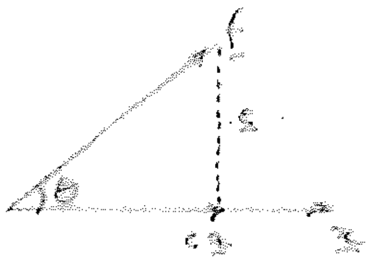
Signal representation by Fourier series (CTFS)

Signals as Vectors



$\underline{f}\cdot\underline{x} = |\underline{f}|\cdot|\underline{x}|\cos\theta$

inner or scalar product

$\rightarrow \underline{f} = c\underline{x} + \underline{e}$

$|c\underline{x}| = c|\underline{x}| = |\underline{f}|\cos\theta$ (1)

Error in the approximation of $\underline{f} \simeq c\underline{x}$

$\underline{e} = \underline{f} - c\underline{x}$

* When the projection of $\underline{f}$ onto $\underline{x}$ is orthogonal, the error is the smallest.

From (1): $c = \frac{1}{|\underline{x}|^2}\,\underline{f}\cdot\underline{x}$ $\quad (c|\underline{x}|^2 = |\underline{f}||\underline{x}|\cos\theta = \underline{f}\cdot\underline{x})$

orthogonality ⇔ scalar product $= 0$

Switching to signals

Problem: approximating a signal in terms of another signal,

$f(t) \simeq c\cdot x(t) \qquad t_1 \le t \le t_2$

error in the approximation

$e(t) = f(t) - c\cdot x(t)$

Best approximation: The one that minimizes the energy of the error over the interval $[t_1, t_2]$

$$E_e = \int_{t_1}^{t_2} e^2(t)\,dt = \int_{t_1}^{t_2} [f(t) - c\,x(t)]^2\,dt \qquad (2)$$

* To find the best approximation we have to find the value of c that minimizes (2) ⇒

$$\frac{dE_e}{dc} = 0 \rightarrow c = \frac{1}{E_x}\int_{t_1}^{t_2} f(t)\,x(t) \quad ; \quad E_x = \int_{t_1}^{t_2} x^2(t)\,dt$$

vectors	signals
$\underline{f} \approx c\, \underline{x}$	$f(t) \simeq c \cdot x(t)$
$c = \frac{1}{\lvert x \rvert^2} \underbrace{\underline{f} \cdot \underline{x}}$	$c = \frac{1}{E_x} \underbrace{\int_{t_1}^{t_2} f(t)\, x(t)\, dt}$ (3)
inner product between vectors	inner product between signals ↓ Area under the product of the two signals

⇒ When a function is approximated by another function (or signal)

$$f(t) \simeq c\, x(t)$$

Then the optimal value of c that minimizes the energy of the error $e(t) = f(t) - c\,x(t)$ is given by (3).

Note:

1. $\int_{t_1}^{t_2} f(t) \cdot x(t)\, dt = \langle f, x \rangle$ ⇒ inner product between (two) signals

2. $c\, x(t)$ ⇒ projection of the signal $f(t)$ onto the signal $x(t)$

$\langle f, x \rangle = 0 \iff f(t)$ and $x(t)$ are <u>orthogonal</u>

<u>Complex signals</u>

Let $f(t), x(t) \in \mathbb{C}$ (complex functions)

+ Energy of a complex function:

$$E_x = \int_{t_1}^{t_2} \lvert x(t) \rvert^2 dt$$

$$E_e = \int_{t_1}^{t_2} \lvert f(t) - c\,x(t) \rvert^2 dt$$

Recall

$$|u+v|^2 = (u+v)(u^*+v^*) = |u|^2 + |v|^2 + u^* v + v^* u$$

After some manipulations:

$$c = \frac{1}{E_x} \int_{t_1}^{t_2} f(t)\, x^*(t)\, dt$$ ⇒ scalar, or inner, product between complex functions

Note:

1. The inner product among complex signals reduces to (2c) for real signals;

2. Orthogonality still corresponds to vanishing of the inner product.

⇒ The analogy between signals and vectors provides a very intuitive graphical representation of the concepts of inner product, projection, error of approximation for signals.

<u>Energy of the sum of orthogonal signals</u>

a) Vectors: given $\underline{x}, \underline{y}$: $\underline{x} \perp \underline{y}$ ⇒ $|\underline{x}+\underline{y}|^2 = |\underline{x}|^2 + |\underline{y}|^2$

b) Signals: given $x(t), y(t) \in \mathbb{C}$

let: $z(t) = x(t) + y(t)$

$$E_z = \int_{t_1}^{t_2} |z(t)|^2 dt = \int_{t_1}^{t_2} |x(t)+y(t)|^2 dt =$$

$$= \int_{t_1}^{t_2} |x(t)|^2 dt + \int_{t_1}^{t_2} |y(t)|^2 dt + \int_{t_1}^{t_2} x^*(t)\, y(t)\, dt + \int_{t_1}^{t_2} x(t)\, y^*(t)\, dt$$

⇒ $E_z = E_x + E_y$ iif $x(t) \perp y(t)$

<u>orrelation</u> ⇒ measures the similarity among signals

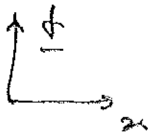
Vectors

$\underline{f}_2$ is more similar to $\underline{x}$ than is $\underline{f}_1$ ⇒ a good index for similarity is $c_i = \cos\theta_i$

Let: $\rho = \cos\theta = \dfrac{1}{|\underline{f}|\cdot|\underline{x}|}\cdot \underline{f}\cdot\underline{x}$ <u>correlation coefficient</u>

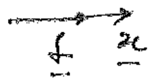
$$-1 \le \rho \le 1$$

$\rho = 0$



The vectors are orthogonal → no similarity ⇒ "complete strangers"

$\rho = 1$



The vectors are aligned in the same direction → the similarity is maximal. "best friends"

$\rho = -1$

The vectors are aligned along opposite directions → the dissimilarity is maximized ⇒ "worst enemies"

Signals: $f(t), x(t)$

Correlation coefficient: $\rho = \dfrac{1}{\sqrt{E_f \cdot E_x}} \displaystyle\int_{-\infty}^{+\infty} f(t)\,x(t)\,dt$

Using Schwarz inequality one can show that:

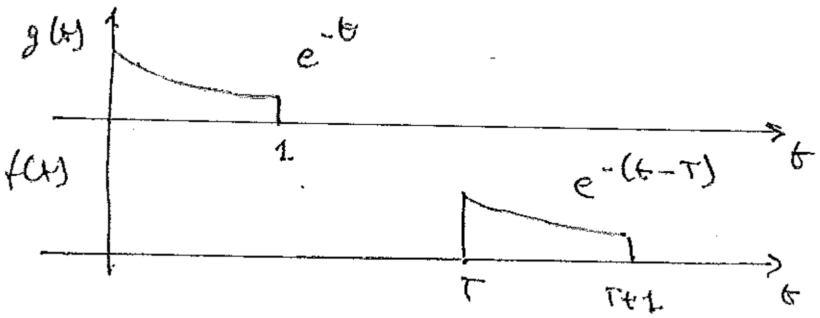
$$-1 \le \rho \le 1.$$

Complex signals

$$\rho = \frac{1}{\sqrt{E_f \cdot E_x}} \int_{-\infty}^{+\infty} f(t)\, x^*(t)\, dt$$

Application to signal detection: pattern matching

Correlation function



Time shift

For $T > 1$

$$\int_{-\infty}^{+\infty} g(t)\, f(t)\, dt = 0 \quad \text{because of the time shift}$$

⇒ To detect similar signals irrespectively of their time delay we can calculate their correlation coefficient for various values of the delay ⇒ cross-correlation

$$\psi_{f,g} \equiv \int_{-\infty}^{+\infty} f^*(\tau)\, g(\tau - t)\, d\tau$$

↓ delay

⇒ in the example: $\psi_{f,g}$ is maximum for $t = T$

* We can detect both the degree of similarity and the relative delay!

⇒ example

Autocorrelation

$$g(t) = f(t)$$

$$\psi_{f,g}(t) = \int_{-\infty}^{+\infty} f(\tau)\, g(\tau - t)\, d\tau = \int_{-\infty}^{+\infty} f(\tau)\, f(\tau - t)\, d\tau$$

$$\psi_f(t) = \int_{-\infty}^{+\infty} f(\tau)\, f(\tau - t)\, d\tau$$

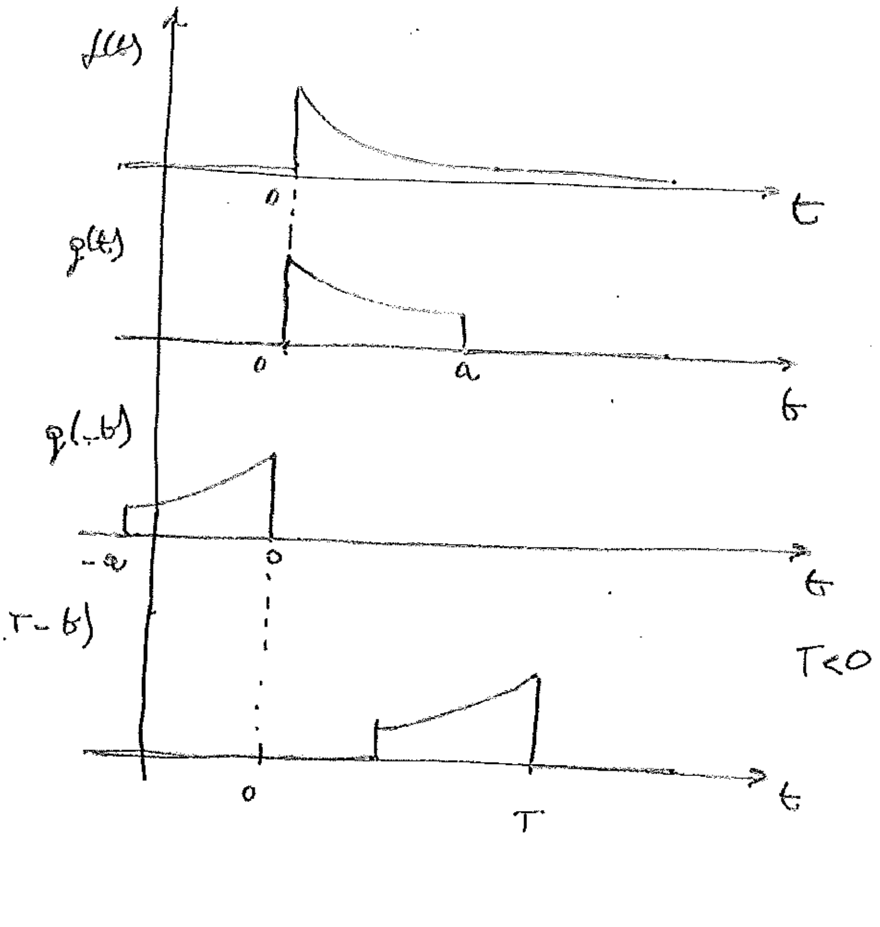
- Related to power spectra (Fourier transform)
- Very important for stochastic processes

Correlation vs Convolution

Convolution: $f(t) * g(t) = \int_{-\infty}^{+\infty} f(\tau)\, g(t-\tau)\, d\tau$

$$\Rightarrow \psi_{f,g}(t) = f(t) * g(-t)$$

$\Rightarrow$ The $g(t)$ signal goes through time reversal in the case of convolution.



BIBLIOGRAPHY 127

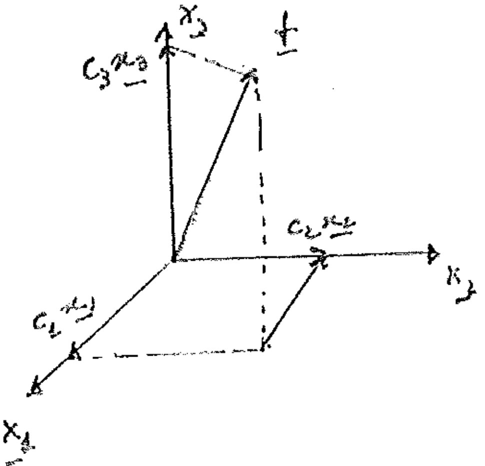
[144] S. Zeki, D.J. McKeefry, A. Bartels, and R.S.J. Frackowiak. Has a new color area been discovered? *Nature neuroscience*, 1(5):335, 1998.

128 BIBLIOGRAPHY

Signal representation by orthogonal signal sets

Goal: To represent a signal as a sum of orthogonal signals

a) Orthogonal vector space



$\underline{f} = c_1 \underline{x}_1 + c_2 \underline{x}_2 + c_3 \underline{x}_3$

$c_1\underline{x}_1, c_2\underline{x}_2, c_3\underline{x}_3$: projections of $\underline{f}$ on the axis

$$c_i = \frac{\underline{f}\cdot\underline{x}_i}{|\underline{x}_i|^2} \qquad i = 1,2,3$$

$\left\{ \frac{\underline{x}_1}{|\underline{x}_1|}, \frac{\underline{x}_2}{|\underline{x}_2|}, \frac{\underline{x}_3}{|\underline{x}_3|} \right\}$ form a <u>complete basis</u> for three-dimensional vectors

⇒ each vector lying in this space can be represented without error by a linear combination of such basis vectors.

The coefficients of the L.C. are the projections of the vector $\underline{f}$ onto the normalized basis.

b) Orthogonal signal space

First: we need to define a <u>orthogonal set of signals</u> that can be used as the basis.

Orthogonality ⇔ inner product = 0

⇒ orthogonality of a real signal set $\{x_1(t), x_2(t), \dots, x_N(t)\}$ over the time interval $[t_1, t_2]$:

$$\int_{t_1}^{t_2} x_n(t)\, x_m(t)\, dt = \begin{cases} 0 & \text{if } n \neq m \\ E_m & \text{if } n = m \end{cases}$$

<u>Orthonormal basis</u>: $E_m = 1 \quad \forall m$

Note: an orthonormal basis can be obtained by dividing $x_m(t)$ by $\sqrt{E_m}$ $\forall m$

ctors of the basis $\{x_1(t), \dots, x_N(t)\}$ permit to represent signal living in these space without error

signal that does not live in the same space:

$$f(t) \simeq c_1 x_1(t) + c_2 x_2(t) + \dots + c_m x_m(t) = $$
$$= \sum_{i=1}^{N} c_i x_i(t)$$

approximation error:

$$e(t) = f(t) - \sum_{i=1}^{N} c_i x_i(t)$$

we can show that the energy of the error signal is minimized here:

$$c_i = \frac{\int_{t_1}^{t_2} f(t)\, x_i(t)\, dt}{\int_{t_1}^{t_2} x_i^2(t)\, dt} \qquad i = 1, \dots, N$$

correspondingly:

$$E_e = \int_{t_1}^{t_2} f^2(t)\, dt - \sum_{i=1}^{N} c_i^2 E_{m}$$

ote: $E_e \downarrow$ when $i \uparrow$

f $E_e \to \emptyset$ when $i \to +\infty \Rightarrow$ the orthogonal signal set is <u>complete</u>

n this case:

1) $$f(x) = \sum_{i=1}^{+\infty} c_i x_i(t) \qquad t_1 \le t \le t_2$$

2) The energy of $f(t)$ is equal to the sum of the energies of its orthogonal components.

If $\{x_m(t)\}$ is a complete set then

$$(1) \qquad f(t) = \sum_{1}^{+\infty} c_i x_i(t)$$ $\Rightarrow$ generalized Fourier series of $f(t)$ with respect to the set $\{x_m(t)\}$

When $\varepsilon_i \to \phi$ for $i \to +\infty$

$\Rightarrow \{x_m(t)\}$ is called a set of basis functions or basis signals

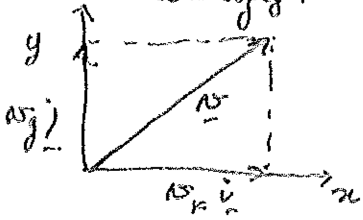
. We will consider energy signals.

The equality (1) is an equality with respect to energy: the energy of the difference signal goes to ϕ as the number of terms in the right-hand side goes to $+\infty$.
This means that there can be local discontinuities where the function and the sum are different.

PARSEVAL'S THEOREM

$$\int_{t_1}^{t_2} f^2(t)\,dt = \sum_{1}^{+\infty} c_m^2 \, \varepsilon_m$$

Geometric analogy:



$$|\underline{v}|^2 = |v_x|^2 + |v_y|^2$$

generalization to complex signals

orthogonality:

$$\int_{t_1}^{t_2} x_m(t)\, x_n^*(t)\, dt = \begin{cases} \phi & \text{if } m \neq n \\ E_n & \text{if } m = n \end{cases}$$

The set $\{x_n(t)\}$ is complete (i.e. forms a basis set) then

$$\begin{cases} f(t) = \sum_{1}^{\infty} {}_n\, C_n\, x_n(t) \\ C_n = \frac{1}{E_n} \int_{t_1}^{t_2} f(t)\, x_n^*(t)\, dt \end{cases}$$

$\therefore$ C_n does not depend on C_m $\forall m \neq n$

Trigonometric Fourier series

The signal set:

$$\{1, \cos \omega_0 t, \cos(2\omega_0 t), \dots \cos(n\omega_0 t), \dots; \sin(\omega_0 t), \sin(2\omega_0 t), \dots \sin(n\omega_0 t)\}$$

is orthogonal over any interval of duration $T_0 = 2\pi/\omega_0$:

- $$\int_{T_0} \cos(n\omega_0 t)\cos(m\omega_0 t)\,dt = \begin{cases} \phi & n \neq m \\ \frac{T_0}{2} & n = m \neq 0 \end{cases}$$

- $$\int_{T_0} \sin(n\omega_0 t)\sin(m\omega_0 t)\,dt = \begin{cases} 0 & n \neq m \\ \frac{T_0}{2} & n = m \neq 0 \end{cases}$$

- $$\int_{T_0} \sin(n\omega_0 t)\cos(m\omega_0 t)\,dt = 0 \quad \text{always}, \ \forall n, m$$

$\Rightarrow$ Trigonometric set

It can be shown that a trigonometric set is complete

$$\Rightarrow \quad f(t) = a_0 + \sum_{1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t); \quad t_1 \le t \le t_1 + 1$$

$\omega_0 = \frac{2\pi}{T_0}$: fundamental

$n\omega_0$: n^{th} harmonic $(n \in I)$

Fourier coefficients

$$a_n = \frac{1}{E_n} \langle f(t), x_n(t) \rangle = \frac{\int_{t_1}^{t_1+T_0} f(t)\cos(n\omega_0 t)\,dt}{\int_{t_1}^{t_1+T_0} \cos^2(n\omega_0 t)\,dt}$$

$x_n(t) \to \cos(n\omega_0 t)$

but:

1) $\int_{t_1}^{t_1+T_0} \cos^2(n\omega_0 t)\,dt = \frac{T_0}{2}$ 2) $n = 0 \Rightarrow \int^{t_1+T_0} dt = T_0$

$$a_n = \frac{2}{T_0} \int_{t_1}^{t_1+T_0} f(t) \cos(n\omega_0 t)\, dt$$

$n = 0 \rightarrow \quad a_0 = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} f(t)\, dt \Rightarrow$ mean value of the signal over the interval

$$b_n = \langle f(t), x_n(t) \rangle = \int_{t_1}^{t_1+T_0} f(t) \sin(n\omega_0 t)\, dt \cdot \frac{1}{\int_{t_1}^{t_1+T_0} \sin^2(n\omega_0 t)\, dt}$$

$x_n(t) \downarrow \sin(n\omega_0 t)$

but:

$$\int_{t_1}^{t_1+T_0} \sin^2(n\omega_0 t)\, dt = \begin{cases} \frac{T_0}{2} & \text{for } n \neq 0 \\ \phi & \text{for } n = 0 \end{cases}$$

$$\Rightarrow b_n = \frac{2}{T_0} \int_{t_1}^{t_1+T_0} f(t) \sin(n\omega_0 t)\, dt \qquad n \neq 0$$

summary:

$$f(t) = a_0 + \sum_{1}^{+\infty} a_n \cos(n\omega_0 t) + \sum_{1}^{+\infty} b_n \sin(n\omega_0 t) \Rightarrow \text{signal reconstruction}$$

$$\left.\begin{aligned} a_0 &= \frac{1}{T_0} \int_{t_1}^{t_1+T_0} f(t)\, dt \\ a_n &= \frac{2}{T_0} \int_{t_1}^{t_1+T_0} f(t) \cos(n\omega_0 t)\, dt \qquad n = 1, 2, \ldots \\ b_n &= \frac{2}{T_0} \int_{t_1}^{t_1+T_0} f(t) \sin(n\omega_0 t)\, dt \qquad n = 1, 2, \ldots \end{aligned}\right\} \text{signal analysis or decomposition or projection over the basis}$$

ompact Trigonometric Fourier series

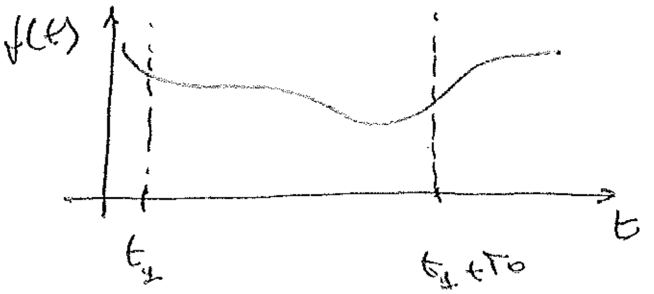
$$a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) = C_n \cos(n\omega_0 t + \theta_n)$$

here:

$$C_n = \sqrt{a_n^2 + b_n^2} \qquad (C_0 = a_0)$$

$$\theta_n = tg^{-1}\left(-\frac{b_n}{a_n}\right)$$

eriodicity of the Trigonometric Fourier series



⇒ we have focused on the interval $\overline{t_1, t_1+T_0}$

⇒ The Fourier series that we have derived is only representative of the behavior of f(t) over such an interval!

uestion: is there any conditions for which the Fourier series is representative of the behavior of the whole signal? $(-\infty \le t \le +\infty)$

t's consider the Fourier series and let $\varphi(t)$ be the corresponding function:

$$\varphi(t) = C_0 + \sum_{n=1}^{+\infty} C_n \cos(n\omega_0 t + \theta_n) \qquad \forall t$$

ote: $\varphi(t) = f(t)$ for $t_1 \le t \le t_1 + T_0$

but extends over the whole temporal axis

ence:

$$\varphi(t+T_0) = C_0 + \sum_{n=1}^{+\infty} C_n \cos(n\omega_0 (t+T_0) + \theta_n)$$

since : $\omega_0 = \frac{2\pi}{T_0} \rightarrow \omega_0 T_0 = 2\pi$

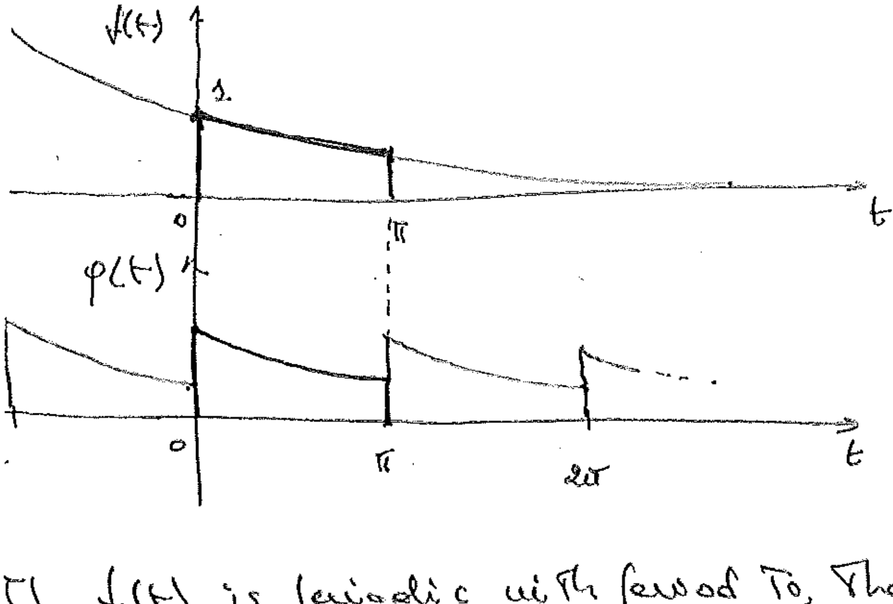
$$\varphi(t+T_0) = C_0 + \sum_{1}^{+\infty} C_n \cos\Big[\big(n\omega_0 t + \underbrace{2n\pi}_{n\omega_0 T_0}\big) + \theta_n\Big] =$$

$$= C_0 + \sum_{1}^{+\infty} C_n \cos\left(n\omega_0 t + \theta_n\right) = \varphi(t) \quad \forall t$$

$$\Rightarrow \boxed{\varphi(t+T_0) = \varphi(t)} \quad \forall t$$

The Fourier series is periodic of period T_0 (the period of its fundamental).

The signal $f(t)$ and the Fourier series $\varphi(t)$ are equal on the interval $t_1 \le t \le t_1+T_0$. Outside such interval, $\varphi(t)$ repeats itself periodically with period T_0.

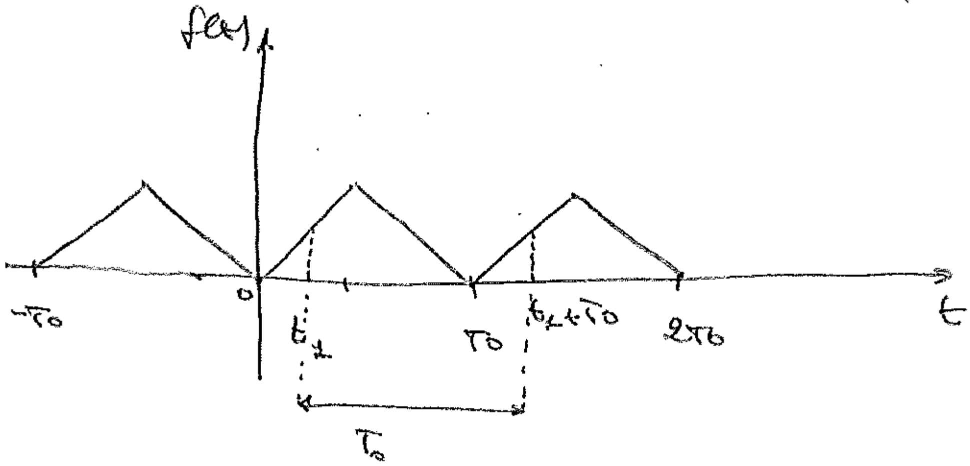


$f(t) = e^{-t/2}$

If $f(t)$ is periodic with period T_0, then the Fourier series representing $f(t)$ over T_0 is also representative of $f(t)$ on the whole temporal axis.

$f(t+T_0) = f(t) \quad \Rightarrow \quad f(t) \equiv \varphi(t)$

Consequence: we can use any interval T_0 to evaluate the Fourier series of the signal $f(t)$



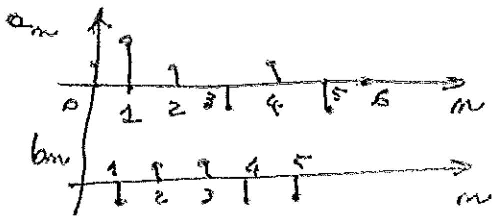
NOTE :

f is a periodic real valued function: $f(t) \in \mathbb{R}$

, Fourier coefficients : $a_n, b_n \Leftrightarrow$ Fourier series (CTFS)

$\downarrow$

discrete values



First golden rule of the F. Transform :

- The F. Transform of a periodic signal is discrete

periodicity in Time domain $\Rightarrow$ sampling in Fourier domain

We will see that it is also true going the other way around :

sampling in Time $\rightarrow$ periodization in Fourier domain

(The F. Transform of a discrete-Time signal is periodic)

Summary

$f(t) \in \mathbb{R}$, $f(t+T_0) = f(t)$

$a_n = \frac{2}{T_0}\int_{T_0} f(t) \cos(n\omega_0 t)\, dt \qquad n = 1, 2, \dots$

$b_n = \frac{2}{T_0}\int_{T_0} f(t) \sin(n\omega_0 t)\, dt \qquad n = 1, 2, \dots$

$a_0 = \frac{1}{T_0}\int_{T_0} f(t)\, dt$

Fourier coefficients of the series

a. spectrum

$$) = C_0 + \sum_{1}^{\infty} C_n \cos(n\omega t + \theta_n) \quad \forall t$$

periodic signal can be expressed as the sum of sinusoids at

quency ϕ (mean value or DC component), ω_0, $2\omega_0$, ...

amplitude of the n^{th} sinusoid is C_n

phase " " " " is θ_n

as a function of n → amplitude spectrum } frequency spectra

" " " " " → phase spectrum } of $f(t)$

identify:

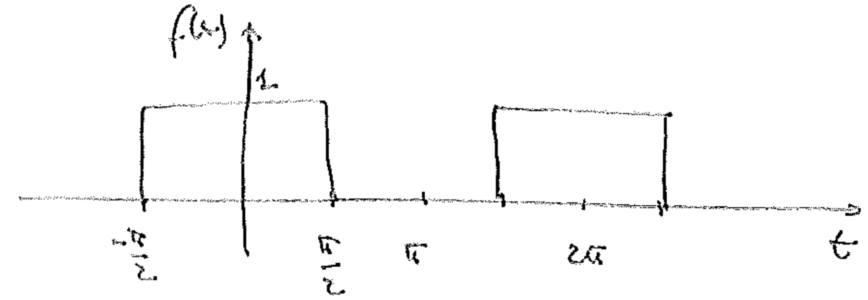
$f(t)$ → Time domain

$\{C_n, \theta_n\}$ → frequency domain

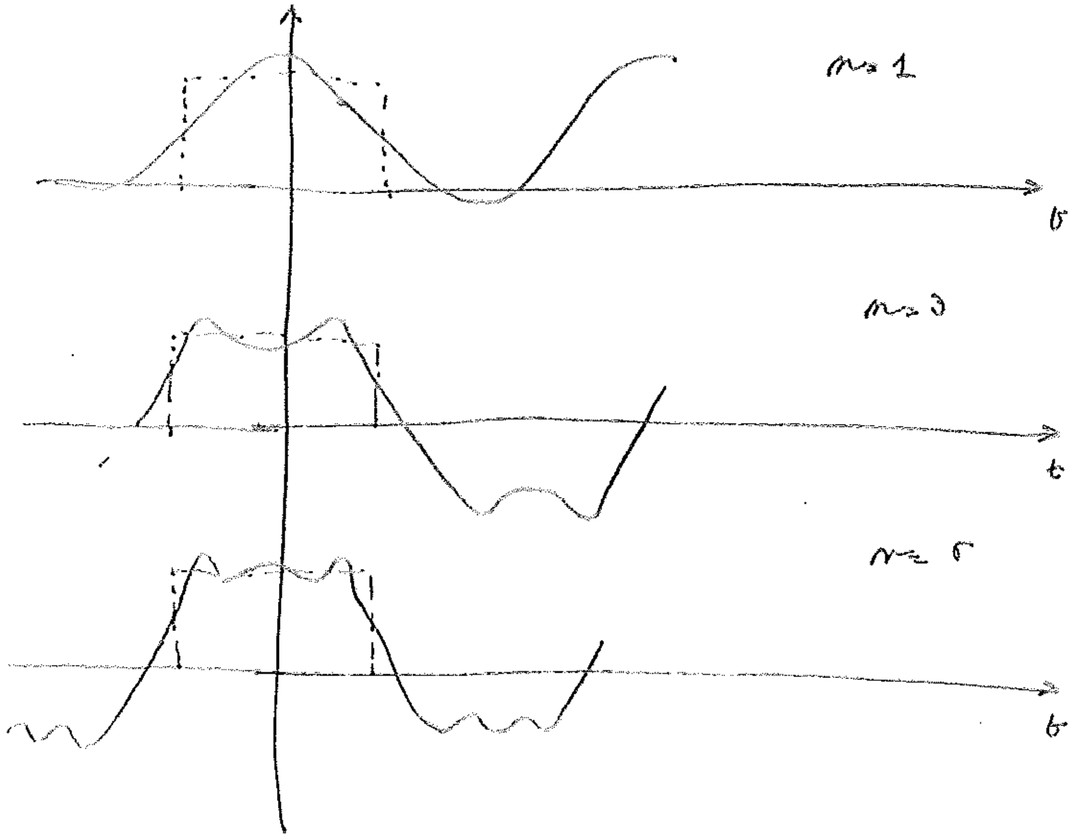
:

$f(t)$ is even periodic → only cosine Terms

$f(t)$ is odd periodic → only sine Terms ave shaping



$$f(t) = \frac{1}{2} + \frac{2}{\pi}\left(\cos(t) - \frac{1}{3}\cos(3t) + \frac{1}{5}\sin(5t) - \frac{1}{7}\sin(7t)\right)$$



Gibbs phenomenon: oscillations around points of discontinuity

A large number of harmonics is needed to represent sharp discontinuities

⇒ The more the signal is sharp, the more frequency components are needed to provide a "good" approximation of the signal.

ote: for a constant signal, only one component is needed

uivalently: the rate of decay of the Fourier spectrum (amplitude) epends on the sharpness of the signal.

teleoperazione, gli algoritmi di controllo del contatto con le superfici, gli algoritmi di compensazione del ritardo di trasmissione, o le strutture software in grado di supportare un sistema di chirurgia robotica o di garantirne il corretto funzionamento.

ULTERIORI CREDITI per altre attività e prova finale (36 CFU)

Docenti referenti in ambito informatico e principali ambiti di competenza
Carlo Combi: basi di dati e sistemi informativi clinici
Vincenzo Manca: bioinformatica
Vittorio Murino: elaborazioni di bioimmagini
Gloria Menegaz: elaborazioni di bioimmagini
Paolo Fiorini: robotica chirurgica

Docenti interessati in ambito medico (elenco parziale e in fase di aggiornamento)
Giancesare Guidi, Andrea Sbarbati, Michele Tansella, Pier Franco Pignatti, Giorgio Berton, Alberto Fenzi

Theorem: it can be shown that if the k-1 first derivatives of a periodic signal $f(t)$ are continuous and the k-th derivative is discontinuous, then its amplitude spectrum decays with frequency at least as rapidly as $1/n^{k+1}$

Exponential Fourier series

Theorem: The set of exponentials $e^{jn\omega_0 t}$ $(n = 0, \pm 1, \pm 2, \dots)$ is orthogonal over every interval of duration $T_0 = 2\pi/\omega_0$:

$$\int_{T_0} e^{jm\omega_0 t} \left(e^{jn\omega_0 t}\right)^* dt = \int_{T_0} e^{j(m-n)\omega_0 t} dt = \begin{cases} 1 \cdot T_0 = T_0 & \text{for } m = n \\ 0 & \text{otherwise} \end{cases}$$

The set is complete.

- A signal $f(t)$ can be expanded over a period T_0 as:

$$f(t) = \sum_{n=-\infty}^{+\infty} D_n e^{jn\omega_0 t}$$

$$D_n = \frac{1}{T_0} \int_{T_0} f(t) e^{-jn\omega_0 t}$$

This is equivalent to the Fourier series representation.

Demonstration:

Euler's relation: $\cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$

$$f(t) = C_0 + \sum_{1}^{+\infty} C_n \cos\left(n\omega_0 t + \theta_n\right) =$$

$$= C_0 + \frac{1}{2} \sum_{1}^{+\infty} C_n \left(e^{j(n\omega_0 t + \theta_n)} + e^{-j(n\omega_0 t + \theta_n)}\right) =$$

$$= C_0 + \frac{1}{2} \sum_{1}^{+\infty} \left(C_n e^{j\theta_n} e^{jn\omega_0 t} + C_n e^{-j\theta_n} e^{-jn\omega_0 t}\right) =$$

$$= C_0 + \sum_{1}^{\infty} \underbrace{\left(\frac{C_n}{2} e^{j\theta_n}\right)}_{D_n} e^{jn\omega_0 t} + \underbrace{\left(\frac{C_n}{2} e^{-j\theta_n}\right)}_{D_{-n}} e^{-jn\omega_0 t} =$$

$$= C_0 + \sum_{-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

$n=0 \quad D_n = \frac{C_0}{2} + \frac{C_0}{2} = C_0 \Rightarrow D_0 = C_0$

$\Rightarrow f(t) = \sum_{-\infty}^{+\infty} D_n e^{-jn\omega_0 t}$

We can also show that: $D_n = \frac{1}{2}(a_n - jb_n)$

Summary:

$$\begin{cases} f(t) = \sum_{-\infty}^{+\infty} D_n e^{-jn\omega_0 t} \\ D_n = \frac{1}{2}(a_n - jb_n) = \frac{1}{2} C_n e^{jn\omega_0 t} \end{cases}$$

Exponential Fourier spectra

$D_n \in \mathbb{C} \Rightarrow \begin{cases} |D_n| \to \text{amplitude spectra} \\ \measuredangle D_n \to \text{phase spectra} \end{cases}$ $\quad D_n = |D_n| e^{\measuredangle D_n}$

$$D_n = \frac{1}{T_0} \int_{T_0} f(t) e^{-jn\omega_0 t} dt$$

Implications:

1. $\rightarrow D_0 = \frac{1}{T_0} \int_{T_0} f(t) dt = a_0 = C_0$

2. $\begin{cases} D_n = \frac{1}{2} C_n e^{j\theta_n} \\ D_{-n} = \frac{1}{2} C_n e^{-j\theta_n} \end{cases} \Rightarrow D_{-n} = D_n^*$

$|D_n| = \frac{1}{2}|C_n| \qquad \forall n \neq 0$

$\begin{cases} \measuredangle D_n = \theta_n \\ \measuredangle D_{-n} = -\theta_n \end{cases} \Rightarrow \begin{cases} D_n = |D_n| e^{j\theta_n} \\ D_{-n} = |D_n| e^{-j\theta_n} \end{cases}$

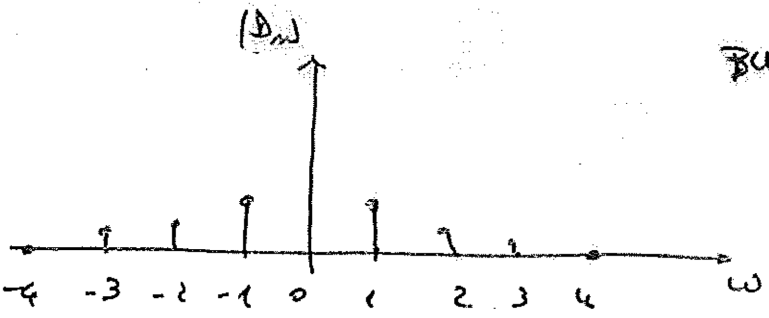
Note:

- $|D_n| = |D_{-n}|$ → The amplitude spectrum is an <u>even</u> function of ω

- $\angle D_n = -\angle D_{-n}$ → The phase spectrum is an <u>odd</u> function of ω

<u>BANDWIDTH</u> of a signal

Difference between the highest and the lowest frequencies of a signal.



BW = 3

<u>Parseval's theorem</u>

$$f(t) = C_0 + \sum_{1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

↓

power signal

$$P_n = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} C_n^2 \cos^2(n\omega_0 t + \theta_n)\,dt =$$

$$= \lim \frac{1}{T} C_n^2 \frac{1}{2} \int_{-T/2}^{T/2} \left[1 + \cos\left[2(n\omega_0 t + \theta_n)\right]\right] dt =$$

$(\cos 2\theta_n \ \sin 2\theta_n)$ $\quad (\cos 2\theta_n + \sin(2\theta_n)) = 0$

$$= \lim_{T\to\infty} \frac{1}{2T} C_n^2 \left[t + \frac{1}{2}\sin(2n\omega_0 t + 2\theta_n)\right]_{-T/2}^{T/2} =$$

$$= \lim_{T\to\infty} \frac{C_n^2}{2T}\cdot T + \lim_{T\to\infty} \frac{1}{4T}\left(\sin(n\omega_0 T + 2\theta_n) - \sin(n\omega_0 T + 2\theta_n)\right) =$$

$$= \frac{C_n^2}{2} + \lim_{T\to\infty} \frac{1}{4T}\left[\sin(2n\pi + 2\theta_n) - \sin(2n\pi + 2\theta_n)\right] = \frac{C_n^2}{2}$$

$P_f = C_0^2 + \frac{1}{2}\sum_{1}^{+\infty} C_n^2$ ⇒ The power of a periodic signal is equal to the sum of the powers of its Fourier components.

For the exponential Fourier series:

$$f(t) = \sum_{-\infty}^{+\infty} D_n e^{jn\omega_0 t}$$

Power of the single component:

$$P_n = \lim_{T\to\infty} \frac{1}{2T}\int_{-T/2}^{T/2} |D_n e^{jn\omega_0 t}|^2 dt = \lim_{T\to\infty} \frac{1}{2T}\int_{-T/2}^{T/2} |D_n|^2 dt = \lim_{T\to\infty} \frac{1}{2T}|D_n|^2 \cdot T = \frac{|D_n|^2}{2}$$

$$\to P_n = \frac{|D_n|^2}{2}$$

$$\Rightarrow \boxed{P_f = \frac{1}{2}\sum_{-\infty}^{+\infty} |D_n|^2}$$

Real signals: $f(t) \in \mathbb{R} \Rightarrow |D_{-n}| = |D_n|$

$$\to P_f = D_0^2 + 2\sum_{+1}^{+\infty} D_n^2$$

Limitations of the Fourier series

1. It can be used only for periodic signals
   Note: real signals are not periodic as they don't start at $-\infty$;
   Solution ⇒ other forms of F. transforms (F. integral)

2. It is suitable only to asymptotically stable systems.
   It cannot handle so easily unstable or marginally stable systems ⇒ Laplace Transform.

Continuous Time Signal Analysis : <u>The Fourier Transform</u>

$$\begin{cases} F(\omega) = \displaystyle\int_{-\infty}^{+\infty} f(t)\, e^{-j\omega t}\, dt & : \text{analysis} \\ f(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty} F(\omega)\, e^{j\omega t}\, dt & : \text{synthesis} \end{cases}$$

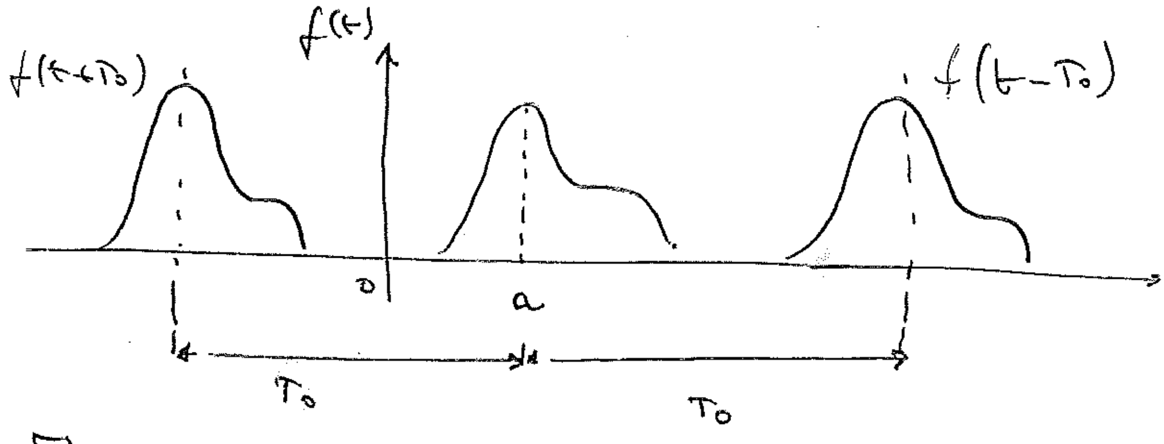
$f(t)$: a-periodic signal.

Sufficient condition :

$$\int_{-\infty}^{+\infty} |f(t)|\, dt < +\infty \qquad \text{Dirichlet's condition}$$

Derivation.

$f(t) \rightarrow$ build a new signal by periodization with period T_0, and let it be $f_{T_0}(t)$



$T_0 \to \infty$ The spacing among the pulses increases and in the limit $T_0 \to \infty$, we find the original function

$$\lim_{T_0 \to \infty} f_{T_0}(t) = f(t)$$

But: $f_{T_0}(t)$ is periodic by construction → it can be expressed as a Fourier series:

$$\Rightarrow f_{T_0}(t) = \sum_{n=-\infty}^{+\infty} D_n e^{jn\omega_0 t} \qquad (1)$$

$$D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f_{T_0}(t) e^{-jn\omega_0 t} dt \qquad \omega_0 = \frac{2\pi}{T_0}$$

Since:

$$\int_{-T_0/2}^{T_0/2} f_{T_0}(t) e^{-jn\omega_0 t} dt = \int_{-\infty}^{+\infty} f(t) e^{-jn\omega_0 t} dt$$

Then:

$$D_n = \frac{1}{T_0} \int_{-\infty}^{+\infty} f(t) e^{-jn\omega_0 t} dt$$

But, by definition:

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt$$

$$\Rightarrow \qquad D_n = \frac{1}{T_0} F(\omega)\Big|_{\omega = n\omega_0} = \frac{1}{T_0} F(n\omega_0)$$

⇒ putting this result into (1):

$$f_{T_0}(t) = \sum_{n=-\infty}^{+\infty} \frac{1}{T_0} F(n\omega_0) e^{jn\omega_0 t}$$

Note that when $T_0 \to \infty \Rightarrow \omega_0 = \frac{2\pi}{T_0} \to 0 \Rightarrow$ let $\Delta\omega = \omega_0$

rewritten up:

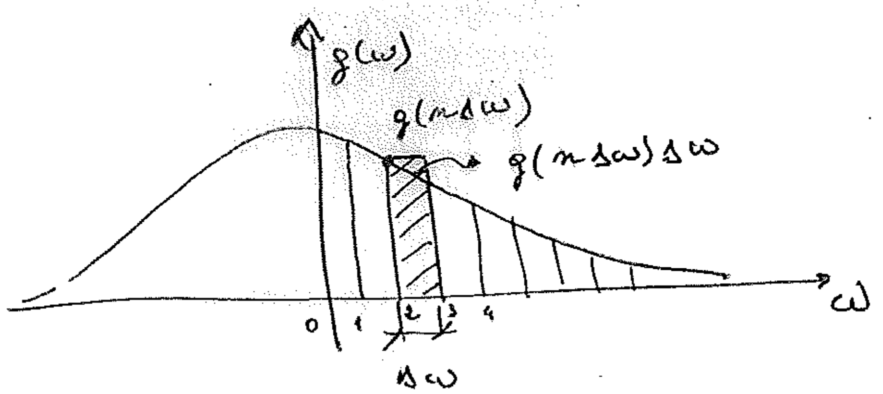
$$f_{T_0}(t) = \sum_{n=-\infty}^{+\infty} \frac{\Delta\omega}{2\pi} F(n\Delta\omega t)\, e^{j(n\Delta\omega)t}$$

thus:

$$f(t) = \lim_{T_0 \to \infty} f_{T_0}(t) = \lim_{\Delta\omega \to \phi} \sum_{n=-\infty}^{+\infty} \frac{\Delta\omega}{2\pi} F(n\Delta\omega t)\, e^{j(n\Delta\omega)t} =$$

$$= \lim_{\Delta\omega \to \phi} \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} F(n\Delta\omega)\,\Delta\omega\, e^{j(n\Delta\omega)t}$$

when $\Delta\omega \to \phi \Rightarrow \sum_{-\infty}^{+\infty} g(n\Delta\omega)\Delta\omega \to \int_{-\infty}^{+\infty} g(\omega)\, d\omega$



Then:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{j\omega t}\, d\omega$$

Summary:

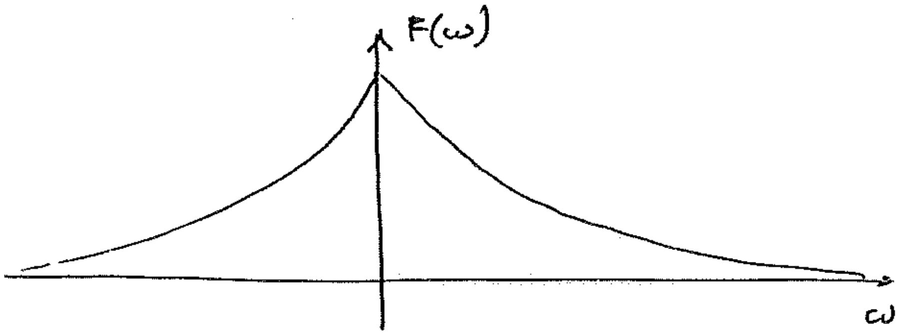
$$\begin{cases} f(t) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{+\infty} F(\omega)\, e^{j\omega t}\, d\omega & \rightarrow \text{synthesis} \\[2ex] F(\omega) = \displaystyle\int_{-\infty}^{+\infty} f(t)\, e^{-j\omega t}\, dt & \Rightarrow \text{analysis or} \end{cases}$$

<u>Fourier integral</u> ~ Continuous Time FT (CTFT)

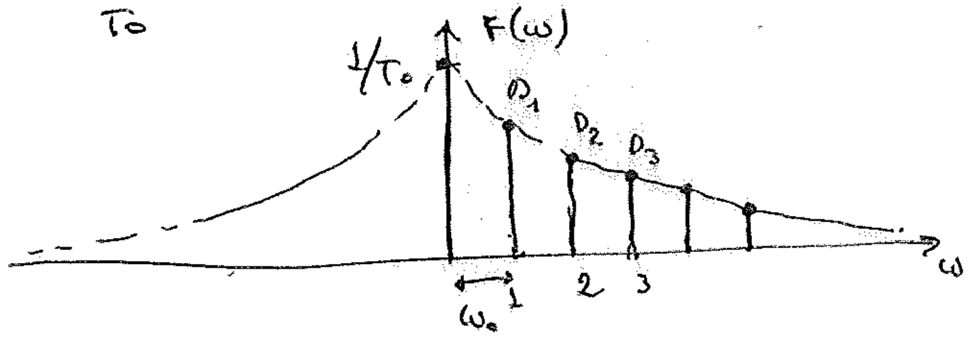
Observations

1. $D_n = \frac{1}{T_0} F(n\omega_0)$

Let $F(\omega)$ be the F. Transform of a given signal $f(t)$.

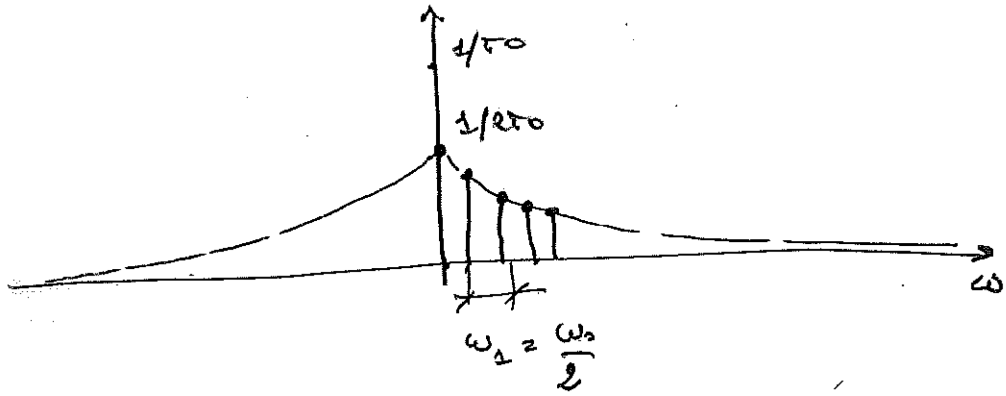


$\omega_0 = \frac{2\pi}{T_0} \Rightarrow$



$T_1 = 2T_0 \rightarrow \omega_1 = \frac{\omega_0}{2} \rightarrow D_n^1 = \frac{1}{T_1} F(n\omega_1) = \frac{1}{2T_0} F(n\omega_1)$

$\Downarrow$



$\Rightarrow$ When $T_0 \uparrow$, $\omega_0 \downarrow \Rightarrow$ The sampling of the spectrum becomes denser and the amplitude decreases.

In the limit $T_0 \rightarrow \infty$, we have ∞ density of zero-amplitude components.

Dirichlet's condition

$$\begin{cases} f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)\, e^{j\omega t}\, d\omega \\ F(\omega) = \int_{-\infty}^{+\infty} f(t)\, e^{-j\omega t}\, dt \end{cases}$$

$$|F(\omega)| = \left|\int_{-\infty}^{+\infty} f(t)\, e^{-j\omega t}\, dt\right| \leq \int_{-\infty}^{+\infty} |f(t)|\, dt$$

$$\Rightarrow \boxed{\int_{-\infty}^{+\infty} |f(t)|\, dt < +\infty}$$ $\Rightarrow$ <u>sufficient</u> condition for the existence of the FT

Note: This is <u>not</u> a necessary condition. Ex: $f(t) = \frac{\sin(t)}{t}$

Pragmatically: the existence of a signal is a sufficient condition for the existence of its FT!

<u>Notes</u>

- $F(\omega) \in \mathbb{C} \Rightarrow F(\omega) = |F(\omega)|\, e^{j\measuredangle F(\omega)}$

$$F(-\omega) = \int_{-\infty}^{+\infty} f(t)\, e^{j\omega t}\, dt$$

$$\Rightarrow \text{if } f(t) \in \mathbb{R} \Rightarrow \boxed{F(-\omega) = F^*(\omega)}$$

or, equivalently:

$$\begin{cases} |F(-\omega)| = |F(\omega)| \rightarrow \text{even function} \\ \measuredangle F(-\omega) = -\measuredangle F(\omega) \rightarrow \text{odd} \end{cases}$$

- $F(\omega) \Leftrightarrow$ <u>spectral density</u> of $f(t)$ in ω

Properties of the FT

Symmetry

$$f(t) \leftrightarrow F(\omega)$$

$$F(t) \leftrightarrow 2\pi f(-\omega)$$

Proof

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(x)\, e^{jxt}\, dx$$

$t \to -t$

$$2\pi f(-t) = \int_{-\infty}^{+\infty} F(x)\, e^{-jxt}\, dx$$

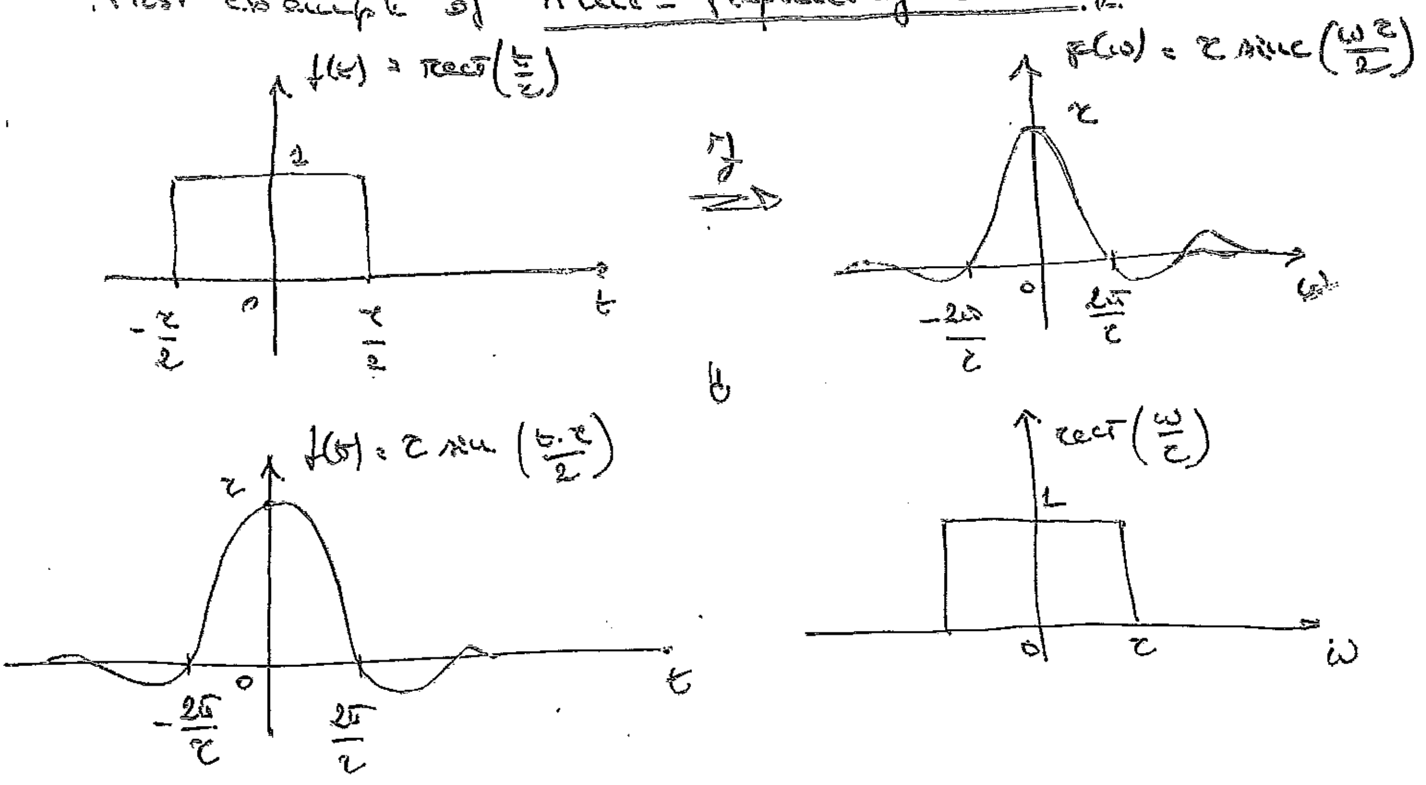
changing $t \to \omega$

$$\to f(-\omega) = \int_{-\infty}^{+\infty} F(x)\, e^{-jx\omega}\, dx$$

changing $x \to t$

$$\to f(-\omega) = \int_{-\infty}^{+\infty} F(t)\, e^{-j\omega t}\, dt \quad \text{cvd}$$

First example of Time-frequency duality:



.. Scaling

$$f(t) \leftrightarrow F(\omega)$$

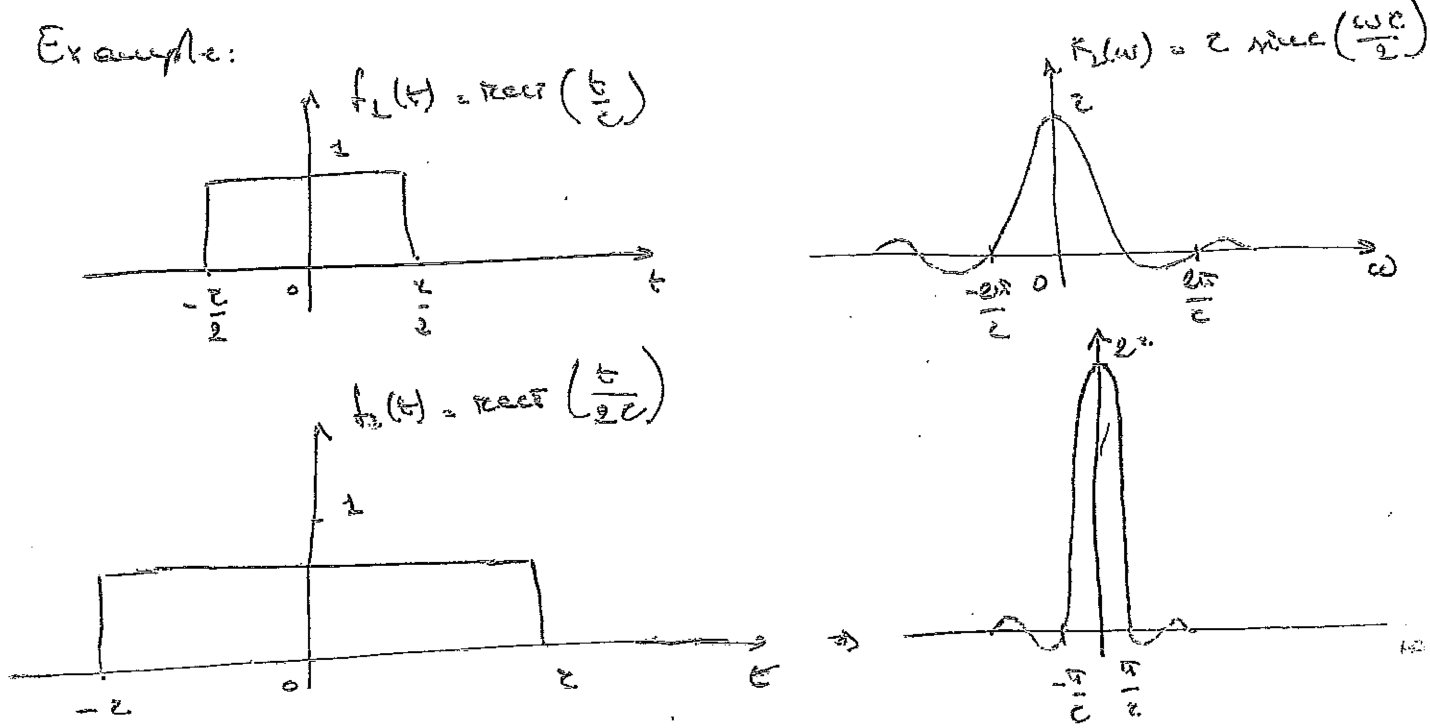
$$f(at) = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

$$\mathcal{F}\{f(at)\} = \int_{-\infty}^{+\infty} f(at)\, e^{-j\omega t}\, dt = \qquad x = at \Rightarrow t = \frac{x}{a}, \quad dt = \frac{dx}{a}$$

$$= \int_{-\infty}^{+\infty} f(x)\, e^{-j\omega \frac{x}{a}}\, \frac{1}{a}\, dx \qquad \text{for } a>0$$

if $a<0 \Rightarrow \int_{+\infty}^{-\infty} \Rightarrow$ sign changes

Example:



$$a = \frac{1}{2} \Rightarrow F_2(\omega) = 2\, F_1(2\omega) = 2\tau\, sinc(\omega\tau)$$

⇒ <u>stretching in time</u> domain corresponds to <u>shrinking in frequency</u> domain and viceversa.

Specifically:

compression in time → expansion in frequency

(and viceversa)

3. Time-shift

$$f(t-t_0) \Leftrightarrow e^{-j\omega t_0} F(\omega)$$

$$\mathcal{F}\{f(t-t_0)\} = \int_{-\infty}^{+\infty} f(t-t_0)\, e^{-j\omega t}\, dt \qquad (1)$$

let: $\quad u = t - t_0 \rightarrow t = u + t_0$

$$\rightarrow (1) = \int_{-\infty}^{+\infty} f(u)\, e^{-j\omega(u+t_0)}\, du = \int_{-\infty}^{+\infty} e^{-j\omega t_0} f(u)\, e^{-j\omega u}\, du =$$

$$= e^{-j\omega t_0} F(\omega) \quad \text{end}$$

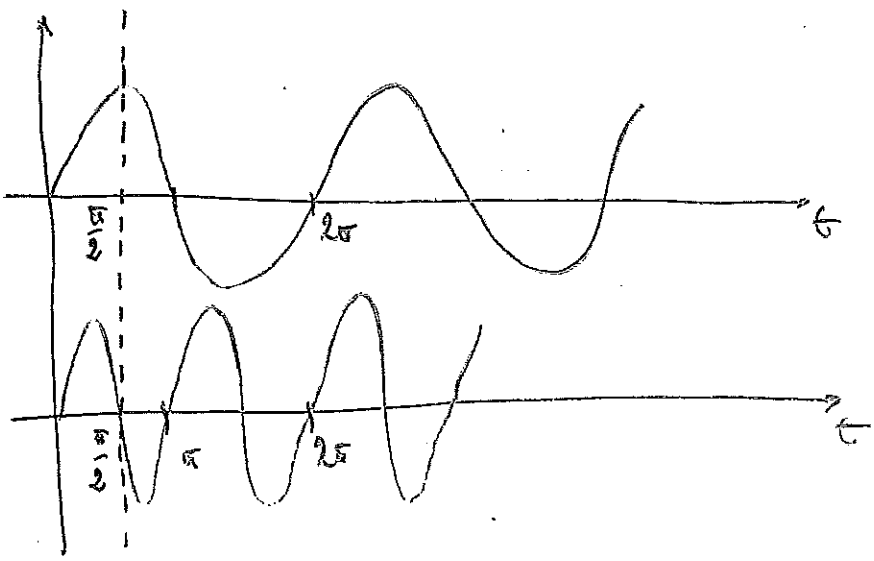
⇒ Delaying the signal by t_0 leaves the amplitude spectrum unaltered and introduces a phase change of $-\omega t_0$. The phase shift is <u>linear</u> with time.

Intuition

$$f(t) = \sin(\omega t) + \sin(2\omega t)$$

$$\rightarrow f(t-t_0) = \sin(\omega t - \omega t_0) + \sin(2\omega t - 2\omega t_0)$$

Let $t_0 = \frac{\pi}{2}$



In order to keep the relative phase unchanged (and thus give rise to the same but shifted signal), the second function must go through a larger individual phase change.

x. Frequency shift

$f(t) \Leftrightarrow F(\omega)$

$f(t)\, e^{j\omega_0 t} \Leftrightarrow F(\omega - \omega_0)$ → dual of time-shift

$$\mathcal{F}\{f(t) e^{j\omega_0 t}\} = \int_{-\infty}^{+\infty} f(t)\, e^{-j(\omega-\omega_0)t}\, dt = F(\omega - \omega_0) \quad \text{cvd}$$

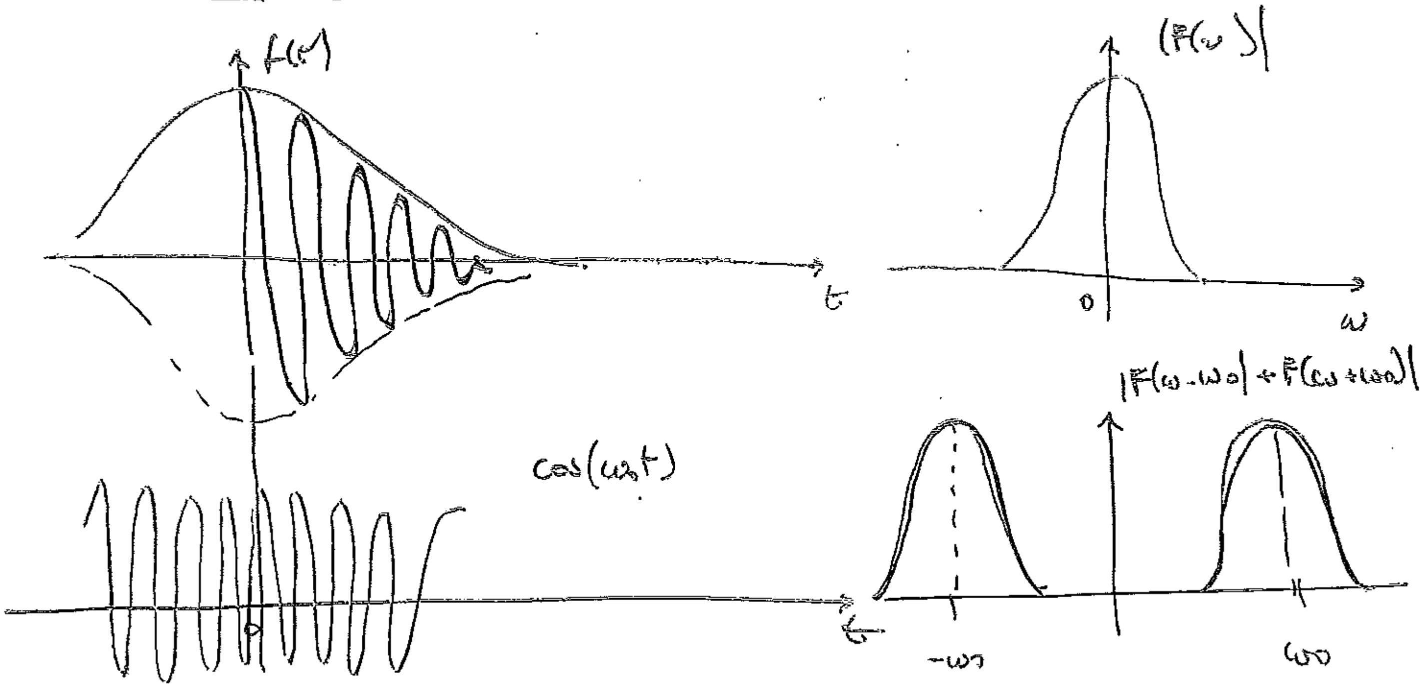
Consequence:

$f(t)\cos(\omega_0 t) \Leftrightarrow F(\omega - \omega_0) + F(\omega + \omega_0)$

proof: $\cos(\omega_0 t) = e^{j\omega_0 t} + e^{-j\omega_0 t}$

$\rightarrow f(t)\cos(\omega_0 t) = f(t) e^{j\omega_0 t} + f(t) e^{-j\omega_0 t}$

⇒ <u>modulation</u>



5. Convolution

$f_1(t) \leftrightarrow F_1(\omega)$ and $f_2(t) \leftrightarrow F_2(\omega)$ Then

$f_1(t) * f_2(t) \Leftrightarrow F_1(\omega) F_2(\omega)$

Proof:

$f_1, f_2 \in \mathbb{R}$

$$f_1 * f_2 = \int_{-\infty}^{+\infty} f_1(\tau) f_2(t-\tau) d\tau$$

$$\Rightarrow \mathcal{F}\{f_1 * f_2\} = \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} f_1(\tau) f_2(t-\tau) e^{-j\omega t} d\tau =$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_1(\tau) f_2(t-\tau) e^{-j\omega t} d\tau dt =$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_1(\tau) \underbrace{f_2(t-\tau) e^{-j\omega t} dt}_{F_2(\omega) e^{-j\omega\tau}} d\tau =$$

$$= \int_{-\infty}^{+\infty} f_1(\tau) F_2(\omega) e^{-j\omega\tau} d\tau =$$

$$= F_2(\omega) \int_{-\infty}^{+\infty} f_1(\tau) e^{-j\omega\tau} d\tau = F_1(\omega) F_2(\omega) \quad \text{END}$$

Conversely:

$$f_1(t) \cdot f_2(t) \leftrightarrow F_1(\omega) * F_2(\omega)$$

Differentiation and integration

$$\frac{df}{dt} \Leftrightarrow j\omega F(\omega)$$

$$\int_{-\infty}^{t} f(t)\,dt \Leftrightarrow \frac{F(\omega)}{j\omega} + \pi F(0)\,\delta(\omega)$$

See Table 4.1 and Table 4.2).

PARSEVAL THEOREM

$$\boxed{\int_{-\infty}^{+\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |F(\omega)|^2 d\omega}$$

Proof

$$E_f = \int_{-\infty}^{+\infty} |f(t)|^2 dt = \int_{-\infty}^{+\infty} f(t)\cdot f^*(t)\,dt =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t)\cdot\left\{\int_{-\infty}^{+\infty} F(\omega)\,e^{j\omega t}\,d\omega\right\}^* dt =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t)\int_{-\infty}^{+\infty} F^*(\omega)\,e^{-j\omega t}\,d\omega\,dt =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(t)\,F^*(\omega)\,e^{-j\omega t}\,d\omega\,dt =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F^*(\omega)\cdot f(t)\,e^{-j\omega t}\,dt\,d\omega =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} F^*(\omega)\int_{-\infty}^{+\infty} f(t)\,e^{-j\omega t}\,dt\cdot d\omega = \int_{-\infty}^{+\infty} F^*(\omega)F(\omega)\,d\omega =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} |F(\omega)|^2 d\omega$$

CVD

$|F(\omega)|^2 \rightarrow$ energy spectral density (per unit bandwidth)

<u>Energy spectral density</u> & <u>autocorrelation function</u>

$$\psi_f(t) = \int_{-\infty}^{+\infty} f(\tau) f(\tau - t)\, d\tau = f(t) * f(-t)$$

$$\rightarrow \psi_f(-t) = f(-t) * f(t) = \psi_f(t) \quad \rightarrow \text{even function}$$

$f \in \mathbb{R}$

$$\Rightarrow \mathcal{F}\{\psi_f(t)\} = \mathcal{F}\{f(t) * f(-t)\} = \mathcal{F}\{f(t)\} \cdot \mathcal{F}\{f(-t)\} = F(\omega) \cdot F(-\omega) = |F(\omega|$$

$$\Rightarrow \boxed{\mathcal{F}\{\psi_f(t)\} = |F(\omega)|^2}$$

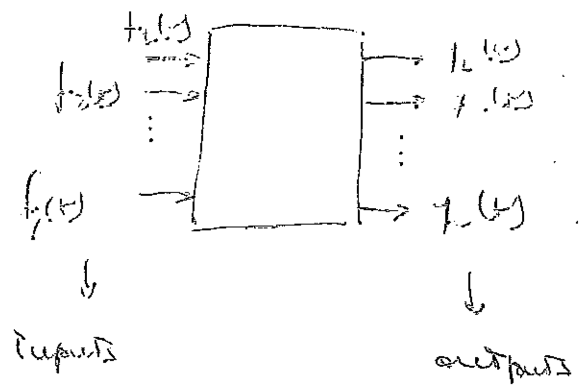
The F-transform of the autocorrelation function is the energy spectral density of the signal.

Intuition: The smoother the signal, the longer the autocorrelation function, the smaller the BW → the lower spectral density (PSD).

<u>Key points</u>

- Duality of the time and frequency domains
- Signal bandwidth is inversely proportional to signal duration
- $\begin{cases} \text{shifting} \rightarrow \text{\& factor} \\ \text{\& factor} \rightarrow \text{shifting} \end{cases}$
- $\begin{cases} \text{convolution in time} \rightarrow \text{product in frequency} \\ \text{product in time} \rightarrow \text{convolution in frequency.} \end{cases}$

System



Mathematical model of the systems: mathematical equations relating the outputs to the inputs.

MIMO : Multiple Input, Multiple Output

SISO : Single input, Single output

Classification of systems

1. Linear and non linear
2. Constant parameters / Time varying parameters
3. Instantaneous (memoryless) / dynamic (with memory)
4. Causal / Non causal
5. Lumped parameters / distributed parameters
6. Continuous time / discrete time
7. Analog / Digital

1. Linear / Non linear

1a. Additivity

$$\begin{aligned} c_1 &\rightarrow e_1 \\ c_2 &\rightarrow e_2 \end{aligned} \Rightarrow c_1 + c_2 \rightarrow e_1 + e_2$$

1b. homogeneity

$$\begin{aligned} c_1 &\rightarrow e_1 \\ a c_1 &\rightarrow a e_1 \end{aligned}$$

Then :

$$c_1 \rightarrow e_1 \quad , \quad c_2 \rightarrow e_2$$

$$k_1 c_1 + k_2 c_2 \rightarrow k_1 e_1 + k_2 e_2 \quad : \quad \underline{\text{superposition}}$$

Contribution to system output:

a) input $f(t)$, $t \geq 0$

b) initial condition of the system ($t=0$)

Linearity ⇒ The system response is the superposition of

a) The system response when the input $= \phi$

⇒ Zero input response

b) The system response when the initial state is zero

⇒ zero state response.

The system is in zero state when all the initial conditions are zero.

⇒ A system output is zero when the input is zero only if the system is in the zero state.

Summary:

$$\boxed{\text{Response} = \text{Zero input response} + \text{zero state response}}$$

↳ decomposition property

Note: all systems that can be described by linear differential equations of n^{th} order are linear systems!

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \dots + a_1\frac{dy}{dt} + a_0 y = b_m\frac{d^m f}{dt^m} + \dots + b_1\frac{df}{dt} + b_0 f$$

Example:

$$\frac{dy}{dt} + 3y(t) = \frac{df(t)}{dt}$$

$f_1 \to y_1$ ⇒ $\frac{dy_1}{dt} + 3y_1(t) = \frac{df_1(t)}{dt}$ (1)

$f_2 \to y_2$ ⇒ $\frac{dy_2}{dt} + 3y_2(t) = \frac{df_2(t)}{dt}$ (2)

Multiplying (1) by k_1 and (2) by k_2

$$k_1 \frac{dy_1}{dt} + k_2 \frac{dy_2}{dt} + 3 y_1 k_1 + 3 k_2 y_2 = k_1 \frac{df_1}{dt} + k_2 \frac{df_2}{dt} \quad (3)$$

Let :

$$f(t) = k_1 f_1 + k_2 f_2$$

$$y(t) = k_1 y_1(t) + k_2 y_2(t)$$

$$(3) \rightarrow \frac{dy}{dt} + 3y = \frac{df}{dt} \qquad \text{CVD}$$

Non linear systems :

- A mathematical model that holds for any input and initial condition might not be found.
- The superposition principle does not hold in general

⇒ difficult to study

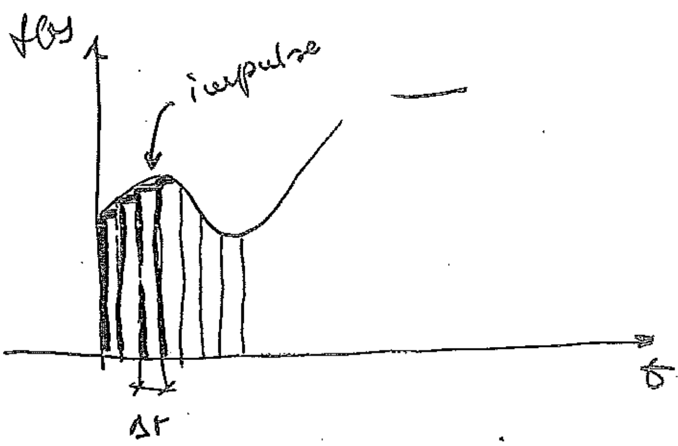
Note :

- All biological systems are Non linear BUT
- linearity holds for small variations of the input parameters around a given value and in given conditions.

⇒ NON linear systems are studied in particular conditions (small input values, changes around stable conditions) OR approximating them by linear systems.

Consequences of Linearity

- An input signal can be replaced by the combination of simpler signals (its "components") for which it is easier to calculate the system output. These can be:
  - The harmonics (Fourier decomposition of the signal)
  - set of impulse functions or step functions



$f(t) = \sum$ impulse functions spaced Δt seconds apart

⇒ if the system impulse response is known, the response to the complex stimulus $f(t)$ can be easily evaluated.

2. Time invariant / Time varying (LTIS)

Parameters do not change → Time invariant or constant parameters

$x(t) \to y(t)$

$x(t+\tau) \to y(t+\tau)$

$x \to \boxed{} \to y$

Ex: a linear differential equation whose coefficients do not depend on time corresponds to a LTIS.

Ex. 2, $y(t) = \sin t \cdot f(t-2)$

$y(t+\tau) = \sin(t+\tau) \cdot f(t-2+\tau)$

$\Downarrow$

$\neq \sin(t)$ in general ⇒ Time-varying system.

$(\tau \neq 2\pi)$

3. Instantaneous/Dynamic systems

<u>Instantaneous</u> : The output at time t depends only on the input at time t

$\Rightarrow$ memoryless

Dynamic : otherwise

In dynamic systems, the past and/or future values of the input contribute to the output at the current instant

Classification of dynamic systems:

- If $y(\bar{t})$ depends on $x(t)$ for $\bar{t}-T \le t \le \bar{t}$ it is said to be <u>finite memory</u>
- If $y(\bar{t})$ depends on $x(t)$ for $-\infty \le t \le \bar{t}$ $\Rightarrow$ infinite memory

5. Lumped / Distributed parameters.

The variables depends on both space and time ⇔ systems that are described by partial differential equations.

Ex: $$\frac{dy}{dx} + \frac{dy}{dt} = f(x,y) + d$$

We will not consider this case in this course.

6. Continuous / Discrete time

$x(t)$, $y(t)$ continuous time ⇒ The system is continuous time

$x[k]$, $y[k]$ discrete " ⇒ " " " discrete "

7. Analog / Digital

$x(t)$, $y(t)$ are analog ⇒ The system is analog

$x(t)$, $y(t)$ " digital ⇒ " " " digital

Note: invertibility

A system is invertible if we can recover $x(t)$ from $y(t)$.

Ex: differentiator (causal: $y(t) = 0 \ \forall \ t \le 0$)

$$y(t) = \frac{dx(t)}{dt}$$

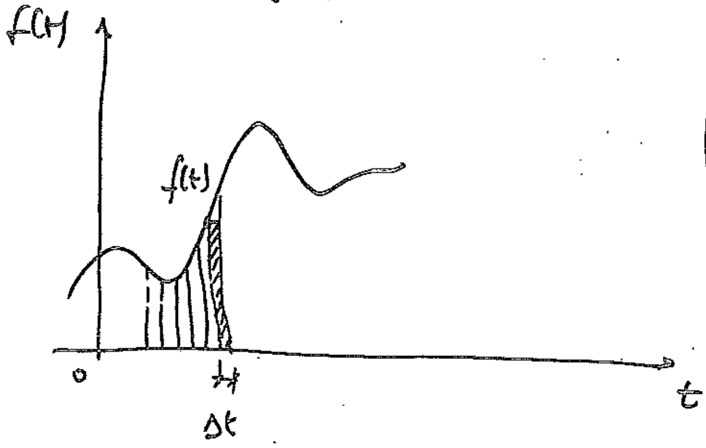
$$x(t) = 3t + 4 \rightarrow y(t) = 3$$

⇒ We can correctly recover $x(t)$ iif we know that $x(0) = 4$.

(See summary pp. 94, Lathi)

Unit impulse response

The $\delta(t)$ function is used to determine the response of a linear system to an arbitrary function $f(t)$.



$\Delta t \to 0 \Rightarrow$ The rectangular pulse $\to \delta(t)$

$\Rightarrow$ The response of the system is the sum of the responses to various impulse components.

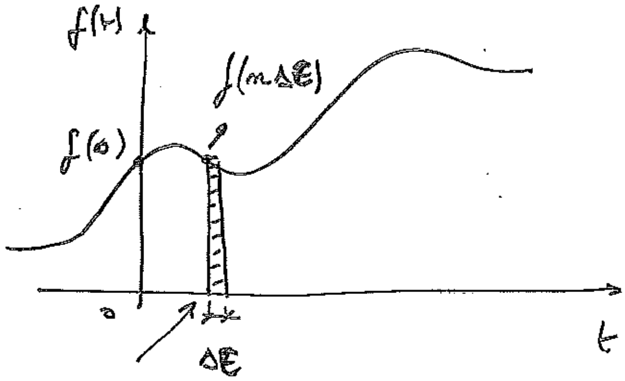
, If the impulse response is known, we can derive the system response to any arbitrary input $f(t)$.

$\boxed{h(t) \text{ impulse response}}$: response of the system to the input impulse $\delta(t)$ applied at $t=0$ with all the initial conditions zero at $t=0^-$.

NOTE: Even though the impulse is vanishing, it changes instantaneously the conditions of the system such that its effect propagates in time beyond $t=0$.
The evolution of the system will be determined by such new initial conditions.

$$\delta(t) \longrightarrow \boxed{h(t)} \longrightarrow h(t) \qquad \text{and} \qquad \delta(t-T) \longrightarrow \boxed{LTIC} \longrightarrow h(t-T)$$

System response to $f(t)$: zero-state response



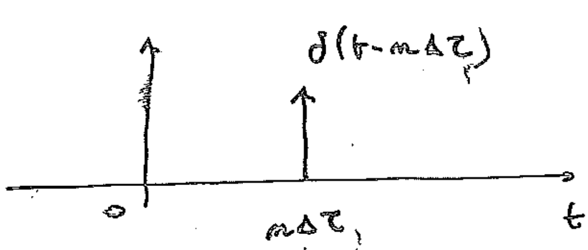
$t = m \cdot \Delta t$

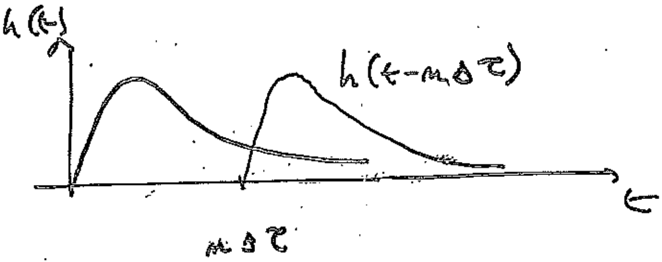
impulse: $f(m\Delta\tau)\,\delta(t - m\Delta\tau)$

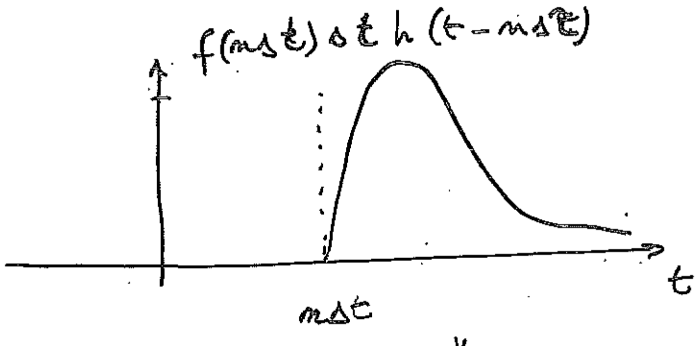
$$\delta(t) \to h(t)$$

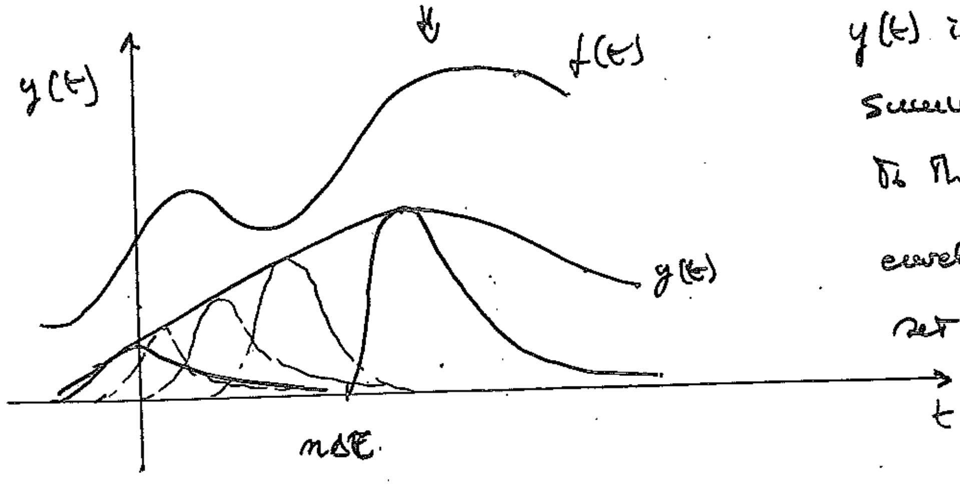
$$\delta(t - m\Delta\tau) \to h(t - m\Delta\tau)$$

$$f(m\Delta\tau)\,\delta(t - m\Delta\tau)\Delta\tau \to f(m\Delta\tau)\,h(t - m\Delta\tau)\,\Delta\tau$$









$y(t)$ is obtained by summing the responses to the impulses as to the envelop of the corresponding set of curves.

Formally:

$$f(t) = \lim_{\Delta t \to \emptyset} \sum_{n=-\infty}^{+\infty} f(n\Delta\tau)\,\delta(t - n\Delta\tau)$$

$$\Rightarrow \; y(t) = \lim_{\Delta t \to \emptyset} \sum_{n=-\infty}^{+\infty} f(n\Delta\tau)\, h(t - n\Delta\tau)\,\Delta\tau$$

$\Rightarrow \Delta\tau \to 0$

$$f(t) = \int_{-\infty}^{+\infty} f(\tau)\,\delta(t-\tau)\,d\tau = f(t)$$ (sampling property of the δ function)

$$\boxed{y(t) = \int_{-\infty}^{+\infty} f(\tau)\, h(t-\tau)\, d\tau}$$ <u>**LTIS**</u>

$\Rightarrow$ The system response to an arbitrary input $f(t)$ is obtained by the <u>convolution</u> of the signal with the impulse response of the system.

$$y(t) = f(t) * h(t)$$

$$f(t) \longrightarrow \boxed{h(t)} \longrightarrow g(t)$$

Fourier domain:

$$\boxed{Y(\omega) = F(\omega)\, H(\omega)}$$

Time-variant systems: $h(t - n\Delta\tau) = h(t, \tau)$

$$\Rightarrow \; y(t) = \int_{-\infty}^{+\infty} f(\tau)\, h(t,\tau)\, d\tau$$

Properties of the convolution integral.

$$f_1(t) * f_2(t) = \int_{-\infty}^{+\infty} f_1(\tau) f_2(t-\tau)\, d\tau$$

1. Commutativity

$$f_1 * f_2 = f_2 * f_1$$

Proof: $$f_2 * f_1 = \int_{-\infty}^{+\infty} f_2(\tau) f_1(t-\tau)\, d\tau \qquad (1)$$

$x = t - \tau$

$dx = -d\tau$ (→ double sign change)

$$(1) = \int_{-\infty}^{+\infty} f_2(t-x) f_1(x)\, dx = f_1 * f_2 \quad \text{cqd}$$

2. Distributivity

$$f_1 * (f_2 + f_3) = f_1 * f_2 + f_1 * f_3$$

3. Associativity

$$f_1 * (f_2 * f_3) = (f_1 * f_2) * f_3$$

4. Shifting

If: $f_1(t) * f_2(t) = c(t)$

Then $f_1(t) * f_2(t-T) = c(t-T)$ and

$f_1(t-T_1) * f_2(t-T_2) = c(t-T_1-T_2)$

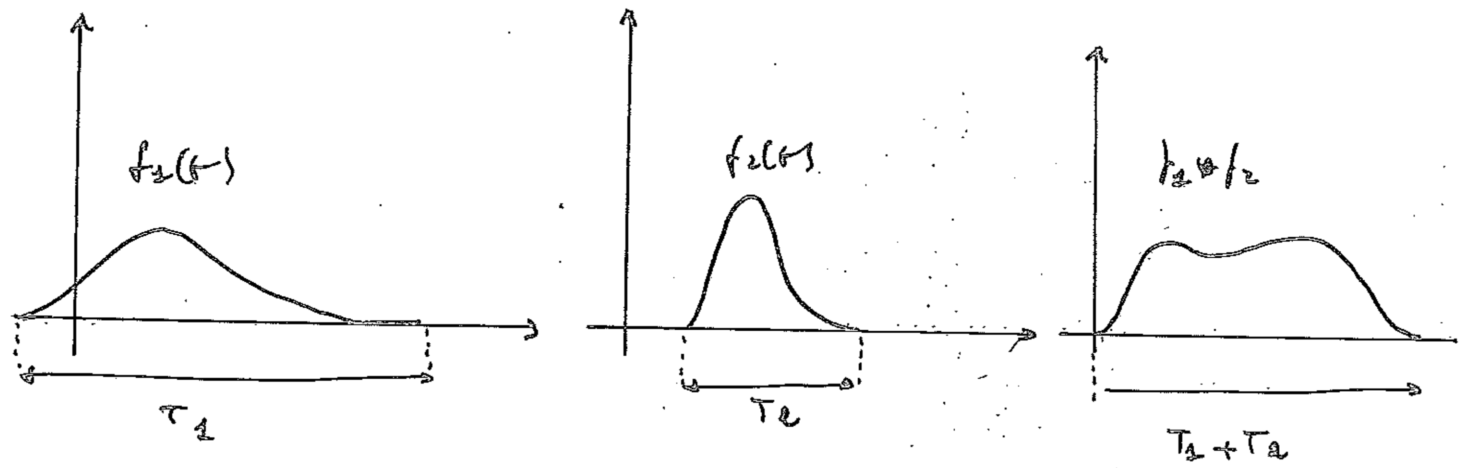
Proof :

$$f_1(t) * f_2(t-T) = \int_{-\infty}^{+\infty} f_1(\tau)\, f_2(\underbrace{t-T}_{\text{shift}}-\tau)\, d\tau = c(t-T)$$

i. Convolution with an impulse

$$f(t) * \delta(t) = f(t) \qquad \text{(sampling property of } \delta(t)\text{)}$$

i. Width



Zero-state response and causality

Causal signal : $f(t) = 0 \quad t < 0$ ($t=0$ is the time when the signal starts)

Causal system: The response cannot begin before the input signal begins.

$\Rightarrow$ The causal system's response to a unit impulse $\delta(t)$ (in $t=0$) cannot begin before $t=0$ $\Rightarrow$ $h(t)$ is a causal signal

$\Rightarrow$ $\boxed{\text{LTIS, causal} \Rightarrow h(t) \text{ causal}}$

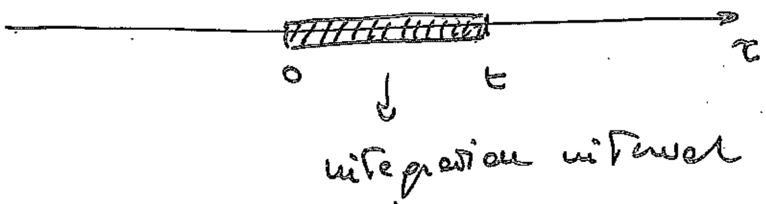
For causal systems:

$$f(t) = 0 \quad \forall t \le 0$$

$$h(t) = 0 \quad \forall t \le 0 \quad \text{and}$$

$$h(t-\tau) = 0 \quad \forall \quad t-\tau \le 0 \rightarrow \forall \tau \ge t$$

$$\Rightarrow \int_{-\infty}^{+\infty} f(\tau)\, h(t-\tau)\, d\tau = \int_{0}^{t} f(\tau)\, h(t-\tau)\, d\tau \qquad t \ge 0$$



(see the convolution Table (pg. 125))

Sistem response to external conditions: Zero input response

Consider a linear differential system:

(1) $$\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1 y + a_0 = b_m \frac{d^m f}{dx^m} + b_{m-1}\frac{d^{m-1}}{dx^{m-1}} + \dots + b_0 f(t)$$

$$\rightarrow \sum_{k=0}^{n} \frac{d^k y}{dx^k} a_k = \sum_{p=0}^{m} b_p \frac{d^p f}{dx^p}$$

$a_n = 1$

$a_i, b_i \rightarrow$ constant

$\frac{d}{dt} \rightarrow D$

1) $\Rightarrow (D^n + a_{n-1} D^{n-1} + \dots + a_1 D + a_0)\, y(t) = (b_m D^m + b_{m-1} D^{m-1} + \dots + b_1 D + b_0) f(t)$

et : $\begin{cases} D^n + a_n^{-1} D^{n-1} + \dots + a_1 D + a_0 = Q(D) \\ b_m D^m + b_{m-1} D^{m-1} + \dots + b_1 D + b_0 = P(D) \end{cases}$

1) can be rewritten as:

$$Q(D)\, y(t) = P(D) f(t)$$

Zero input response: $y(t)$ when $f(t) = 0 \Rightarrow$ let it be $y_0(t)$

Then:

$$Q(D)\, y_0(t) = 0$$

$$(D^n + a_{n-1} D^{n-1} + \dots + a_1 D + a_0)\, y_0(t) = 0$$

Intuition for the solution: eq. (2) can only be verified when $y(t)$ and all its derivatives up to order n are in the same form.

$\Rightarrow$ exponential function (the only one having this property)

$\Rightarrow \quad y_0(t) = Ce^{\lambda t}$

Which value for λ?

If $y_0(t) = ce^{\lambda t}$ is a solution, since

$$D y_0(t) = c\lambda e^{\lambda t}$$

$$D^2 y_0(t) = c\lambda^2 e^{\lambda t}$$

$$\vdots$$

$$D^n y_0(t) = c\lambda^n e^{\lambda t}$$

$\Rightarrow$ replacing in (2)

$$c(\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0)e^{\lambda} = 0$$

$$\Rightarrow \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = 0$$

$$\Rightarrow \boxed{Q(\lambda) = 0}$$

n solutions: $\lambda_1, \lambda_2, \dots, \lambda_n$

$\Rightarrow$ n possible solutions for the system:

$$C_1 e^{\lambda_1 t}, \; C_2 e^{\lambda_2 t}, \dots, C_n e^{\lambda_n t}$$

and thus, due to linearity

$$\boxed{y_0(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + \dots + C_n e^{\lambda_n t}}$$

$C_1, C_2, \dots, C_n$: arbitrary constants determined by n constraints (the auxiliary conditions) on the solution.

$Q(\lambda) \to$ CHARACTERISTIC POLYNOMIAL

$Q(\lambda) = 0$ " equation

λ_i, $i = 1, \dots, n$ " roots of the system, and also

" values and eigenvalues

$e^{\lambda_i t} \to$ " MODES

There is a characteristic mode $\forall$ characteristic root.

The zero input response of the system is a linear combination of the characteristic modes.

Repeated roots

$$(D - \lambda_0)^2 y_0(t) = 0$$

$n = 1$

$e = 1$

characteristic modes: $e^{\lambda_0 t}$, $t e^{\lambda_0 t}$

Proof

$$(D - \lambda_0)^2 y_0 = \frac{d^2 y_0}{dt^2} - 2\lambda_0 \frac{dy}{dt} + \lambda_0^2 y_0$$

1) $y_0(t) = e^{\lambda_0 t}$

$D y_0(t) = \lambda_0 e^{\lambda_0 t}$

$D^2 y_0(t) = \lambda_0^2 e^{\lambda_0 t}$

$$\left(\lambda_0^2 e^{\lambda_0 t} - 2\lambda_0^2 e^{\lambda_0 t} + \lambda_0^2 e^{\lambda_0 t}\right) = 0 \Rightarrow 0 = 0 \quad \text{CVD}$$

2) $te^{\lambda_0 t} = y_0$

$$Dy_0 = t\lambda_0 e^{\lambda_0 t} + e^{\lambda_0 t}$$

$$D^2 y_0 = \lambda_0 e^{\lambda_0 t} + t\lambda_0^2 e^{\lambda_0 t} + \lambda_0 e^{\lambda_0 t} = 2\lambda_0 e^{\lambda_0 t} + t\lambda_0^2 e^{\lambda_0 t}$$

$$D y_0^2 - 2Dy_0 \lambda_0 + \lambda_0^2 y_0 =$$

$$= 2\lambda_0 e^{\lambda_0 t} + t\lambda_0^2 e^{\lambda_0 t} - 2\lambda_0 \left(t\lambda_0 e^{\lambda_0 t} + e^{\lambda_0 t}\right) + \lambda_0^2 t e^{\lambda_0 t} =$$

$$= 2\lambda_0 e^{\lambda_0 t} + t\lambda_0^2 e^{\lambda_0 t} - 2\lambda_0^2 t e^{\lambda_0 t} - 2\lambda_0 e^{\lambda_0 t} + \lambda_0^2 t e^{\lambda_0 t} = 0$$

CVD

Generalization

$$(D - \lambda_0)^k y_0(t) = 0$$

$$\Rightarrow \boxed{y_0(t) = \left(c_1 + c_2 t + \dots + c_k t^{k-1}\right) e^{\lambda t}}$$

Equivalently, if the characteristic polynomial is in the form:

$$\varphi(\lambda) = (\lambda - \lambda_1)^k (\lambda - \lambda_{k+1}) \dots (\lambda - \lambda_n)$$

The characteristic function is:

$$y_0(t) = \left(c_1 + c_2 t + \dots + c_k t^{k-1}\right) + c_{k+1} e^{\lambda_{k+1} t} + \dots + c_n e^{\lambda_n t}$$

since the root $\lambda = \lambda_1$ has multiplicity $= k$.

<u>Complex roots</u>

For <u>real systems</u> with <u>real coefficients</u> in the characteristic zero response of a real system with c.c. characteristic roots:

$$y_0(t) = c_1 e^{(\alpha+j\beta)t} + c_2 e^{(\alpha-j\beta)t}$$

In order for $y_0(t) \in \mathbb{R}$:

$$\begin{cases} c_1 = \frac{c}{2} e^{j\theta} \\ c_2 = \frac{c}{2} e^{-j\theta} \end{cases}$$

in this case:

$$y_0(t) = \frac{c}{2} e^{j\theta} e^{(\alpha+j\beta)t} + \frac{c}{2} e^{-j\theta} e^{(\alpha-j\beta)t} =$$

$$= \frac{c}{2} e^{\alpha t} \left[e^{j(\theta+\beta t)} + e^{-j(\theta+\beta t)} \right] =$$

$$= \frac{c}{2} e^{\alpha t} \cdot 2\cos(\theta+\beta t) = c e^{\alpha t} \cos(\theta+\beta t) \quad \in \mathbb{R} \quad \text{OK}$$

Note:

- The input is assumed to start at $t=0$ $\Rightarrow$ $t=0$ is the reference (ti...
- Initial conditions:

$y(t)$: total response

$y_0(t)$: response with $f(t)=0$

$$\Rightarrow \quad y(0^-) = y_0(0^-)$$

$$\left.\frac{d^n y}{dt^n}\right|_{t=0^-} = \left.\frac{d^n y_0}{dt^n}\right|_{t=0^-} \qquad \forall\, n = 0, 1, \ldots$$

But

$$\left.\frac{d^n y}{dt^n}\right|_{t=0^+} \neq \left.\frac{d^n y_0}{dt^n}\right|_{t=0^+}$$

and

$$\left.\frac{d^n y_0}{dt^n}\right| = \left.\frac{d^n y_0}{dt^n}\right|_{+}$$ as y_0 does not depend on $f(t)$.