Sets, Tuples, Functions

EECS 20

Lecture 3 (January 22, 2001)

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Important Mathematical Objects

1. Sets (unordered collections)
2. Tuples (ordered collections)
3. Functions

SETS

Let Evens = $\{ x \in \text{Nats} \mid \exists\, y \in \text{Nats},\; x = 2 \cdot y \}$.

Let Evens be the set of all $x \in$ Nats such that $x = 2 \cdot y$ for some $y \in$ Nats.

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Constants

Variables

Operators

Quantifiers

Definition

SETS

Set constants : e.g. $\{1, 2, 3\}$

Set operator : $\in$

Set quantifier : $\{ x \mid \ldots \}$

Additional constants can be defined: e.g. Nats

Additional operators on sets

set $\cap$ set	Result: set
set $\cup$ set	set
set \ set	set
set $\subseteq$ set	truth value
set = set	truth value
P (set)	set

Meaning of additional operators can be defined

$\forall$ set x, $\forall$ set y, let $x \cap y = \{ z \mid z \in x \wedge z \in y \}$.

$\forall$ set x, $\forall$ set y, let $x \cup y = \{ z \mid z \in x \vee z \in y \}$.

$\forall$ set x, $\forall$ set y, let $x \setminus y = \{ z \mid z \in x \wedge z \notin y \}$.

$\forall$ set x, $\forall$ set y, let $x \subseteq y \Leftrightarrow (\forall z \mid z \in x \Rightarrow z \in y)$.

$\forall$ set x, $\forall$ set y, let $x = y \Leftrightarrow x \subseteq y \wedge y \subseteq x$.

$\forall$ set x, let $P(x) = \{ y \mid y \subseteq x \}$.

TUPLES

Tuple constants

(2, 7)	2-tuple (or "pair")
(2, 7, 1)	3-tuple (or "triple")
(b, e, r, k, e, l, e, y)	8-tuple

Note: $\{ 2, 7 \} = \{ 7, 2 \}$

$(2, 7) \neq (7, 2)$

Tuple operators

(anything , anything)	Result: pair
(any , any , any)	triple
$\text{tuple}_{\text{number}}$	any

Examples: $(2, 7, 1)_2 = 7$

$\forall$ pair x, $x = (x_1, x_2)$

Additional operators on tuples

pair = pair | Result: truth value

triple = triple | truth value

$\forall$ pair x, $\forall$ pair y, let $x = y \Leftrightarrow x_1 = y_1 \wedge x_2 = y_2$.

$\forall$ triple x, $\forall$ triple y, let $x = y \Leftrightarrow x_1 = y_1 \wedge x_2 = y_2 \wedge x_3 = y_3$.

Additional operators on sets

set × set   Result: set of pairs

set × set × set   set of triples

$\forall$ set x, y,   let $x \times y$ $= \{ u \mid \exists v, \exists w,\ u = (v,w) \wedge v \in x \wedge w \in y \}$

$= \{ (v,w) \mid v \in x \wedge w \in y \}$.

$\forall$ set x, y, z,   let $x \times y \times z$ $= \{ (u,v,w) \mid u \in x \wedge v \in y \wedge w \in z \}$.

FUNCTIONS

Each function has three things:

1. the domain (a set)
2. the range (a set)
3. the graph (for every domain element, a range element)

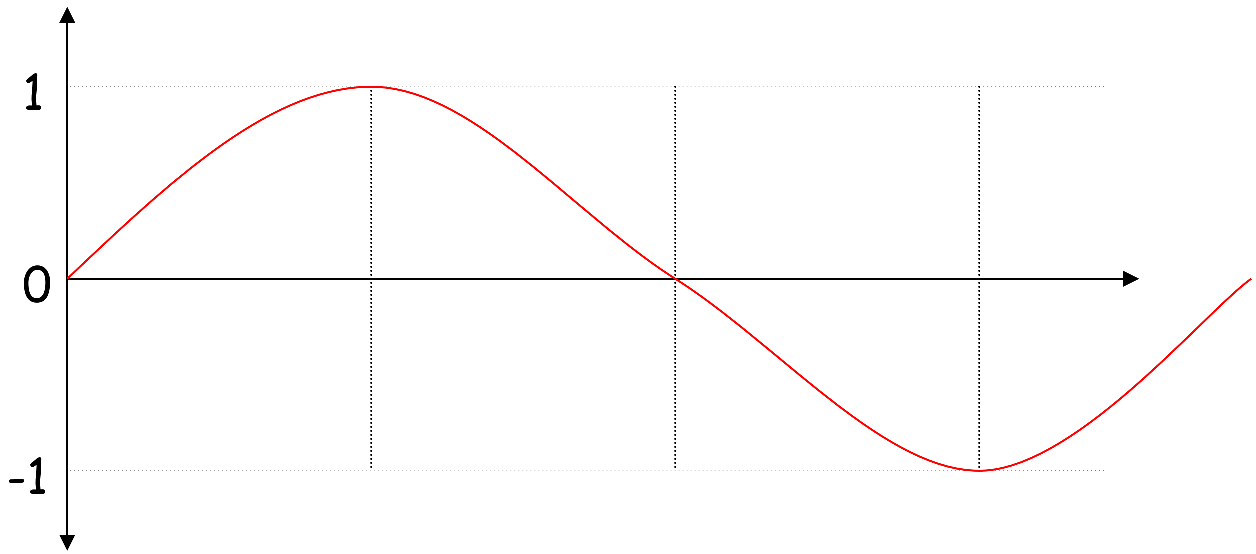
Function constants

sin, cos

1 Domain : Reals .

2 Range : $[-1,1] = \{ x \in \text{Reals} \mid -1 \leq x \leq 1 \}$.

3 Graph : for each real x, the real sin (x) $\in [-1,1]$.



Formally, the graph of a function can be thought of as a set of pairs :

$\{\ (x,y) \in (\ \text{Reals} \times [-1,1]\ )\ |\ \ y = \sin(\ x\ )\ \}$

$\quad = \{\ \dots\ ,\ (0,0)\ ,\ \dots\ ,\ (\pi/2,\ 1)\ ,\ \dots\ ,\ (\pi,0),\ \dots\ ,\ (3\pi/2,-1),\ \dots\ \}$.

All operators are really function constants

!

Domain : Nats .

Range: Nats .

Graph: $\{ (x, y) \in \text{Nats} \times \text{Nats} \mid y = x! \}$

$= \{ (1,1), (2,2), (3,6), (4, 24), \ldots \}$.

All operators are really function constants

$+$

Domain : $\text{Nats}^2 = \text{Nats} \times \text{Nats}$.

Range: Nats .

Graph: $\{ ((x,y) , z) \in \text{Nats}^2 \times \text{Nats} \mid z = x + y \}$

$= \{ ((1,1), 2), ((1,2), 3), \ldots , ((7,5), 12), \ldots \}$.

All operators are really function constants

$\wedge$

Domain : Bools^2 .

Range: Bools .

Graph: { ((true, true), true),
((true, false), false),
((false, true), false),
((false, false), false) } .

If domain and range of a function are finite, then the graph can be given by a table :

x	y	f(x,y)
true	true	true
true	false	false
false	true	false
false	false	false

Operators on functions

	Result:
domain (function)	set
range (function)	set
graph (function)	set of pairs
function (domain element)	range element

Examples:

domain (sin) = Reals

range (sin) = [-1,1]

sin (π) = 0

Function definition

$\boxed{\begin{array}{l} \text{Let } f : \text{Domain} \rightarrow \text{Range such that} \\ \forall\, x \in \text{Domain},\ f(x) = \ \ldots \end{array}}$

Let double : Nats $\rightarrow$ Nats such that

$\forall$ x $\in$ Nats, double (x) = 2$\cdot$ x .

domain (double) = Nats

range (double) = Nats

graph (double) = { (1,2), (2,4), (3,6), (4,8), ... }

Let $\exp : \text{Nats}^2 \to \text{Nats}$ such that

$\forall\, x, y \in \text{Nats},\ \exp(x, y) = x^y$.

domain (exp) = Nats^2

range (exp) = Nats

graph (exp) = { ((1,1) , 1), … , ((2,3) , 8), … }

Additional operators on functions

[set $\rightarrow$ set]	Result:	set of functions
function ⍰ function		function
one-to-one (function)		truth value
onto (function)		truth value

Meaning of additional operators

$\forall$ set x, y, let

$$[\, x \to y \,] = \{\, f \mid \text{domain}(f) = x \wedge \text{range}(f) = y \,\}\,.$$

$\forall$ set x, y, z, $\forall\, f \in [\, x \to y \,]$, $\forall\, g \in [\, y \to z \,]$, let

$g \circ f : x \to z$ such that

$$\forall\, u \in x,\ (g \circ f)(u) = g(f(u))\,.$$

Meaning of additional operators

$\forall$ function f , let

one-to-one (f) $\Leftrightarrow$

$\forall$ x, y $\in$ domain(f), if x $\neq$ y then f(x) $\neq$ f(y) .

$\forall$ function f , let

onto (f) $\Leftrightarrow$

$\forall$ x $\in$ range(f), $\exists$ y $\in$ domain(f), x = f(y) .