Reactive Systems

EECS 20

Lecture 7 (January 31, 2001)

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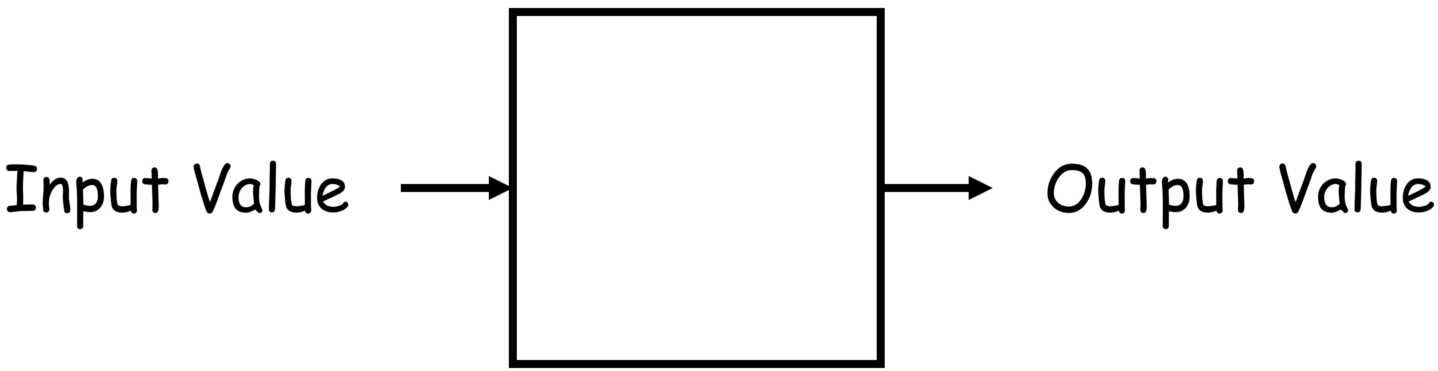
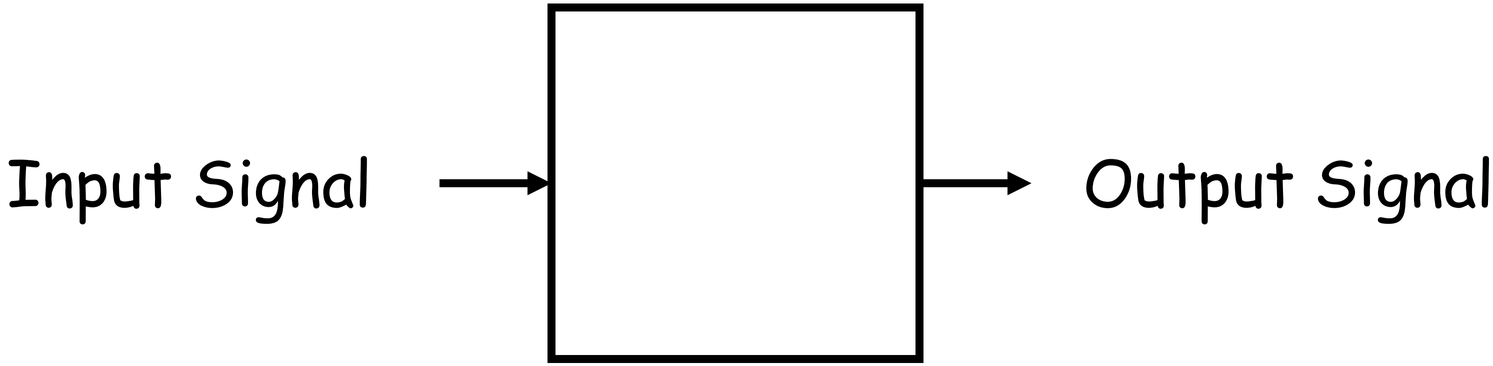
Revised by Matteo Zavatteri and Tiziano Villa, Fall 2020

Transducive or Combinational System



$$transduciveSystem : Values \rightarrow Values$$

Reactive or Sequential System



reactiveSystem : [Time $\rightarrow$ Values] $\rightarrow$ [Time $\rightarrow$ Values]

Reactive Systems

1. Memory-free systems
2. Delays
3. Causality
4. Finite-memory systems
5. Infinite-memory systems

A reactive system

$F : [\, Time \to Values \,] \to [\, Time \to Values \,]$

is memory-free

iff

there exists a transducive system

$f : Values \to Values$

such that

$\forall\, x \in [\, Time \to Values \,],\ \forall\, y \in Time,$

$(\, F\,(x)\,)\,(y) = f\,(\, x\,(y)\,)\,.$

Memory-free Reactive Systems

Normalize
Trunc
Quantize
Negate

The identity system Id

Every constant system Const

Every composition of memory-free systems

Every block-diagram composition of memory-free systems

Every combinational circuit

The Identity System

$$Id : [\, Time \to Values \,] \to [\, Time \to Values \,]$$

such that $\forall\, x \in [\, Time \to Values \,],\ Id(x) = x$.

The Constant Systems

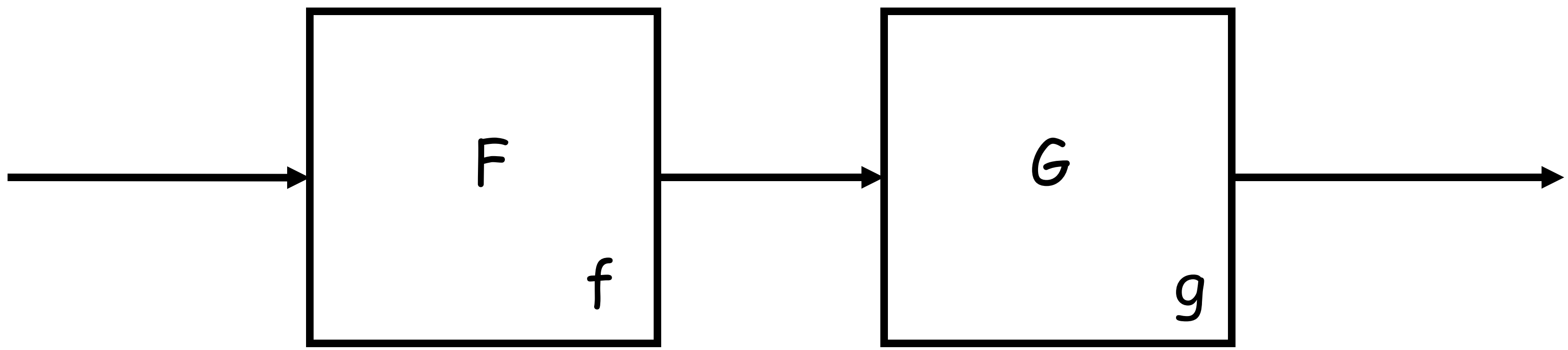
For every constant $const \in Values$,

$$Const : [\, Time \to Values \,] \to [\, Time \to Values \,]$$

such that $\forall\, x \in [\, Time \to Values \,],\ \forall\, y \in Time$,

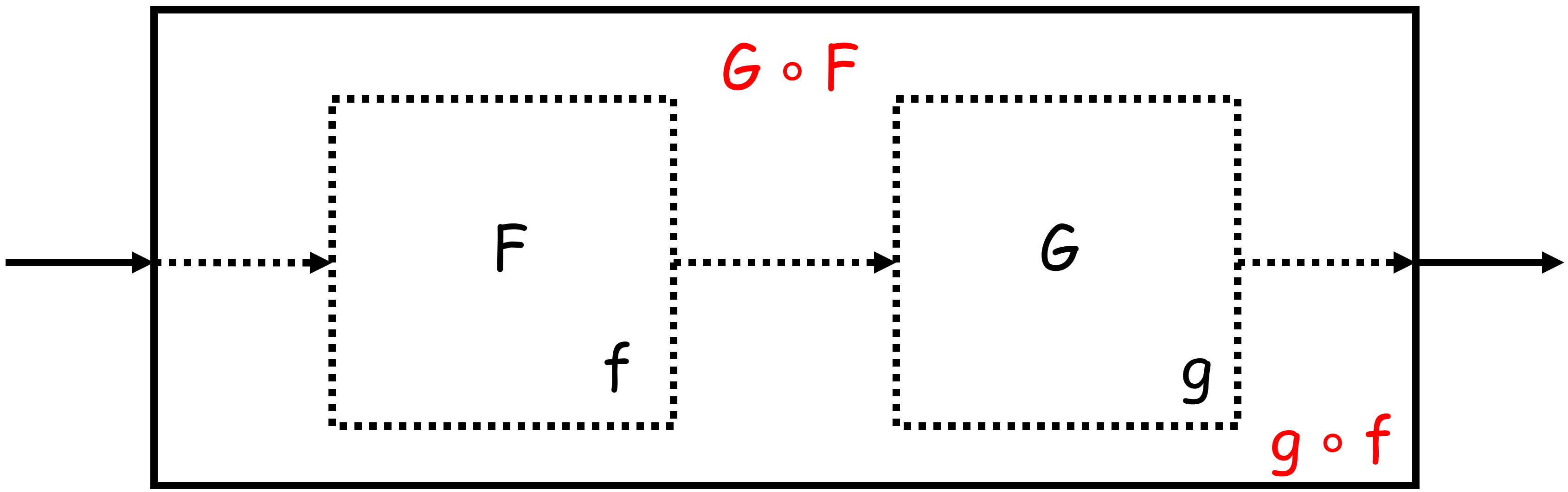
$$(\, Const(x)\,)(y) = const \,.$$

Composition of Memory-free Systems



If F and G are memory-free,

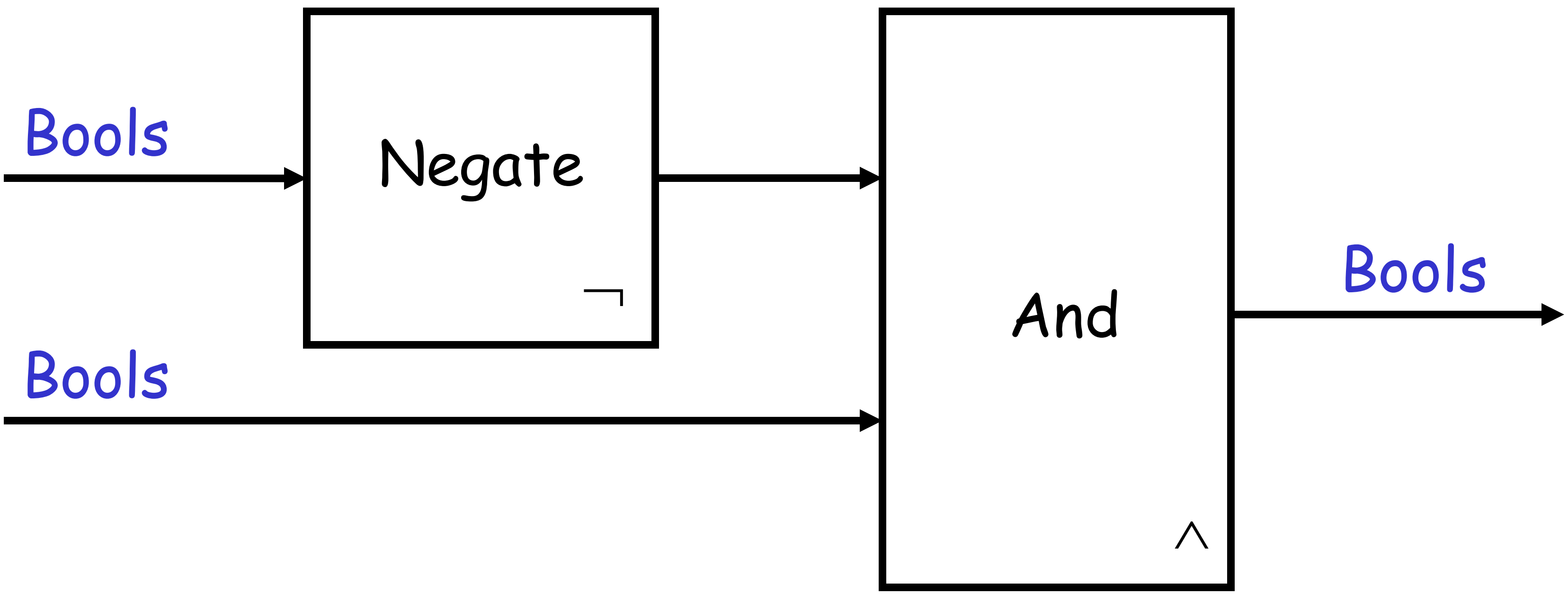
Composition of Memory-free Systems



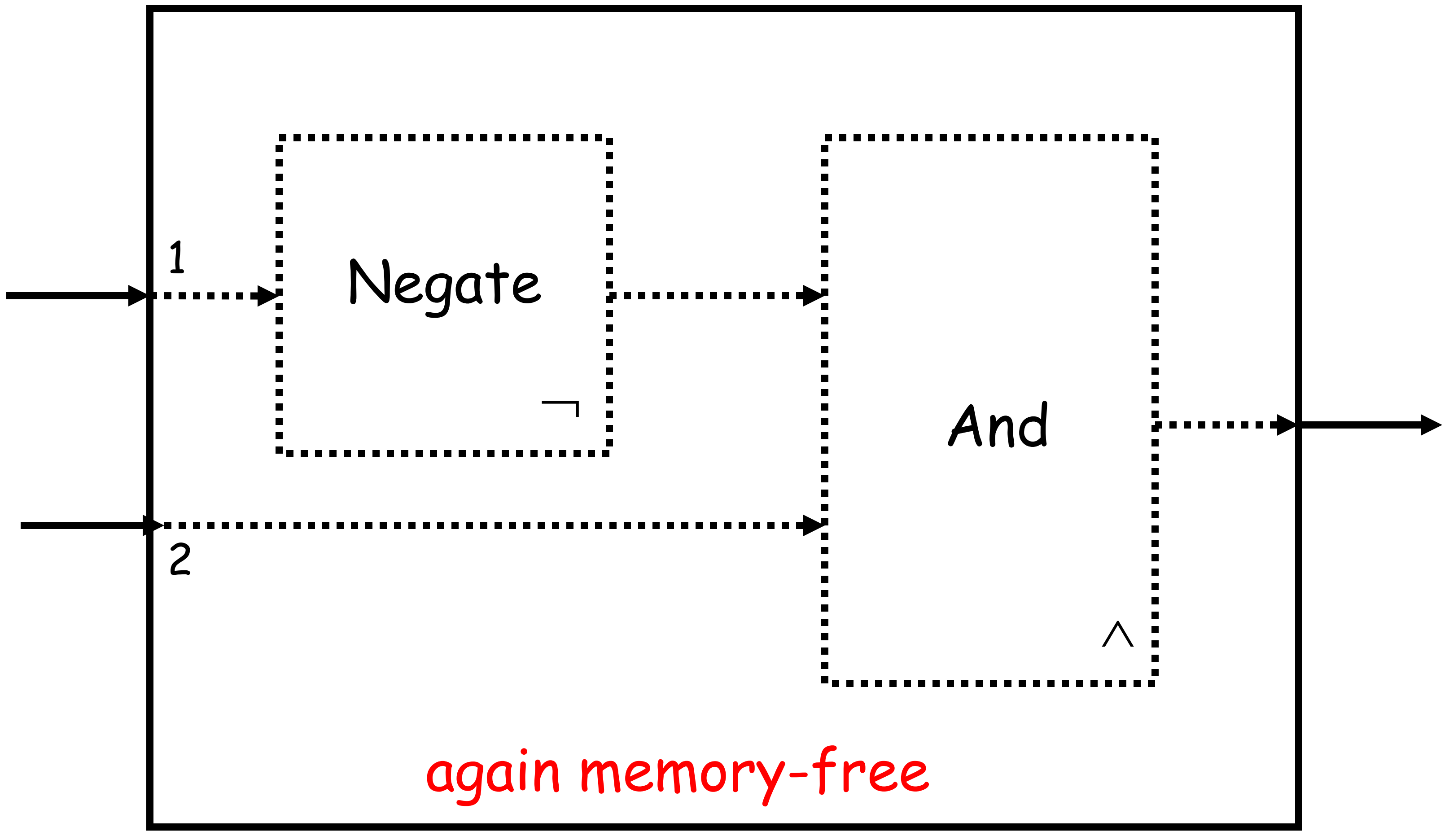
If F and G are memory-free,

then $G \circ F$ is memory-free.

Block-Diagram Composition of Memory-free Systems



Block-Diagram Composition of Memory-free Systems



The Delay System

Delay: [Time $\to$ Values] $\to$ [Time $\to$ Values]

such that $\forall$ x $\in$ [Time $\to$ Values] , $\forall$ y $\in$ Time,

$$(\,\mathrm{Delay}(x)\,)(y) = \begin{cases} ? & \text{if } y < 1 \\ x(y-1) & \text{if } y \geq 1 \end{cases}$$

The Delay System

$$\mathrm{Delay}_c : [\,\mathrm{Time} \to \mathrm{Values}\,] \to [\,\mathrm{Time} \to \mathrm{Values}\,]$$

such that $\forall\, x \in [\,\mathrm{Time} \to \mathrm{Values}\,],\ \forall\, y \in \mathrm{Time},$

$$(\,\mathrm{Delay}(x)\,)(y) = \begin{cases} c & \text{if } y < 1 \\ x(y-1) & \text{if } y \geq 1 \end{cases}$$

Continuous-Time Delay

$\text{Delay}_0\ (\sin) : \ \text{Reals}_+ \ \to \ \text{Reals}$

such that $\forall\, y \in [0,1)\,,\ \ \text{Delay}\,(\sin)\,(y) \ = \ 0$

and $\forall\, y \in [1,\infty)\,,\ \ \text{Delay}\,(\sin)\,(y) \ = \ \sin\,(y-1)\,.$

$\text{Delay}_{11}\ (\text{id}) : \ \text{Reals}_+ \ \to \ \text{Reals}$

such that $\forall\, y \in [0,1)\,,\ \ \text{Delay}\,(\text{id})\,(y) \ = \ 11$

and $\forall\, y \in [1,\infty)\,,\ \ \text{Delay}\,(\text{id})\,(y) \ = \ y - 1\,.$

Discrete-Time Delay

$\mathrm{Delay}_{13}\,(\mathrm{id}) : \ \mathrm{Nats}_0 \ \to \ \mathrm{Nats}_0$

such that $\ \mathrm{Delay}\,(\mathrm{id})\,(0) = 13$

and $\ \forall\, y \in \mathrm{Nats},\ \ \mathrm{Delay}\,(\mathrm{id})\,(y) \ = \ y - 1\,.$

$\mathrm{Delay}_{\mathrm{true}}\,(\mathrm{alt}) : \ \mathrm{Nats}_0 \ \to \ \mathrm{Bools}$

such that $\ \mathrm{Delay}\,(\mathrm{alt}) \ =$

$$\{\,(0,\ \mathrm{true}),\ (1,\ \mathrm{false}),\ (2,\ \mathrm{true}),\ (3,\ \mathrm{false})\ ,\ \ldots\,\}\,.$$

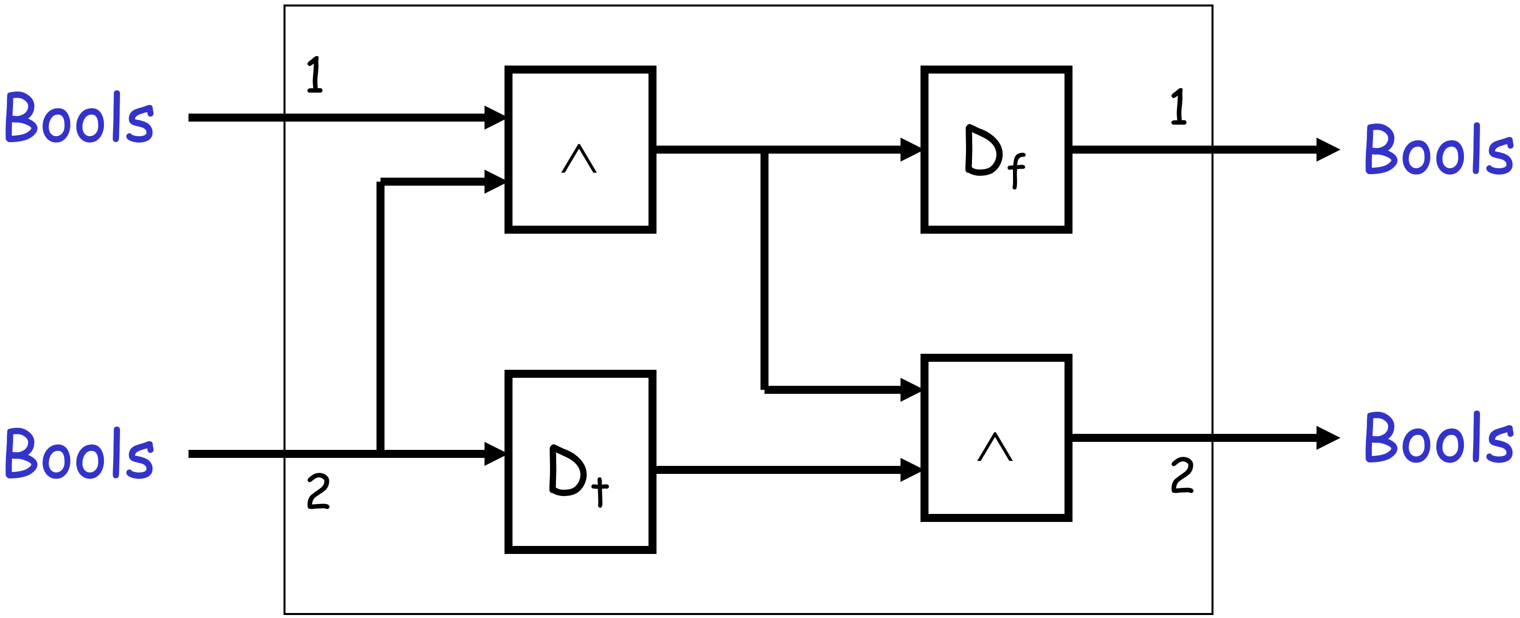
Discrete-time delay over finite set of values :

finite memory

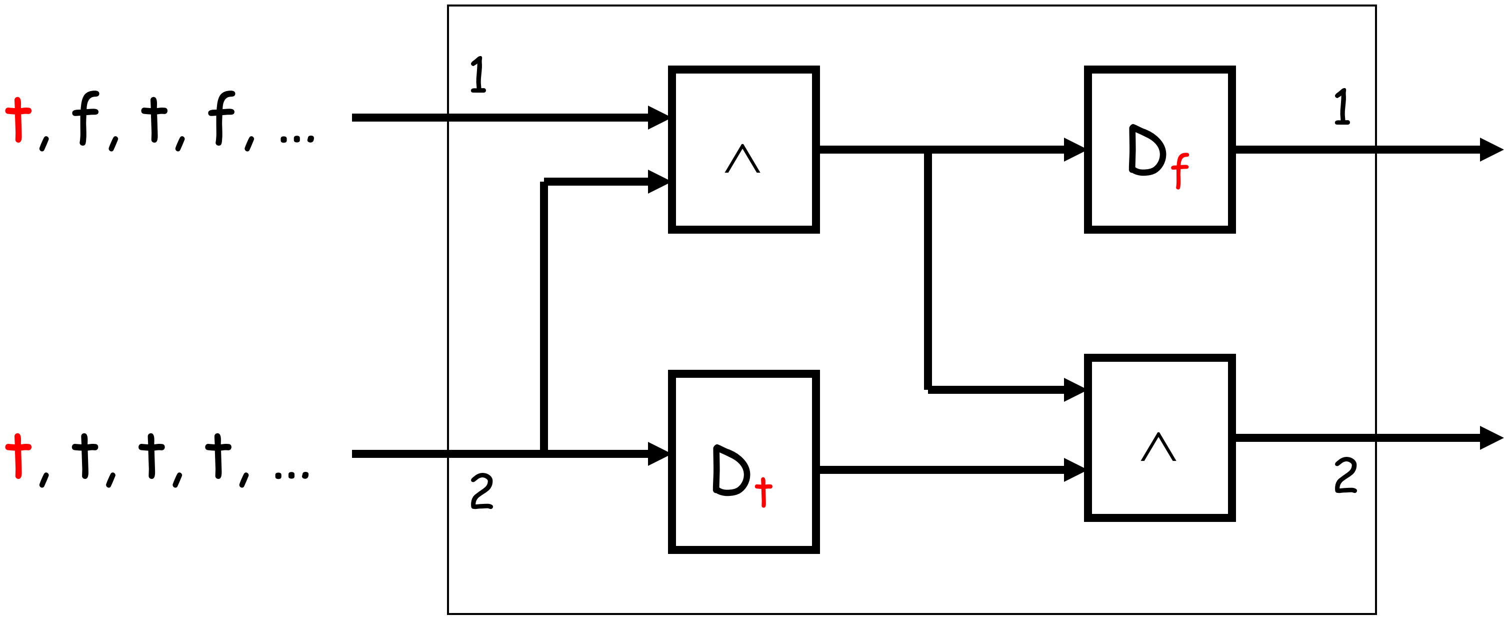
Continuous-time delay, or infinite set of values:

infinite memory

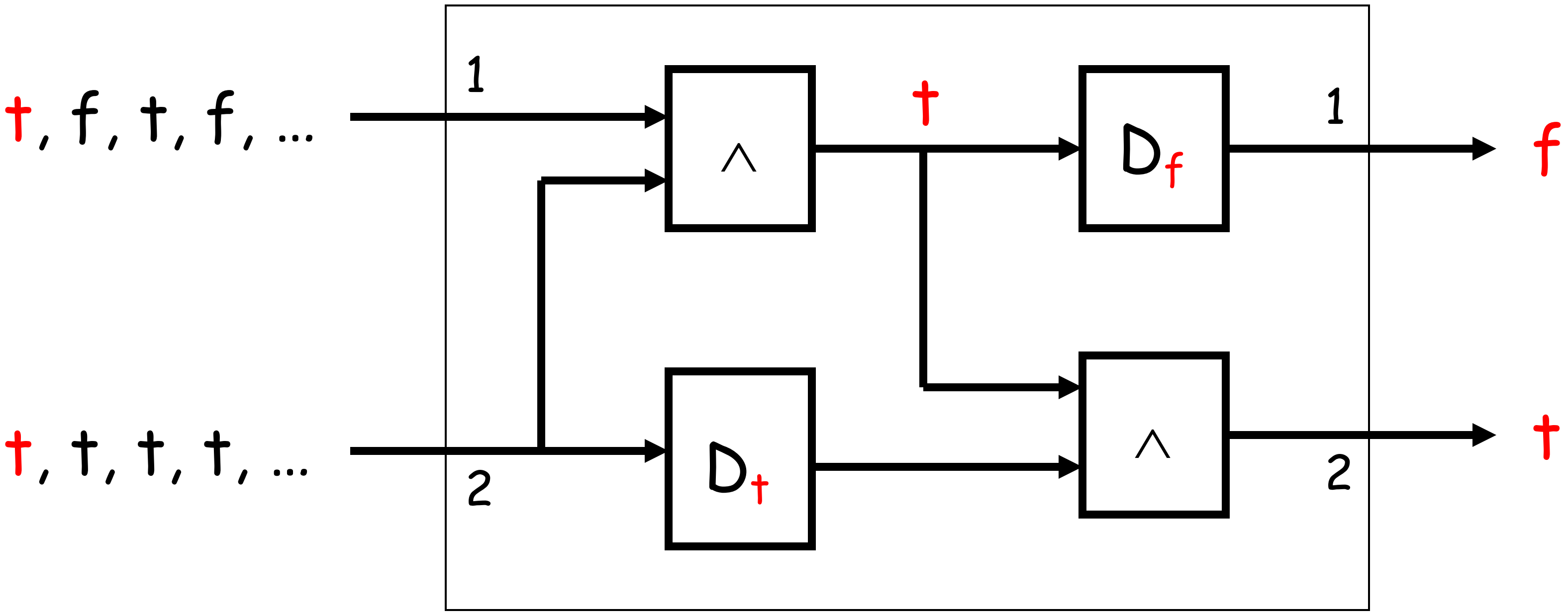
Block Diagram with Delay



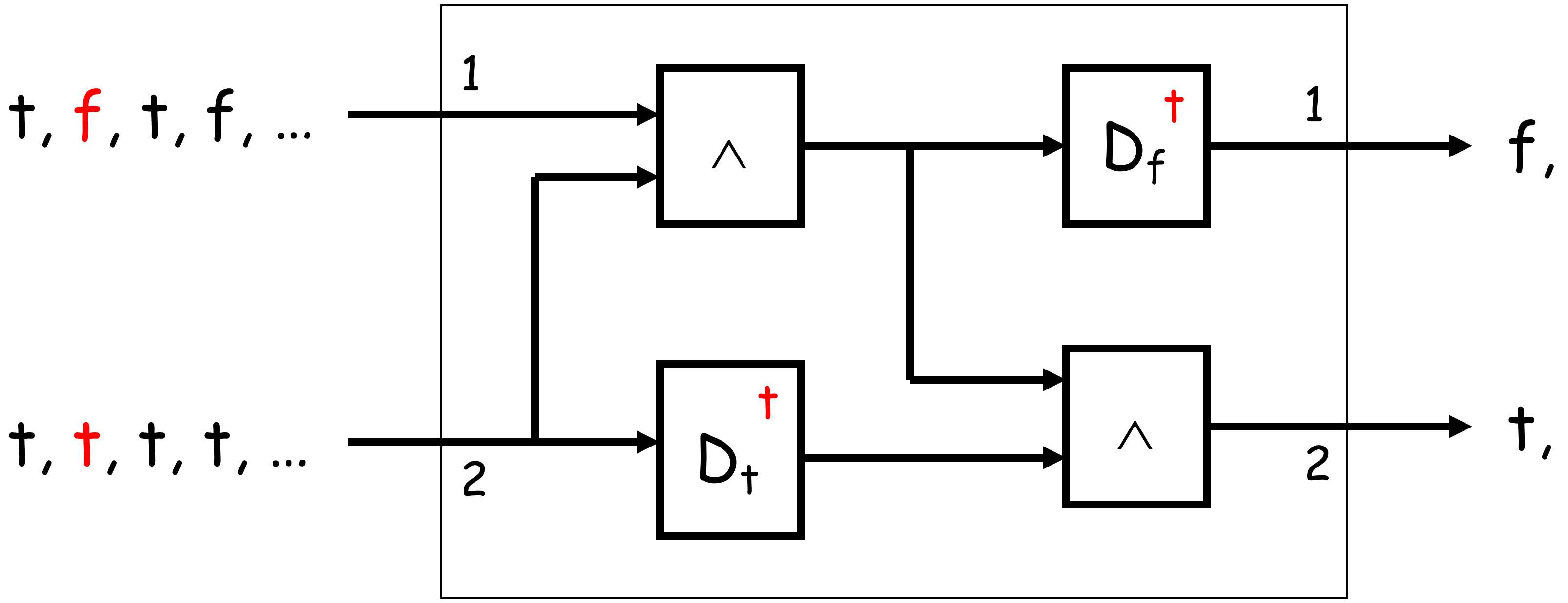
Block Diagram with Delay



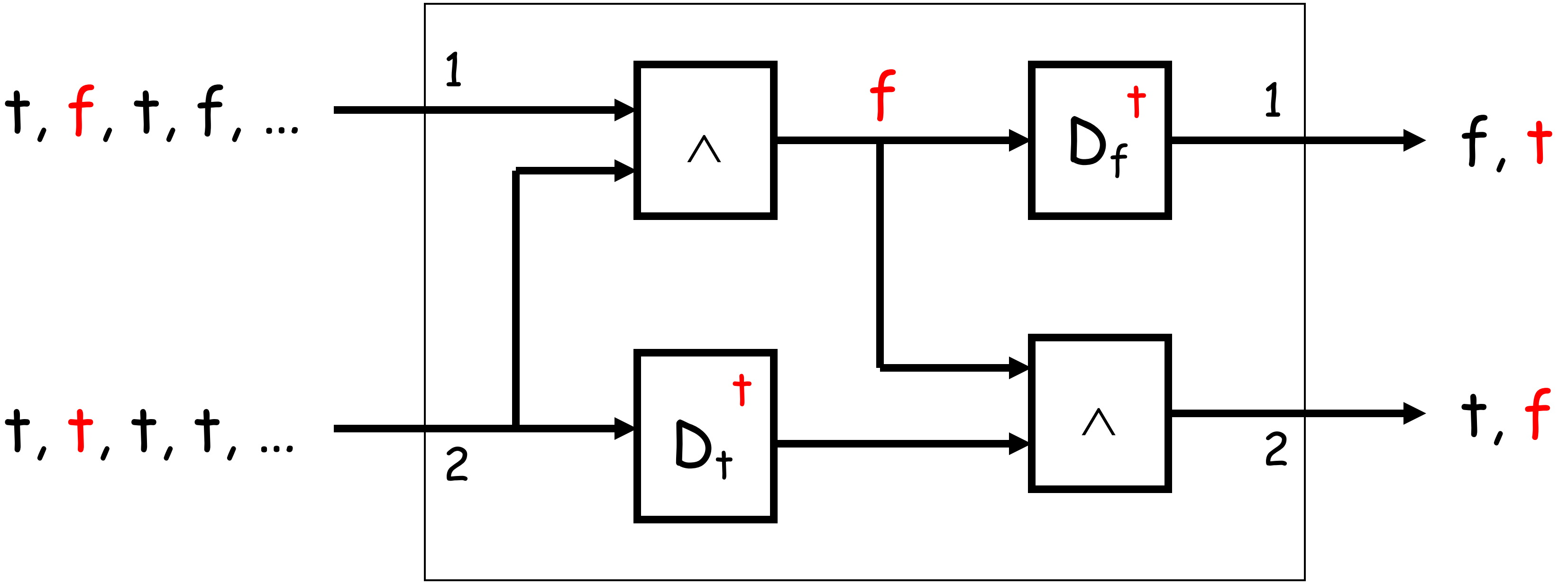
Block Diagram with Delay



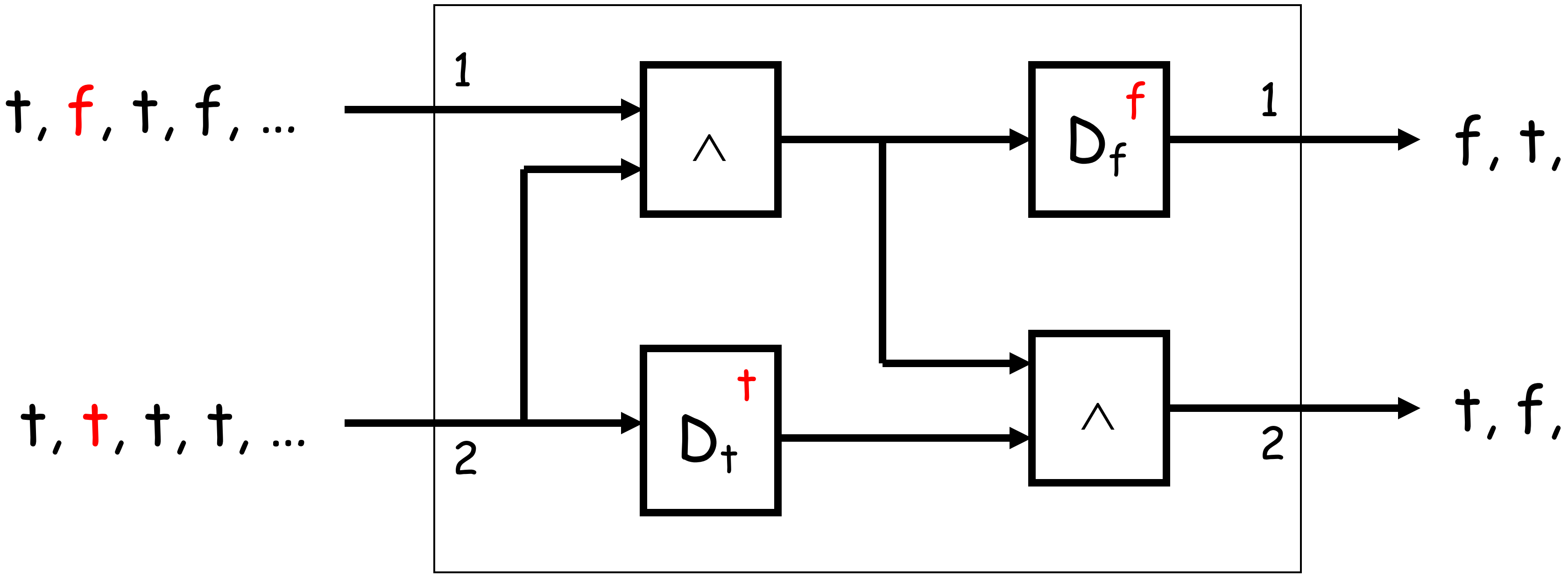
Block Diagram with Delay



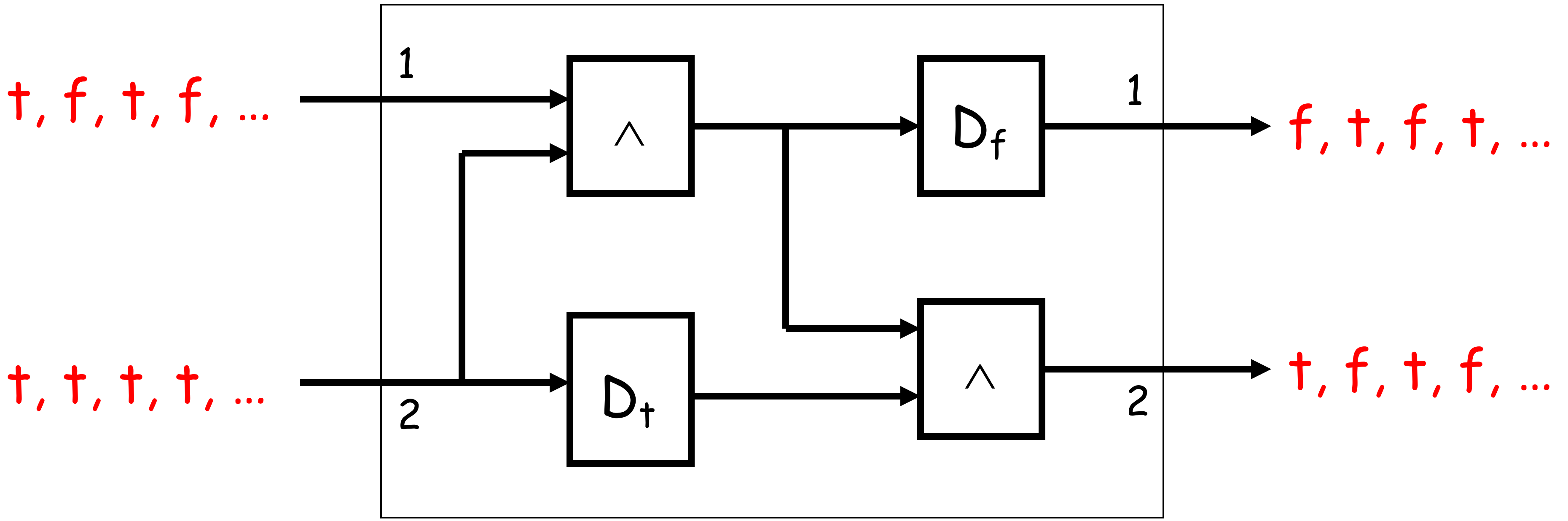
Block Diagram with Delay



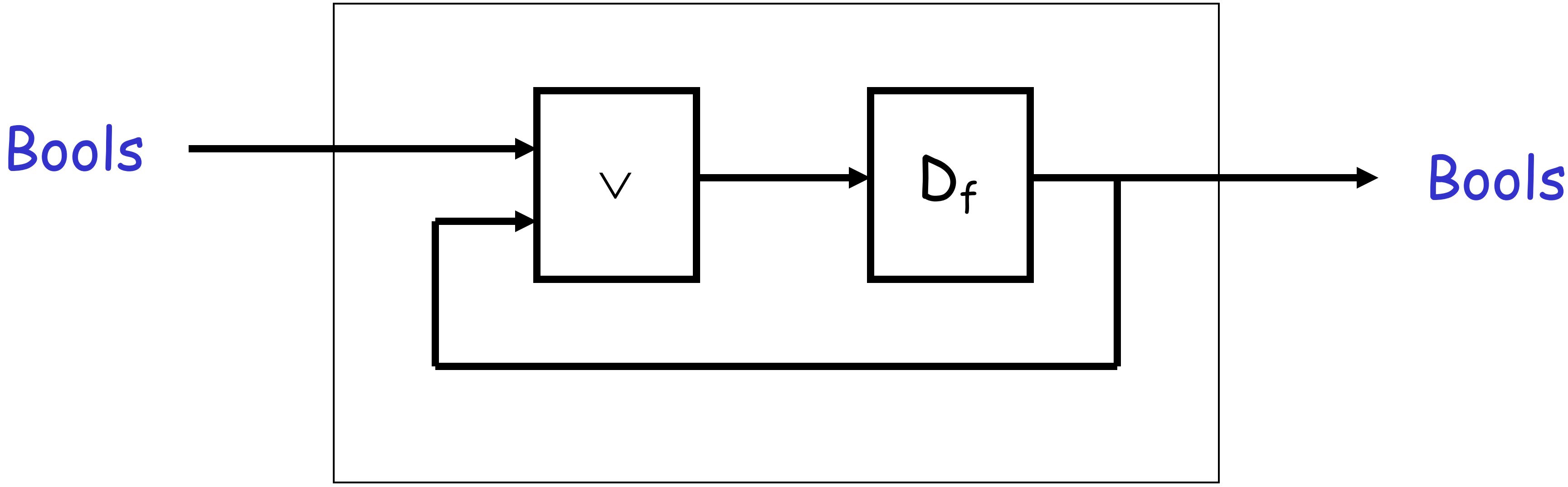
Block Diagram with Delay



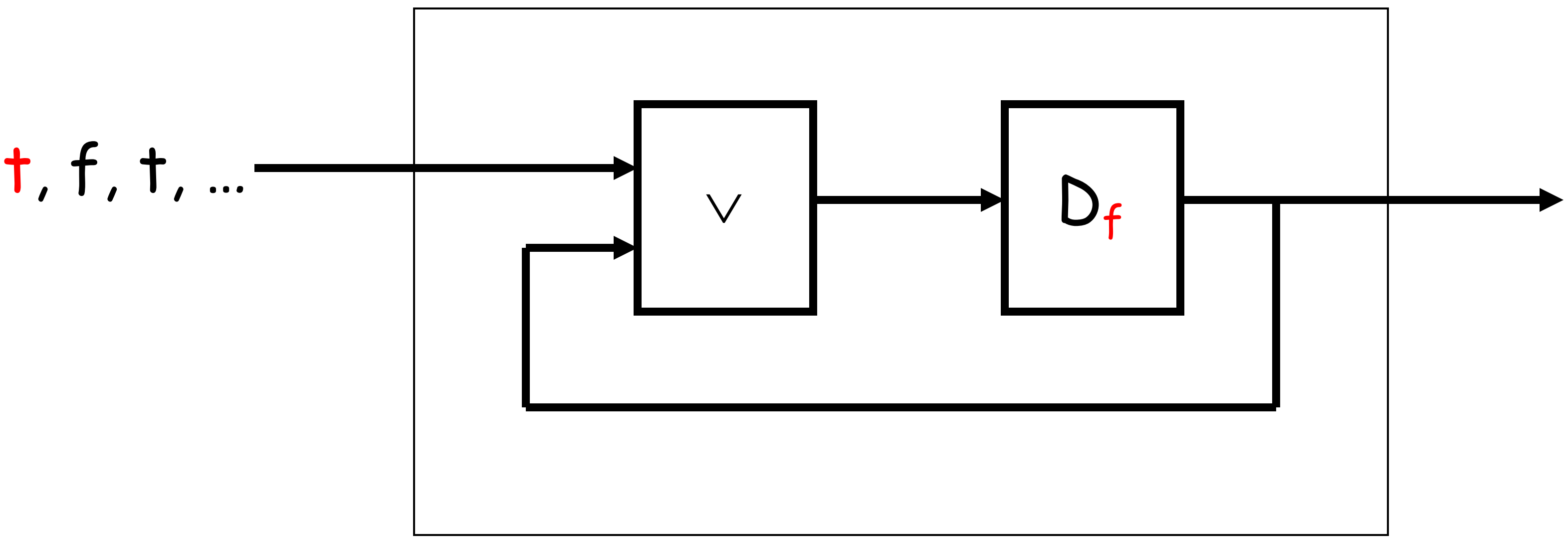
Block Diagram with Delay



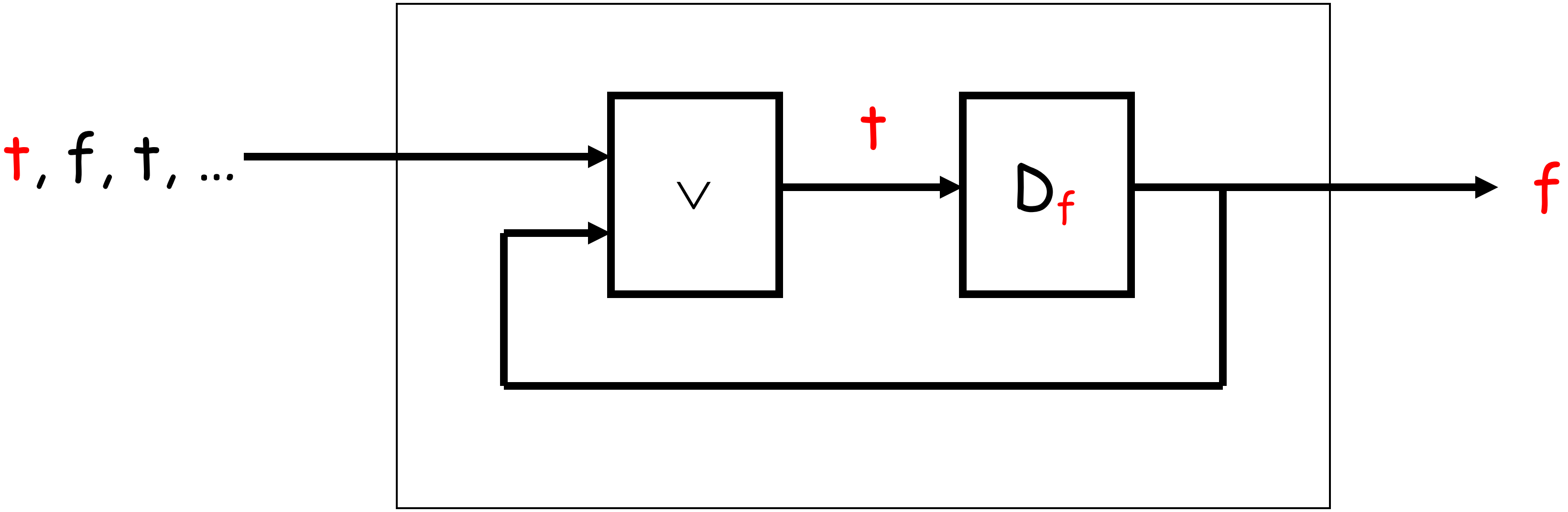
Block Diagram with Cycle



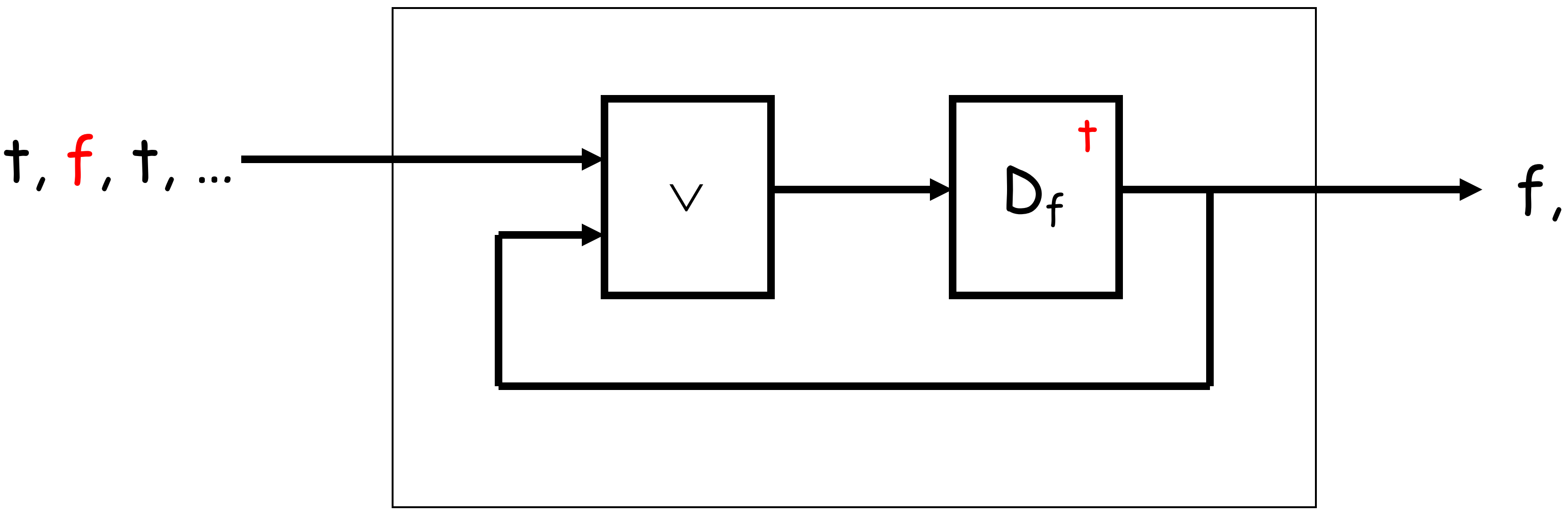
Block Diagram with Cycle



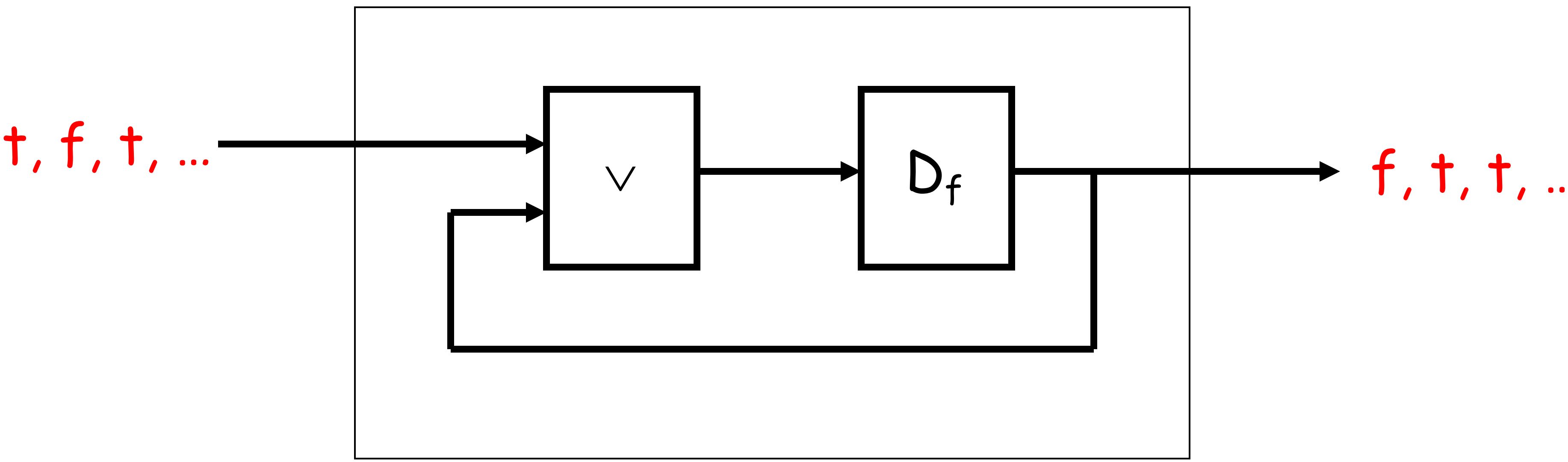
Block Diagram with Cycle



Block Diagram with Cycle



Block Diagram with Cycle



Legal Transducive Block Diagrams

-all components are transducive systems

-no cycles

e.g., combinational circuits

Legal Reactive Block Diagrams

-all components are memory-free or delay systems

-every cycle contains at least one delay

e.g., sequential circuits

Discrete-Time Moving-Average

DiscMovAvg: $[\, Nats_0 \rightarrow Reals \,] \rightarrow [\, Nats_0 \rightarrow Reals \,]$

such that $\forall\, x \in [\, Nats_0 \rightarrow Reals \,],\ \forall\, y \in Nats_0$,

$$(\, DiscMovAvg\,(x)\,)\,(y) = \begin{cases} ? & \text{if } \ y < 2 \\ 1/3 \cdot (\, x\,(y) + x\,(y-1) + x\,(y-2)\,) & \text{if } \ y \geq 2 \end{cases}$$

Discrete-Time Moving-Average

$\text{DiscMovAvg}_c : [\,\text{Nats}_0 \to \text{Reals}\,] \to [\,\text{Nats}_0 \to \text{Reals}\,]$

such that $\forall\, x \in [\,\text{Nats}_0 \to \text{Reals}\,],\ \forall\, y \in \text{Nats}_0$,

$$(\,\text{DiscMovAvg}\,(x)\,)\,(y) = \begin{cases} c & \text{if } y < 2 \\ 1/3 \cdot (\,x\,(y) + x\,(y-1) + x\,(y-2)\,) & \text{if } y \geq 2 \end{cases}$$

Continuous-Time Moving-Average

$\text{ContMovAvg}_c : [\text{Reals}_+ \to \text{Reals}] \to [\text{Reals}_+ \to \text{Reals}]$

such that $\forall\, x \in [\text{Reals}_+ \to \text{Reals}]$, $\forall\, y \in \text{Reals}_+$,

$$(\text{ContMovAvg}(x))(y) = \begin{cases} c & \text{if } y < 3 \\ 1/3 \cdot \int_{y-3}^{y} x(t)\, dt & \text{if } y \geq 3 \end{cases}$$

Continuous-Time Moving-Average

$\text{ContMovAvg}_C : [\,\text{Reals}_+ \to \text{Reals}\,] \to [\,\text{Reals}_+ \to \text{Reals}\,]$

such that $\forall\, x \in [\,\text{Reals}_+ \to \text{Reals}\,],\ \forall\, y \in \text{Reals}_+ ,$

$$(\,\text{ContMovAvg}\,(x)\,)\,(y) = \begin{cases} C & \text{if } y < 3 \\ 1/3 \cdot \displaystyle\int_{y-3}^{y} x\,(t)\,dt & \text{if } y \geq 3 \end{cases}$$

Integral is a quantifier
that binds the variable t.

Discrete-Time Moving-Average

DiscMovAvg : [$Nats_0 \to$ Reals] $\to$ [$Nats_0 \to$ Reals]

such that $\forall$ x $\in$ [$Nats_0 \to$ Reals] , $\forall$ y $\in$ $Nats_0$,

$$(\text{DiscMovAvg}(x))(y) = \begin{cases} c & \text{if } y < 2 \\ 1/3 \cdot \sum_{t=y-2}^{y} x(t) & \text{if } y \geq 2 \end{cases}$$

Sum is a quantifier that binds the variable t.

A Noncausal Reactive System

$$\text{Predict: } [\,\text{Time} \to \text{Bins}\,] \to [\,\text{Time} \to \text{Bins}\,]$$

such that $\forall\, x \in [\,\text{Time} \to \text{Bins}\,],\ \forall\, y \in \text{Time},$

$$(\,\text{Predict}\,(x)\,)\,(y) = \begin{cases} 1 & \text{if } \exists\, z \in \text{Time},\ x\,(z) = 1 \\ 0 & \text{if } \forall\, z \in \text{Time},\ x\,(z) = 0 \end{cases}$$

A reactive system

$F : [\ Time \rightarrow Values\ ] \rightarrow [\ Time \rightarrow Values\ ]$

is causal (or implementable)

iff

$\forall\ x, y \in [\ Time \rightarrow Values\ ],\ \forall\ z \in Time,$

if $(\ \forall\ t \in Time,\ t \leq z \Rightarrow x(t) = y(t)\ )$

then $(\ F(x)\ )(z) = (\ F(y)\ )(z)$.