TOPOLOGIA E GEOMETRIA DIFFERENZIALE

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a.a. 2011/12

Prova scritta del 20 febbraio 2012

① Dire se (in $\mathbb{R}^3$) $X = \frac{\partial}{\partial x}$, $Y = \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$

individuano una distribuzione integrabile e, in caso affermativo, trovare $\omega \in \Lambda^1(\mathbb{R}^3)$ il cui nucleo individui la distribuzione stessa.

② Sia $T = z \frac{\partial}{\partial y} \otimes dx \otimes dz$ $\quad X = x \frac{\partial}{\partial x}$

(in $\mathbb{R}^3$) Calcolare $\mathcal{L}_X T$

③ Su $\mathbb{R}^2 \cong \mathbb{C}$, costruire un campo vettoriale avente pti critici in $z=1$, $z=2$, $z=3$ di indici rispettivi 1, 2, 3. [fare uso dell'analisi complessa...]

④ Determinare $\pi_1(\mathbb{T}^2, *)$ utilizzando il teorema di Seifert-Van Kampen

$*$ pto base

⑤ Sia $(SO(3),\ g = \text{metrica di Killing-Cartan})$

$\langle X, Y \rangle = -\mathrm{Tr}(X Y)$

$X, Y \in \mathfrak{so}(3)$

Calcolare l'operatore di curvatura associato alla connessione di Levi-Civita = connessione di Cartan

e determinare la curvatura sezionale. Commentare...

si ricordi $[\hat{e}_i, \hat{e}_j] = \varepsilon_{ijk} \hat{e}_k \qquad \langle \hat{e}_i, \hat{e}_j \rangle = \ldots = 2\delta_{ij}$

rot. infinitesima attorno all'asse e_i

Tempo a disposizione 1h 15m. Le risposte vanno adeguatamente giustificate.

in $\mathbb{R}^3$

Topogo
20/2/2012

(1) $X = \frac{\partial}{\partial x}$

$Y = \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$

X e Y commutano $\Rightarrow$ def una distr. integrabile.

$\omega = \alpha\, dx + \beta\, dy + \gamma\, dz$

$\omega(X) = 0 \Rightarrow \alpha = 0$

$\omega(Y) = 0 \Rightarrow \beta = -\gamma$

$\omega = \beta\, dy - \beta\, dz$

controllo a posteriori: $d\omega = d\beta \wedge dy - d\beta \wedge dz$

$\omega \wedge d\omega = -\beta\, dz \wedge d\beta \wedge dy$

$\qquad - \beta\, dy \wedge d\beta \wedge dz$

$= -\beta\beta_x\, dz \wedge dx \wedge dy - \beta\beta_x\, dy \wedge dx \wedge dz$

$= -\beta\beta_x\, dx \wedge dy \wedge dz + \beta\beta_x\, dx \wedge dy \wedge dz$

$= 0 \quad \forall \beta.$

② Calcolare

$$\mathcal{L}_{\underbrace{x\frac{\partial}{\partial x}}_{X}}\Big(\overbrace{z\frac{\partial}{\partial y}\otimes dx\otimes dz}^{T}\Big)$$

$$x\frac{\partial}{\partial x}\frac{\partial}{\partial y}f = x\frac{\partial^2}{\partial x\partial y}f$$

$$\frac{\partial}{\partial y}\Big(x\frac{\partial}{\partial x}f\Big) = x\frac{\partial^2}{\partial x\partial y}f$$

$$\Big[x\frac{\partial}{\partial x},\frac{\partial}{\partial y}\Big]=0$$

$$= \overset{=0}{\mathcal{L}_X(z)}\frac{\partial}{\partial y}\otimes dx\otimes dz + z\,\underbrace{\mathcal{L}_X\Big(\frac{\partial}{\partial y}\Big)}_{=[x\frac{\partial}{\partial x},\frac{\partial}{\partial y}]=0}\otimes dx\otimes dz$$

$$+\; z\frac{\partial}{\partial y}\otimes \underbrace{\mathcal{L}_X(dx)}_{=}\otimes dz + z\frac{\partial}{\partial y}\otimes dx\otimes \underbrace{\mathcal{L}_X dz}_{=0}$$

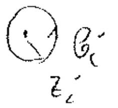
$$= \qquad d\,\mathcal{L}_X(x) = d\,X(x) = d\Big\{x\frac{\partial}{\partial x}(x)\Big\} = dx$$

$$= z\frac{\partial}{\partial y}\otimes dx\otimes dz \qquad \Rightarrow \qquad \mathcal{L}_X T = T$$

③ Campo vettoriale su $\mathbb{C}$, con poli critici in $z=1$, $z=2$, $z=3$, di indice, rispettivamente, 1, 2, 3

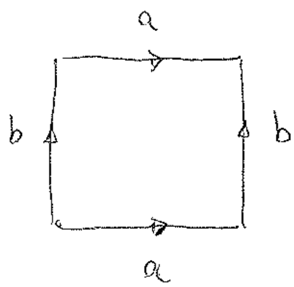
Sol. $$\bar V(z) = P(z) = (z-1)(z-2)^2(z-3)^3$$

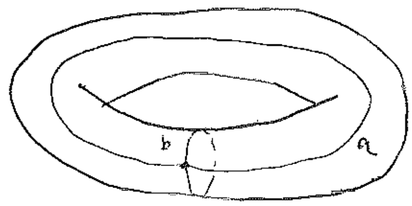
$$\operatorname{ind}_{z_i} \bar V = \frac{1}{2\pi i}\int_{\mathcal{C}_i} \frac{P'}{P}\, d\zeta = \frac{1}{2\pi i}\int_{\mathcal{C}_i} d\log P$$

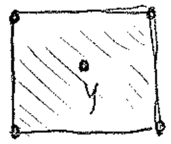


④ Dimostrare che $\pi_1(\underbrace{S^1\times S^1}_{\mathbb{T}^2}) = \mathbb{Z}^2$ utilizzando il teorema di Seifert-van Kampen

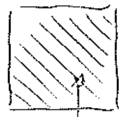
Cenno:

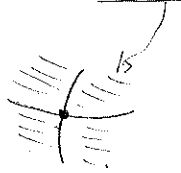




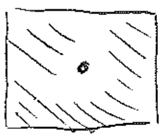
$\bar U =$ 

$(= \mathbb{T}^2 \setminus \{y\}) \simeq S^1 \vee S^1$

$\bar V =$ 



$\bar U \cup V = \mathbb{T}^2$

$\bar U \cap \bar V =$  $\sim S^1$

$\pi_1(S^1) = \mathbb{Z}$

$\pi_1(\bar U) = \mathbb{Z} * \mathbb{Z}$

ma $aba^{-1}b^{-1} = 1$ 

$\Rightarrow \pi_1(\mathbb{T}^2) = \mathbb{Z} * \mathbb{Z}$ abelianizzato $= \mathbb{Z}^2$

(5) $so(3)$ = matr. anti-simmetriche 3×3

$\langle X, Y\rangle = -\mathrm{Tr}(XY)$ metrica di Killing-Cartan

Data $\nabla_{X^\#} Y^\# = \frac{1}{2}[X^\#, Y^\#]$ $X^\#$: campo vett. inv. a sinistra su $SO(3)$ indotto da $X \in so(3)$

(connessione di Cartan)

Calcolare

$\langle R(X^\#, Y^\#) Z^\#, W^\#\rangle = \ldots = \frac{1}{4}[[X^\#, Y^\#], Z^\#]$

$[e_i, e_j] = \epsilon_{ijk} e_k$

Determinare la curvatura sezionale di $SO(3)$

$\langle e_i e_j\rangle = -\mathrm{Tr}(e_i e_j) = 2\delta_{ij}$

$R(\hat{e}_i, \hat{e}_j, \hat{e}_i, \hat{e}_j) = \frac{1}{4}\|[\hat{e}_i, \hat{e}_j]\|^2 = \frac{1}{4}\|\epsilon_{ijk}\hat{e}_k\|^2 = \frac{1}{4}\cdot 4 = 1$

$\mathcal{K} = \frac{R}{A^2}$ $A^2 = \|\hat{e}_i\|^2\|\hat{e}_j\|^2 = 4$

ds^2_{K-C} ∝ prop. alla metrica standard su S^3 $SO(3) = SU(2)/\mathbb{Z}_2$

$\mathcal{K} = \frac{1}{4}$

$\hat{e}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & +1 & 0 \end{pmatrix}$

$[\hat{e}_1, \hat{e}_2] = \hat{e}_3$

$$\underbrace{\overset{\hat{e}_1}{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}}\overset{\hat{e}_2}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}}}_{\begin{pmatrix} 0 & 0 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

$\hat{e}_2\hat{e}_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\hat{e}_1\hat{e}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ $\mathrm{Tr}(\ ) = -2$

ecc.