~~A Generalization (Pfaff systems)~~

Let Δ_k be a distribution of order k in M, $\dim M = n$ given by the intersection of kernels of $n-k$ 1-forms

$$\Delta_k : \begin{cases} \omega_1 = 0 \\ \omega_2 = 0 \\ \omega_{n-k} = 0 \end{cases}$$

The differential form version of the Frobenius theorem states that Δ_k is integrable if and only if

$$(\heartsuit) \qquad \boxed{d\omega_j \wedge \omega_1 \wedge \dots \wedge \omega_{n-k} = 0} \qquad \forall\, j = 1, \dots, n-k$$

$\dim M = n = 3$, $k = 2$, we recover the previous condition

Us check the necessity of $(\heartsuit)$, for $n = 4$, $k = 2$, for simplicity.

If $\Delta = \Delta_2$ is integrable, let (x_1, x_2, ξ_1, ξ_2) be a local coord. system such that the integral submanifolds are given by $\xi_1 = c_1$, $\xi_2 = c_2$ (and described by coordinates x_i)

Then ω_1 is of the form smooth functions

$$\omega_1 = f_1^{(1)}(x, \xi)\, d\xi_1 + f_2^{(1)}(x, \xi)\, d\xi_2$$

the kernel of ω_1, containing, at each point $\frac{\partial}{\partial x^i}$, $i = 1, 2$.)

Similarly $\omega_2 = f_1^{(2)}(x, \xi)\, d\xi_1 + f_2^{(2)}(x\, \xi_2)\, d\xi_2$

e intersection of the two kernels is $\langle \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2} \rangle$, the tangent space of the integral submanifold