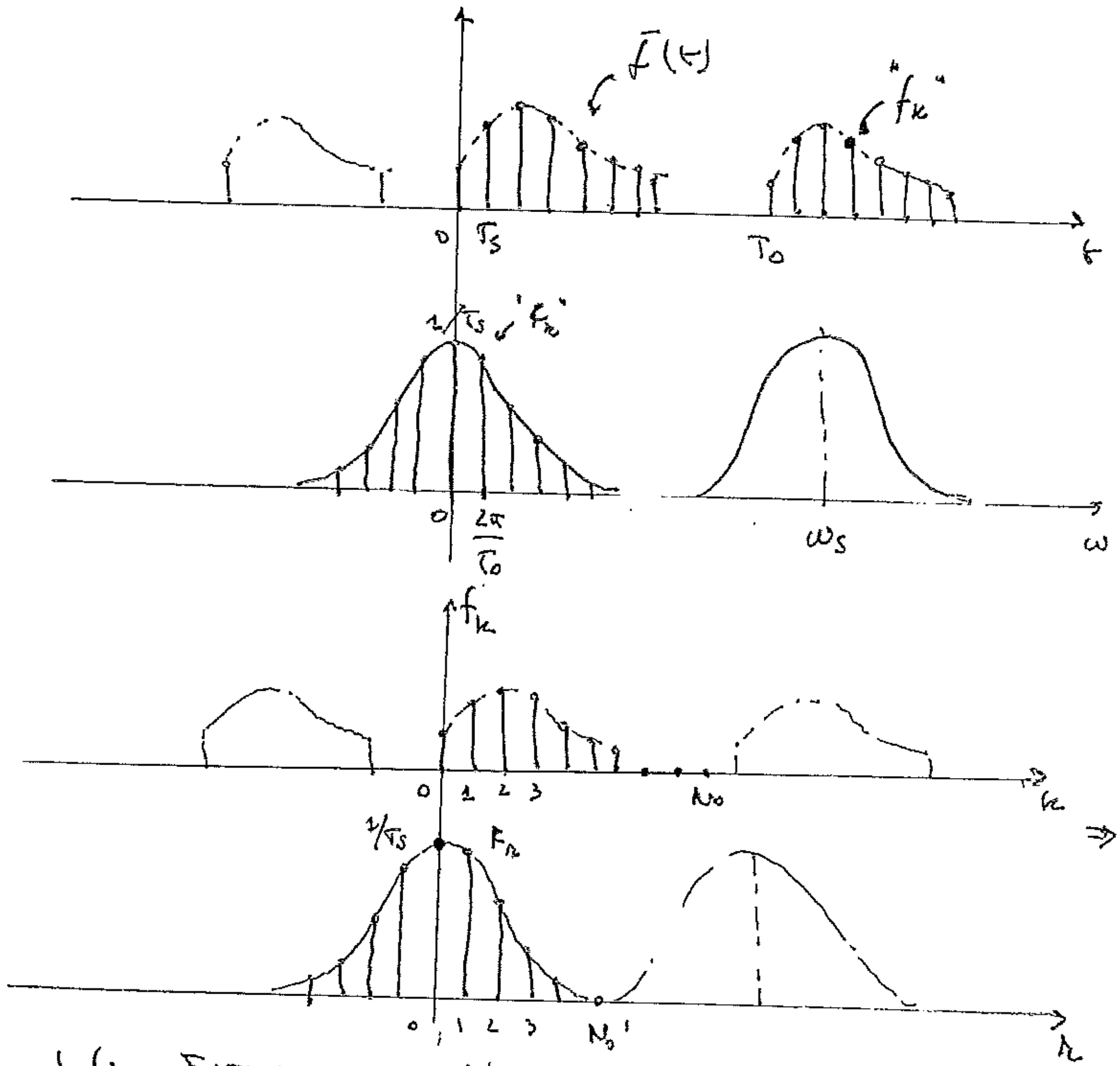
Note: The sampling interval must be sufficiently small in both time and frequency domains



⇒ N_0 must be <u>sufficiently large</u> to represent the signal adequately.

DISCRETE FOURIER TRANSFORM

Consider the two sequences of samples obtained by sampling the signal in time and in frequency domains respectively:



No periodic ⇒ sequences

Let's define two new variables

$$f_k = T_s f(kT_s) = \frac{T_o}{N_o} f(kT_s)$$

$$F_r = F(r\omega_o) \quad ; \quad \omega_o = \frac{2\pi}{T_o} = 2\pi \cdot F_o$$

We want to prove that the following relations hold:

$$\begin{cases} F_r = \sum_{k=0}^{N_o-1} f_k e^{-jr\Omega_o k} \\ f_k = \frac{1}{N_o} \sum_{r=0}^{N_o-1} F_r e^{jr\Omega_o k} \end{cases}$$

DTFT

(We refer to book notations, such that:

$\bar{f}(t)$: sampled signal (discrete time)

$\rightarrow$ $\bar{F}(\omega)$: F-Transf. of $\bar{f}(t)$ (continuous function of ω, periodic

$$\bar{f}(t) = \sum_{k=0}^{N_0-1} f(kT_s)\,\delta(t-kT_s) \qquad (1)$$

$$\delta(t-kT_s) \xrightarrow{\mathcal{F}} e^{-j\omega kT_s}$$

$\Rightarrow$ Taking the F-Transform of both sides of (1)

$$\bar{F}(\omega) = \sum_{k=0}^{N_0-1} f(kT_s)\, e^{-j\omega kT_s} \qquad (2)$$

But: for $|\omega| < \frac{\omega_s}{2}$ $\Rightarrow$ $\bar{F}(\omega) = \frac{1}{T_s} F(\omega)$ $\rightarrow$ ($\bar{F}(\omega)$ is obtained by spectral repetition of $F(\omega)$)

$$\rightarrow F(\omega) = T_s \cdot \bar{F}(\omega)$$

$\rightarrow$ using relation (2):

$$F(\omega) = T_s \sum_{k=0}^{N_0-1} f(kT_s)\, e^{-j\omega kT_s} \qquad (3)$$

But also:

$$F_r = F(r\omega_0)$$

$\rightarrow$ replacing $\omega \leftarrow r\omega_0$ in (3)

$$F_r = T_s \sum_{k=0}^{N_0-1} f(kT_s)\, e^{-jr\omega_0 kT_s}$$

Let now: $\boxed{\Omega_0 = \omega_0 T_s}$ $\rightarrow$ $\boxed{\Omega_0 = \frac{2\pi}{T_0}\cdot T_s = \frac{2\pi}{N_0}}$

Then:
$$F_r = T_s \sum_{k=0}^{N_0-1} f(kT_s)\, e^{-jrk\Omega_0}$$

Remembering that: $f_k = T_s f(kT_s)$
(by definition)

$$F_r = \sum_{k=0}^{N_0-1} f_k e^{-jkr\Omega_0} \qquad \Omega_0 = \frac{2\pi}{N_0} \qquad \text{CVD}$$

Let's now prove the inverse Fourier relationship.

Let's multiply both members by $e^{jm\Omega_0 r}$ and sum over r:

$$\sum_{0}^{N_0-1} F_r e^{jmr\Omega_0} = \sum_{r=0}^{N_0-1} \left\{ \sum_{k=0}^{N_0-1} f_k e^{j(m-k)\Omega_0 r} \right\} =$$

$$= \sum_{0}^{N_0-1} {}_k \left\{ \sum_{0}^{N_0-1} {}_r f_k e^{j(m-k)\Omega_0 r} \right\} =$$

$$= \sum_{0}^{N_0-1} {}_k f_k \left\{ \sum_{0}^{N_0-1} {}_r e^{j(m-k)\Omega_0 r} \right\}$$

But:

$$\sum_{0}^{N_0-1} {}_r e^{j(m-k)\Omega_0 r} = \begin{cases} N_0 & \text{for } m = k \\ 0 & \text{for } m \neq k \end{cases}$$

Thus:

$$\sum_{0}^{N_0-1} {}_r F_r e^{jmr\Omega_0} = \sum_{0}^{N_0-1} {}_k f_k \, \delta(m-k) \cdot N_0 =$$

$$= N_0 \cdot f_m$$

$$\rightarrow f_m = \frac{1}{N_0} \sum_{0}^{N_0-1} {}_k F_r e^{jmr\Omega_0} \qquad \text{CVD}$$

Summary of the DTFT

$$F_r = \sum_{0}^{N_0-1} {}_k f_k e^{-jkr\Omega_0}$$

$$f_k = \frac{1}{N_0} \sum_{0}^{N_0-1} {}_r F_r e^{jkr\Omega_0} \qquad \Rightarrow \begin{cases} F_r = \sum_{0}^{N_0-1} {}_k f_k e^{-j\frac{2\pi}{N_0}kr} \\ f_k = \frac{1}{N_0} \sum_{0}^{N_0-1} {}_r F_r e^{j\frac{2\pi kr}{N_0}} \end{cases}$$

$$\Omega_0 = \frac{2\pi}{\ }$$

Choice of T_s, T_0, N_0

Given the bandwidth B:

$$\omega_s \geq 2B \cdot 2\pi$$

$$f_s \geq 2B \quad \rightarrow \quad T_s = \frac{1}{f_s} \leq \frac{1}{2B}$$

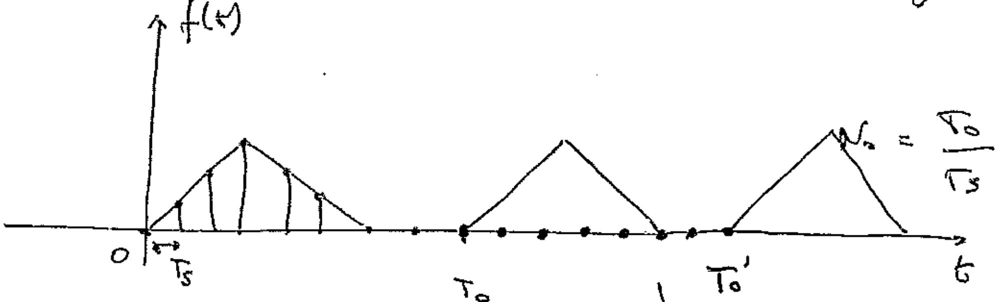
If F_0, or ω_0, is known:

$$N_0 = \frac{T_0}{T_s}$$

Zero-padding

$$f_0 = \frac{1}{T_0} = \frac{1}{N_0 \cdot T_s}$$

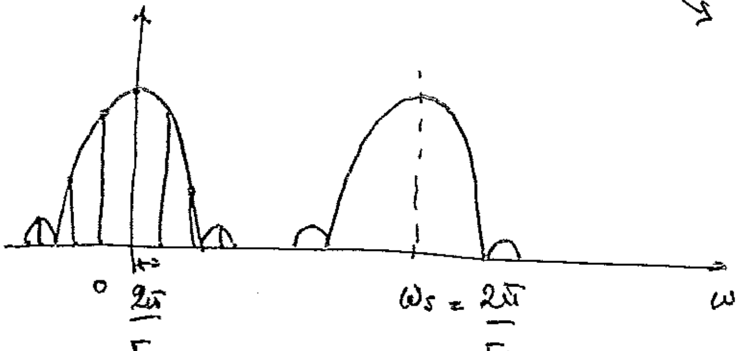
→ if T_s is given, $f_0 \downarrow$ when $T_0 \uparrow$ ⇒ adding more "samples"



→ samples with value = 0

↓

They increase frequency resolution without improving accuracy in signal representation, which depends only on T_s.



Sheet1

Properties of the DFT

the DTFT is basically the F-Transform of a sampled signal repeated periodically

⇒ its properties derive from those of the CTFT.

1. Linearity

$$f_k \leftrightarrow F_r$$
$$g_k \leftrightarrow G_r \quad \Rightarrow \quad a_1 f_k + a_2 g_k \leftrightarrow a_1 F_r + a_2 G_r$$

2. Conjugate symmetry : $F_{N_0-r} = F_r^*$

Hints :

$$F^*(\omega) = F(-\omega) \quad \rightarrow \quad F_r^* = F_{-r}$$

periodicity: $F_{-r} = F_{N_0-r}$ cvd

for real functions we need to compute ⇒ only half of the DFT coefficients

3. Time shifting (Circular shifting)

$$f_{k-m} \leftrightarrow F_r e^{-jr\Omega_0 m}$$

Proof :

$$f_k = \frac{1}{N_0} \sum_{0}^{N_0-1} {}_r \, F_r e^{jkr\Omega_0}$$

$$DFT^{-1}\left\{F_r e^{-jmr\Omega_0}\right\} = \frac{1}{N_0} \sum_{0}^{N_0-1} {}_r \, F_r e^{-jmr\Omega_0} \cdot e^{jkr\Omega_0} =$$

$$= \frac{1}{N_0} \sum_{0}^{N_0-1} {}_r \, F_r e^{j(k-m)r\Omega_0} \triangleq f_{k-m} \quad \text{cvd}$$

4. Frequency shifting

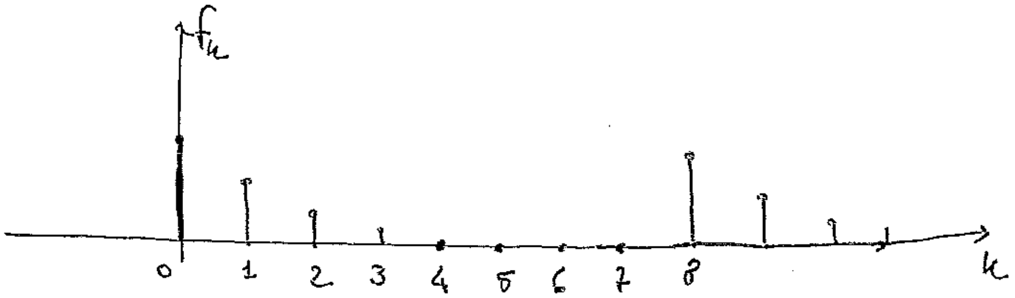
$$f_k e^{jk\Omega_0 m} \leftrightarrow F_{r-m}$$

· Circular (or periodic) convolution

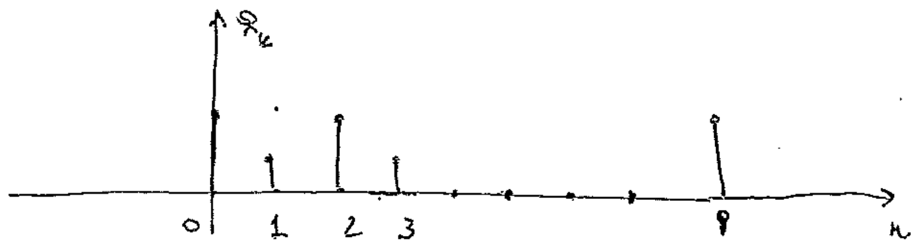
$$f_k \circledast g_k \leftrightarrow F_n \cdot G_n \qquad (f_k, g_k \text{ periodic sequences})$$

$$f_k g_k \leftrightarrow F_n \circledast G_n$$

Where
$$f_k \circledast g_k = \sum_{0}^{N_0-1} {}_m f_m g_{k-m} = \sum_{0}^{N_0-1} {}_m g_m f_{k-m}$$

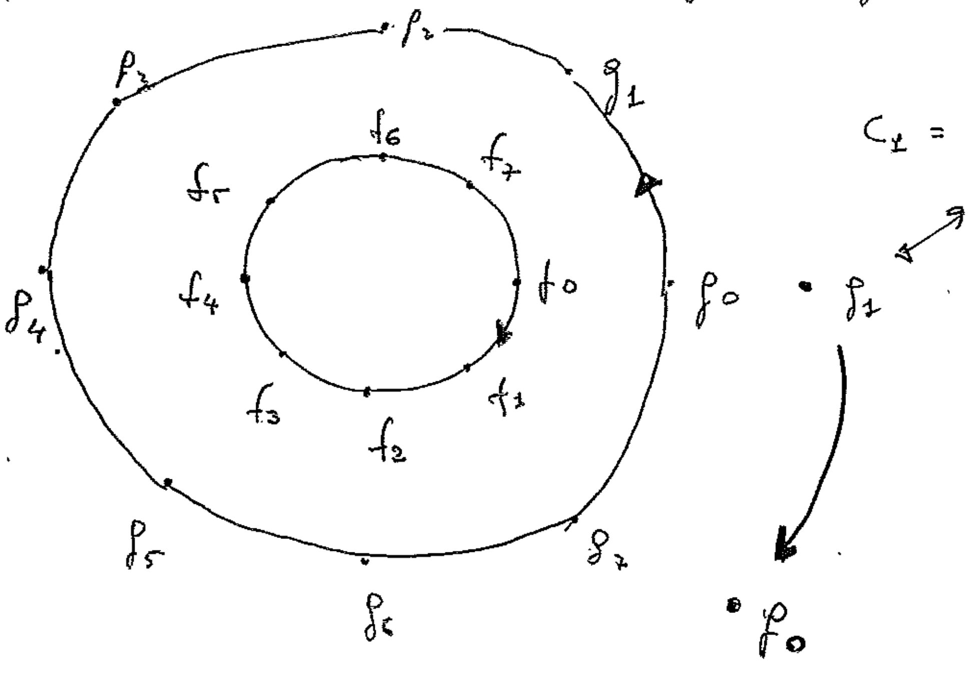


$N_0 = 8$



$$f_k \circledast g_k = \sum_{0}^{N_0-1} {}_m f_m g_{k-m} = \sum_{0}^{7} {}_m f_m g_{k-m}$$

$$C_0 = \sum_{0}^{7} {}_m f_m g_{-m} = f_0 g_0 + f_1 g_7 + f_2 g_6 + f_3 g_5 + f_4 g_4 + f_5 g_3 + f_6 g_2 + f_7 g_1$$



$$C_1 = \sum_{0}^{7} {}_m f_m g_{1-m} =$$

$$= f_0 g_1 + f_1 g_0 + \dots$$

Note: To perform circular convolution f_k and g_k must have the same length.

$f_k \longrightarrow N_1$

$g_k \longrightarrow N_2$

linear convolution: $f_k * g_k = \sum_{m=-\infty}^{+\infty} f_m g_{k-m} \to N_1 + N_2 - 1$

$\Rightarrow$ in order for the circular convolution to have the same number of samples:

$f_k \to$ add $N_2 - 1$ dummy samples

$g_k \to$ " $N_1 - 1$ " "

Filtering

$f(t)$: signal to be filtered

$f_k = T_s f(kT_s) \leftrightarrow F_r = F(r\omega_0)$

$h(t)$: filter

$h_k = T_s h(kT_s) \leftrightarrow H_r = H(r\omega_0)$

$$F_r \longrightarrow \boxed{H_r} \longrightarrow Y_r \qquad Y_r = F_r H_r$$

$$\to y_k = DFT^{-1}(Y_r) = T_s\, y(kT_s)$$

The FAST FOURIER TRANSFORM (FFT)

Reduce the number of computations from N_0^2 to $N_0 \log N_0$.

$$F_r = \sum_{k=0}^{N_0-1} f_k e^{-jkr\Omega_0} \Rightarrow \begin{cases} N_0 \text{ multiplications} \\ N_0 - 1 \text{ additions} \end{cases} \forall F_r$$

$\rightarrow$ for N_0 values of F_r $\rightarrow$ $N_0(N_0-1)$ additions
N_0^2 multiplications

Let's define:

$$W_{N_0} = e^{-j2\pi/N_0} = e^{-j\Omega_0} \qquad \left(W_{N_0}^{kr} = e^{-j\Omega_0 kr}\right)$$

$$\Rightarrow \begin{cases} F_r = \sum_{k=0}^{N_0-1} f_k W_{N_0}^{+kr} \\ f_k = \sum_{r=0}^{N_0-1} F_r W_{N_0}^{-kr} \end{cases}$$

and

$$W_{\frac{N_0}{2}} = e^{-j2\pi/N_0/2} = e^{(-j2\pi N_0)\cdot 2} = W_{N_0}^2$$

Two possible approaches:

1. Decimation in Time
2. " " Frequency

Idea: split the computation of F_r in the computation of the DFTs of the even and odd indexed samples (the first and last $N_0/2$ digits, respectively for 1. and 2.) and iterate the procedure until a 1 point transform is to be calculated.

* Purely implementation related issue *.

Discrete time signals and systems. (Chapter 8)

CT system

$$f(t) \longrightarrow \boxed{\text{LTIC}} \longrightarrow y(t)$$

DT systems

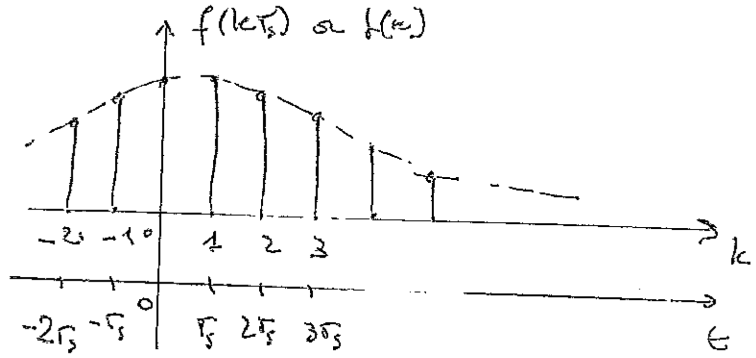
$$f[k] \longrightarrow \boxed{\text{LTID}} \longrightarrow y[k]$$

$$f(t) \longrightarrow \boxed{\text{C/D}} \longrightarrow \boxed{\text{LTID}} \longrightarrow \boxed{\text{D/C}} \longrightarrow y(t)$$

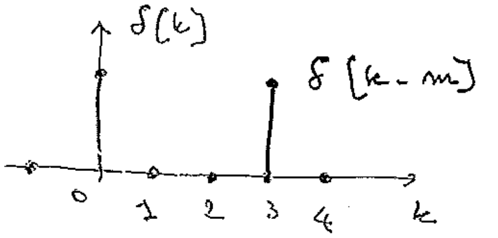
$f(kT_s)$ $\updownarrow$ $f[k]$ $\qquad$ $y[k]$

▷ A DT signal → sequence of numbers

LTID → system that process sequences of numbers.



DT signals



$m = 3$

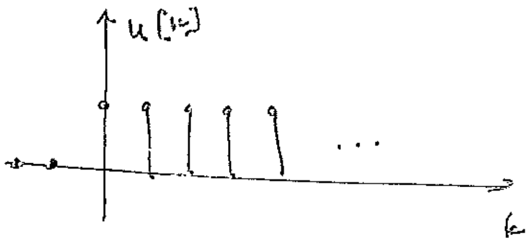
$$\delta[k] = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

* zero-input response:

$$y_0[k] = \sum_{m=-\infty}^{\infty} f[k]\,\delta[m-k]$$

⇒ sum of responses to impulse components

2. Unit step



$$u(k) = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases}$$

3. DT exponential

CT: $e^{\lambda t} = \gamma^t$ $\quad (\gamma = e^\lambda$ or $\lambda = \ln \gamma)$

eg. $e^{-0.3t} = \gamma^t$ with $\gamma = 0,7408 = e^{-0.3}$

DT $\rightarrow$ $e^{k\lambda} = \gamma^k$

<u>Relations to stability issues</u>

$f[k] = e^{k\lambda}$

a. $Re\{\lambda\} < 0 \rightarrow$ exponential decay

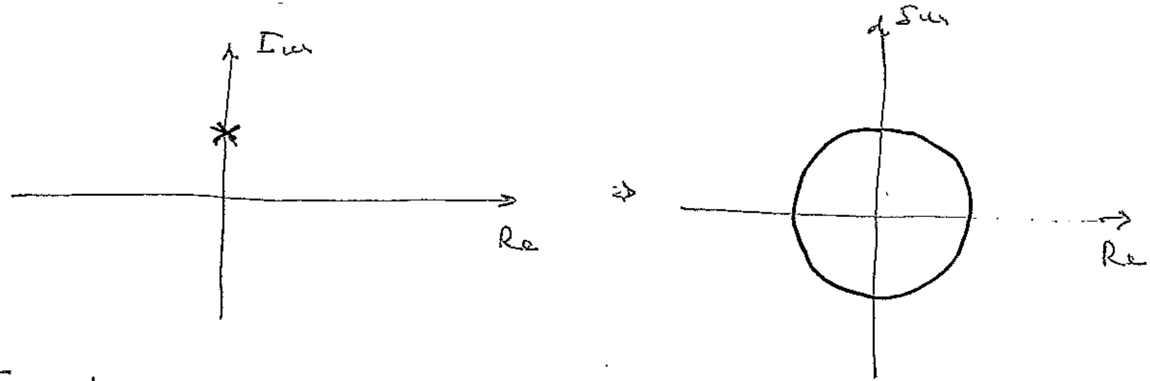
b. $Re\{\lambda\} > 0 \rightarrow$ exponential growth

c. $Re\{\lambda\} = 0 \rightarrow$ oscillation

Consider case c. :

$$\lambda = j\Omega \rightarrow e^{\lambda k} = e^{j\Omega k} \rightarrow \gamma = e^\lambda = e^{j\Omega}$$

$$\rightarrow |\gamma| = 1$$

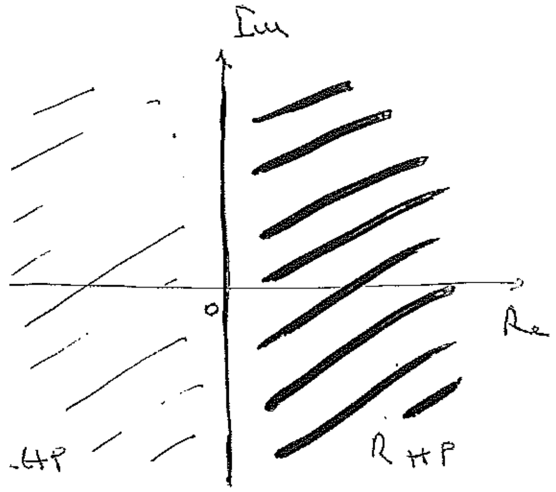


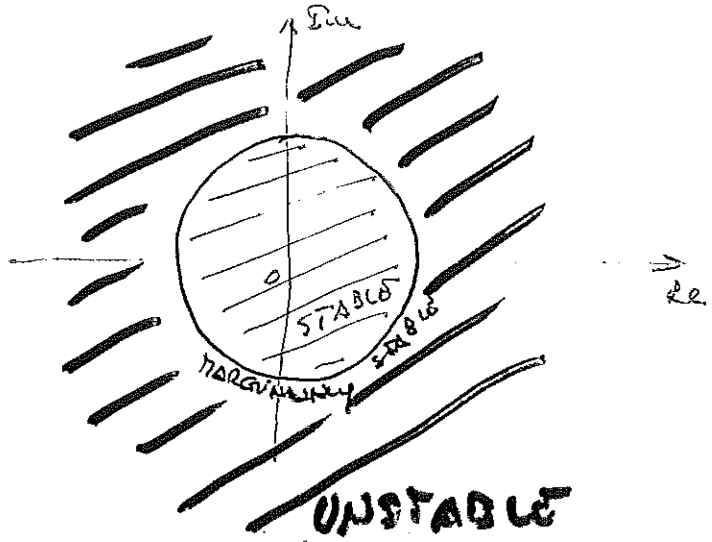
Similarly:

a., b. $\rightarrow$ $\lambda = a + jb \rightarrow \gamma = e^\lambda = e^a \cdot e^{jb} \rightarrow |\gamma| = e^a$

$a < 0 \rightarrow |\gamma| < 1 \rightarrow$ inside the unit circle $\rightarrow$ stable system

$a > 0 \rightarrow |\gamma| > 1 \rightarrow$ outside " " " " $\rightarrow$ unstable system



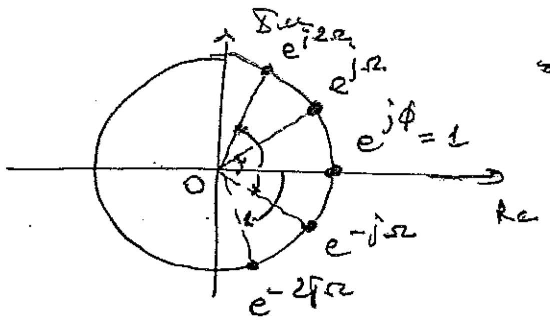


Mapping of the complex plane to the μ plane ?

iscrete time exponential (phasor)

$f[k] = e^{j\Omega k}$

$$\begin{cases} |e^{j\Omega k}| = 1 \\ \measuredangle e^{j\Omega k} = \Omega k \end{cases}$$



$\Rightarrow$ rotating phasor (counterclockwise)

have: $e^{-j\Omega k}$ $\Rightarrow$ phasor rotating clockwise

Using Eulero's formula:

$$\begin{cases} e^{j\Omega k} = \cos(\Omega k) + j\sin(\Omega k) \\ e^{-j\Omega k} = \cos(\Omega k) - j\sin(\Omega k) \end{cases} \Rightarrow$$ The frequency of both is Ω

$\Rightarrow e^{j\Omega k} \Leftrightarrow |\Omega|$

· DT sinusoid

$f[k] = \cos(\Omega k + \theta)$

Ω: frequency [rad/sample]

θ: phase (rad)

Since: $\cos(\Omega k + \theta) = \cos(-\Omega k - \theta) \Rightarrow$ frequency is $|\Omega|$