Mathematical Language

EECS 20

Lecture 2 (January 19, 2001)

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Mathematical Language

Let Evens = $\{ x \mid (\exists y,\ y \in \text{Nats} \wedge x = 2 \cdot y) \}$.

Mathematical Language

Let Evens = { x | (∃ y, y ∈ Nats ∧ x = 2· y) } .

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Definition

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Definition

Constants have meaning

20	a certain number
Berkeley	a certain city
false	a certain truth value

Variables have no meaning

x

y_0

z'

Operators on numbers

number + number	Result:	number
number !		number
number = number		truth value
number ≤ number		truth value

Operators on cities

merge (city, city)	Result: city
population-of (city)	number
has-a-university (city)	truth value

Operators on truth values

truth value $\wedge$ truth value	Result:	truth value
truth value $\vee$ truth value		truth value
$\neg$ truth value		truth value
truth value $\Leftrightarrow$ truth value		truth value
truth value $\Rightarrow$ truth value		truth value

Expressions on constants have meaning

$3 + 20$	Result:	23
$(3! + 2) \cdot 4$		32
$4 \leq$ population-of (Berkeley)		true
$4 \cdot 20 \leq 4 + 20$		false
true $\wedge$ false		false
true $\wedge$ (4 + 20)		not well-formed

Implication

true $\Rightarrow$ true	Result:	true
true $\Rightarrow$ false		false
false $\Rightarrow$ true		true
false $\Rightarrow$ false		true

Expressions on variables have no meaning

$x + 20$	Free variables: x
$(3! + y) \cdot 4$	y
$x \leq y$	x, y

Quantifiers remove free variables from expressions

$x = 0$	Result: free x
$\exists\, x,\ x = 0$	true
$\forall\, x,\ x = 0$	false
$\exists\, y,\ x + 1 = y$	free x
$\forall\, x,\ \exists\, y,\ x + 1 = y$	true
$\forall\, x,\ \exists\, y,\ x \vee y$	true
$\forall\, x,\ x + 7$	not well-formed

Every mathematical expression

1. is not well-formed ("type mismatch"), or
2. contains free variables, or
3. is a definition, or
4. has a meaning (e.g., 20, Berkeley, false).

SETS

Set constants

{ 1, 2, 3 }

{ Atlanta, Berkeley, Chicago, Detroit }

{ 1, 2, 3, 4, … }

Set operator

anything $\in$ set Result: truth value

$2 \in \{ 1, 2, 3 \}$ true

$2 \in \{$ Atlanta, Berkeley $\}$ false

Set quantifier

(∃ x, truth value)	Result: truth value
(∀ x, truth value)	truth value
{ x | truth value }	set

Quantifiers remove free variables from expressions

$\{ x \mid x \leq y \}$	Result: free y
$\{ x \mid x = 1 \vee x = 2 \}$	{ 1, 2 }
$\{ x \mid \exists y,\ x = 2 \cdot y \}$	{ 2, 4, 6, 8, … }
$\{ x \mid x + 7 \}$	not well-formed

Bounded quantification

($\exists$ x $\in$ set, truth value)	Result: truth value
($\forall$ x $\in$ set, truth value)	truth value
{ x $\in$ set | truth value }	set

Meaning of constants can be defined

Let Nats = { 1, 2, 3, 4, … } .

Let Bools = { true, false } .

Define Cities = { Atlanta, Chicago, Berkeley, Detroit } .

Define Ø = { } .

Let Evens = { $x \in$ Nats | $\exists\, y \in$ Nats, $x = 2 \cdot y$ } .

Let Evens be the set of all $x \in$ Nats such that $x = 2 \cdot y$ for some $y \in$ Nats.