Summer School on Formal Methods for Cyber-Physical Systems

Edition 2019: Numerical and Symbolic Methods for Reachability Analysis of Hybrid Systems

DIPARTIMENTO DI INGEGNERIA INFORMATICA AUTOMATICA E GESTIONALE ANTONIO RUBERTI

SAPIENZA UNIVERSITÀ DI ROMA

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What is a hybrid system?

A dynamic system that exhibits both **interacting**

CONTINUOUS dynamic behaviour

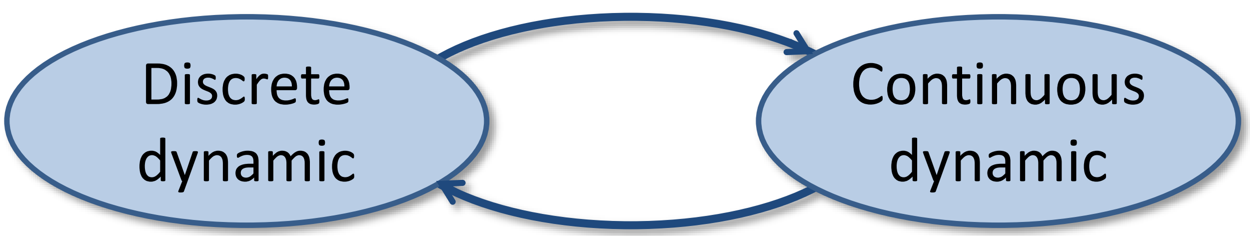
- ***time-driven continuous-state systems***

state takes values in a continuous set and changes as time progresses

DISCRETE dynamic behaviour

- ***event-driven discrete-state systems***

state takes values in a discrete set and changes due to the occurrence of an event



Hybrid behavior arises in …

Continuous systems with phased operation

dynamics inherently hybrid (mechanical systems with collisions, robotics, four-stroke engine, circuits with diodes, biological cell growth and division)

Continuous systems controlled by discrete logic

quantized control of a continuous system (thermostat, chemical plant with valves and pumps)

embedded systems where computational systems modeled as finite-state machines are coupled with plants and controllers modeled by continuous systems

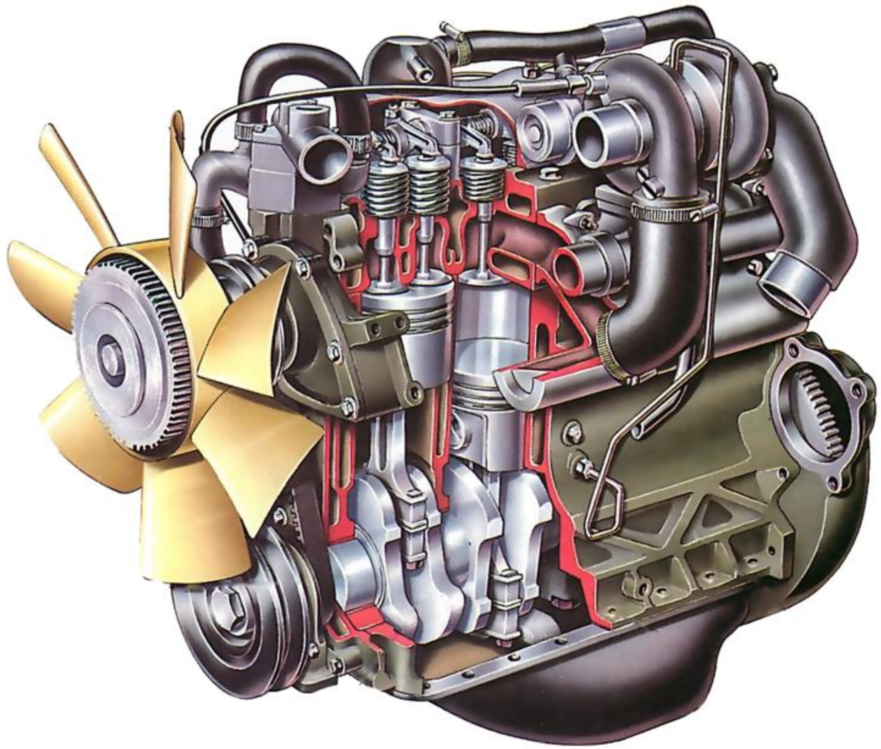
networked control systems where sensors, controllers and actuators are connected by a shared network medium

switching control systems where some supervisor decides which controller in a given set to apply and when to switch to a different one

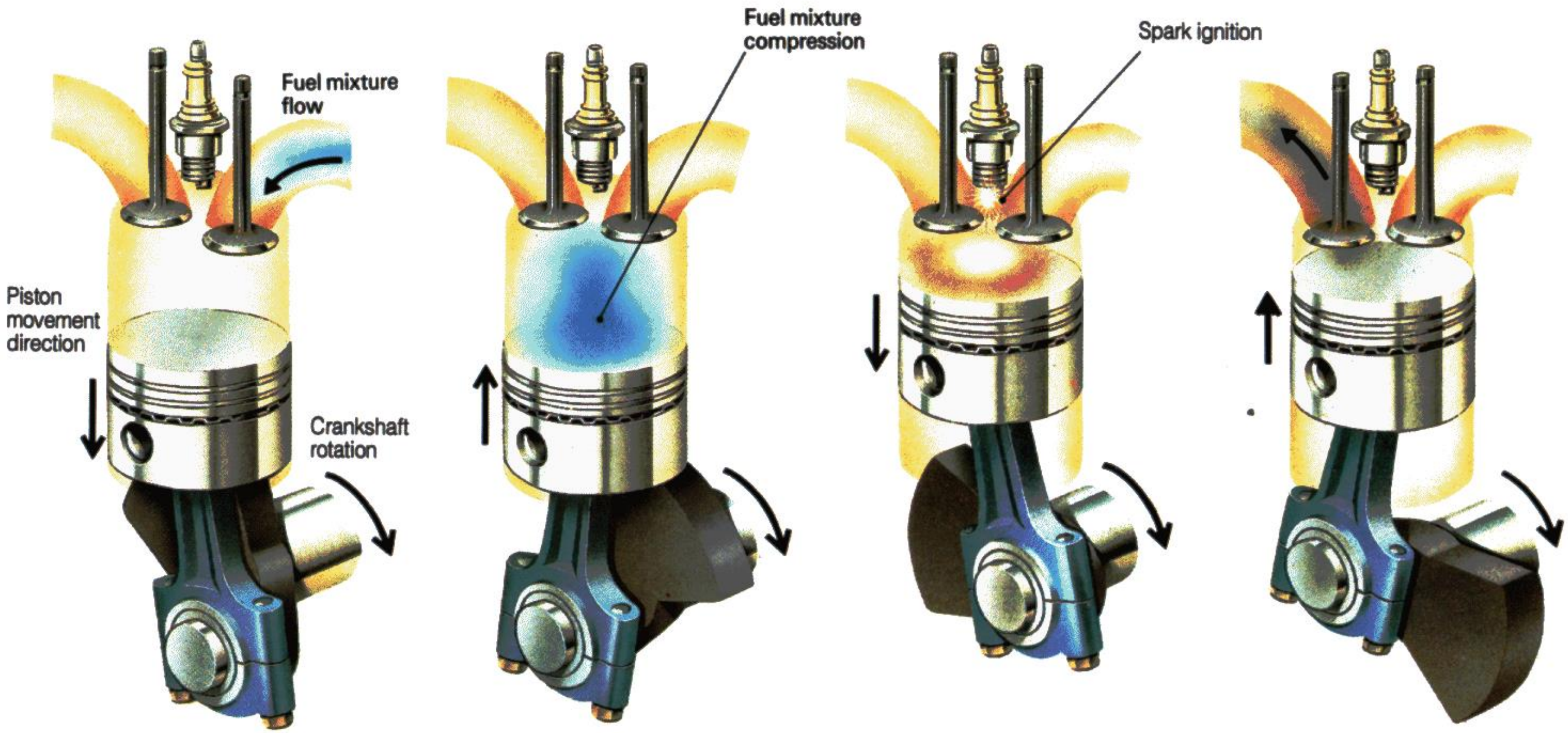
Coordination of multi-agent systems

resource allocation for competing agents with a continuous dynamics (air and ground transportation systems)

Engine control



a four-stroke gasoline engine is naturally modeled using four modes corresponding to the position of the pistons, while combustion and power train dynamics are continuous



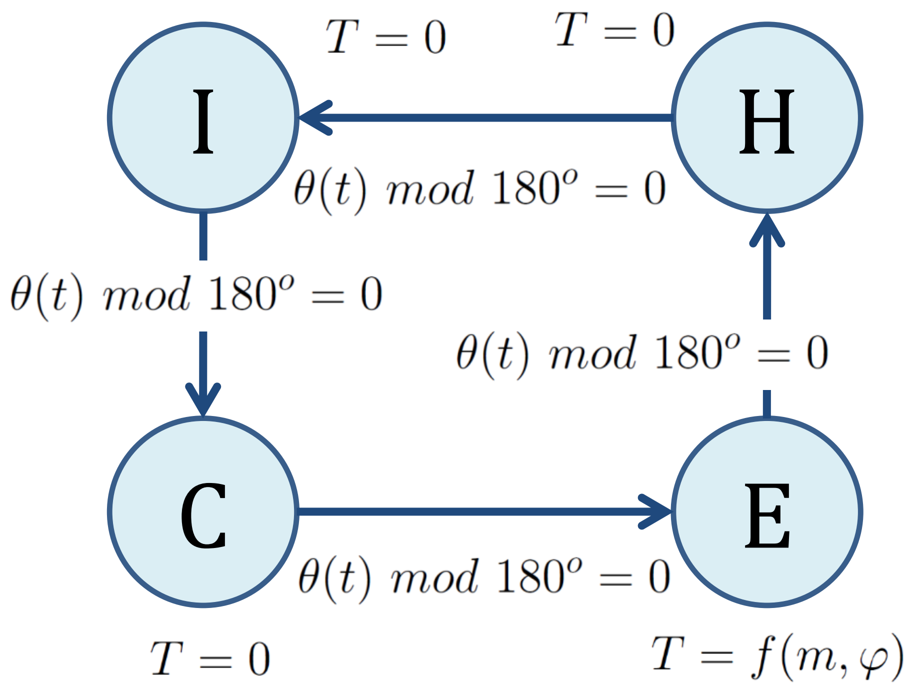
Intake stroke: the piston is descending, the inlet valve is fully open and the exhaust valve is closed

Compression stroke: the piston rises, the inlet and exhaust valves are closed

The **power stroke** drives the piston downloads as the ignited gases expand. The inlet and the exhaust valves are closed

The hot gases in the cylinder escape through the open exhaust valve as the piston rises again for the **exhaust stroke**

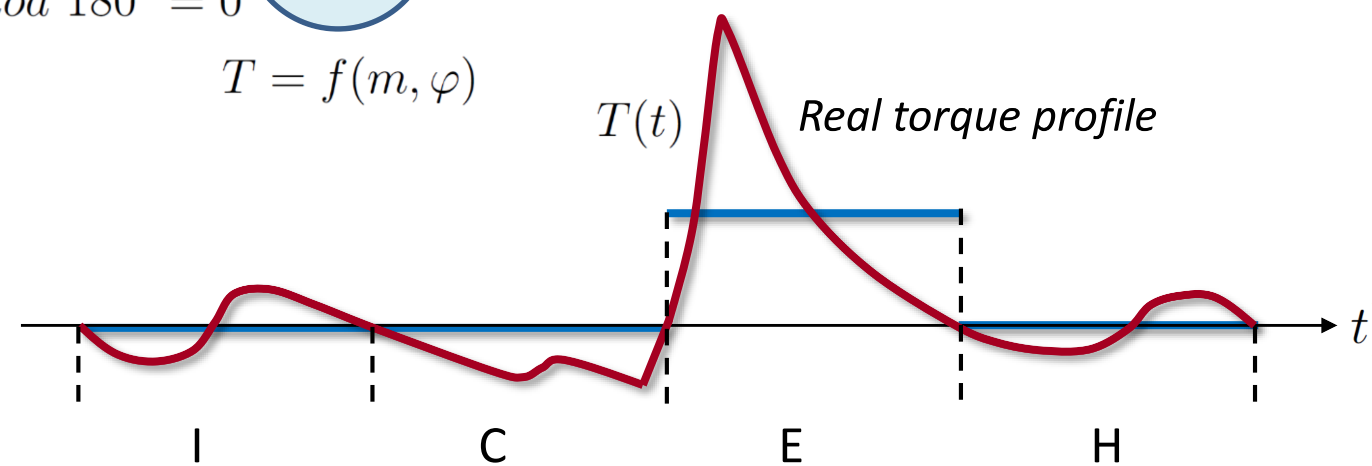
Crankshaft dynamics depends on the torque generated by the pistons



$T(t)$ torque generated by the piston

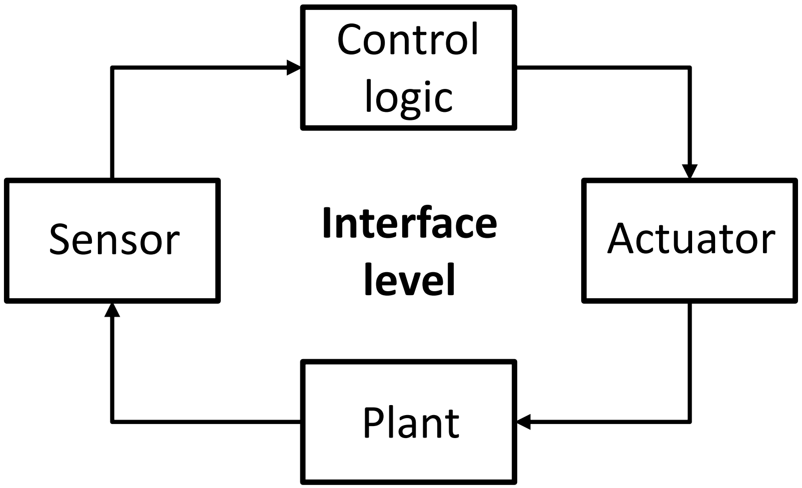
$n(t)$ crankshaft speed

$\theta(t)$ crankshaft angle



Continuous systems controlled by discrete logic **Embedded systems**

• micro-processors with an inherently discrete behavior (e.g. due to finite precision computations and quantization of signals) embedded in a physical device
• integrated with the physical world (continuous environment) through actuators and sensors
• sharing data and resources by a networked architecture (networked embedded systems)



Characteristics of embedded systems

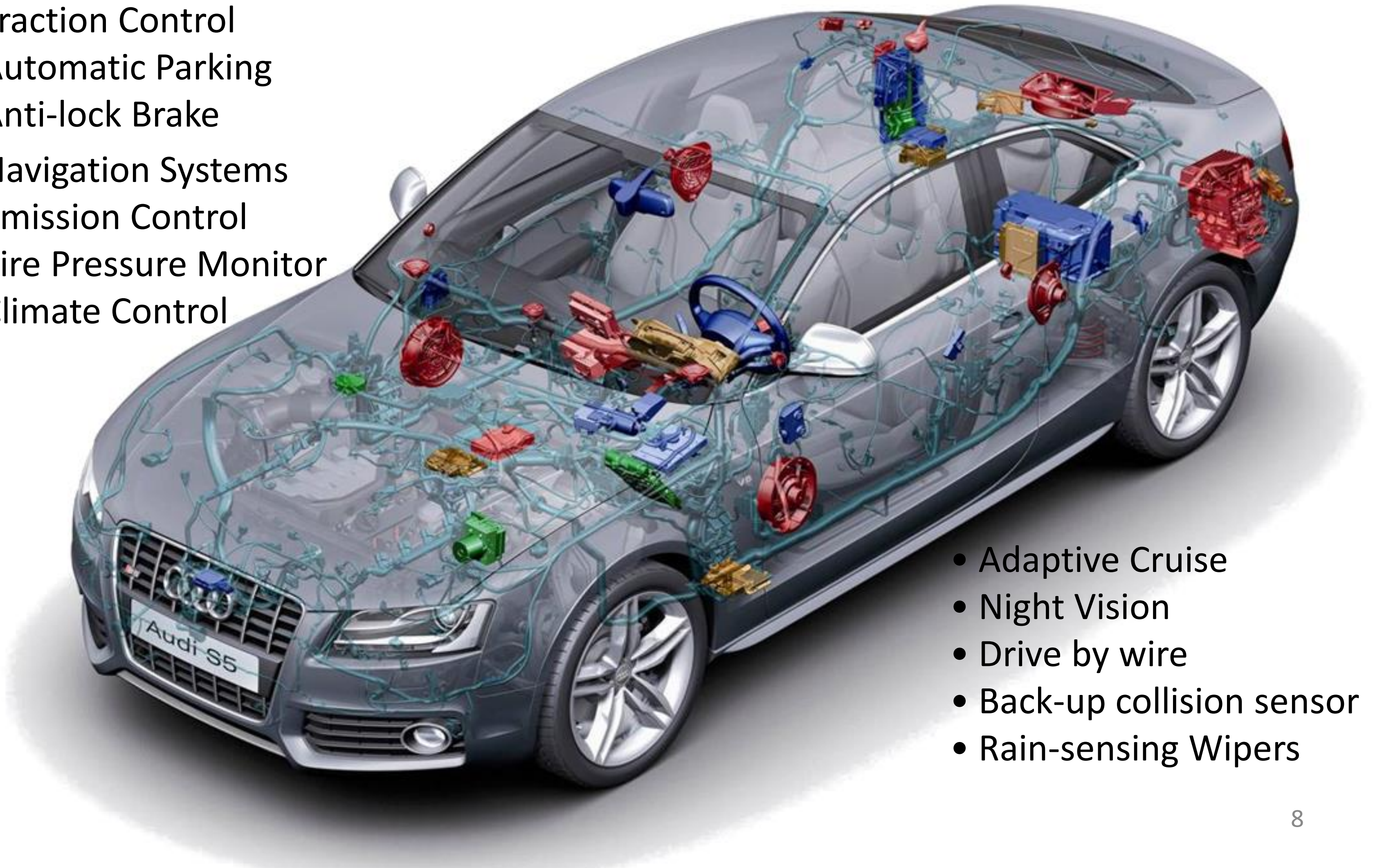
Real-time operation

- Must finish operations by deadlines.
- Hard real time: missing deadline causes failure.
- Soft real time: missing deadline results in degraded performance.

Continuous systems controlled by discrete logic — Embedded systems

Automotive applications

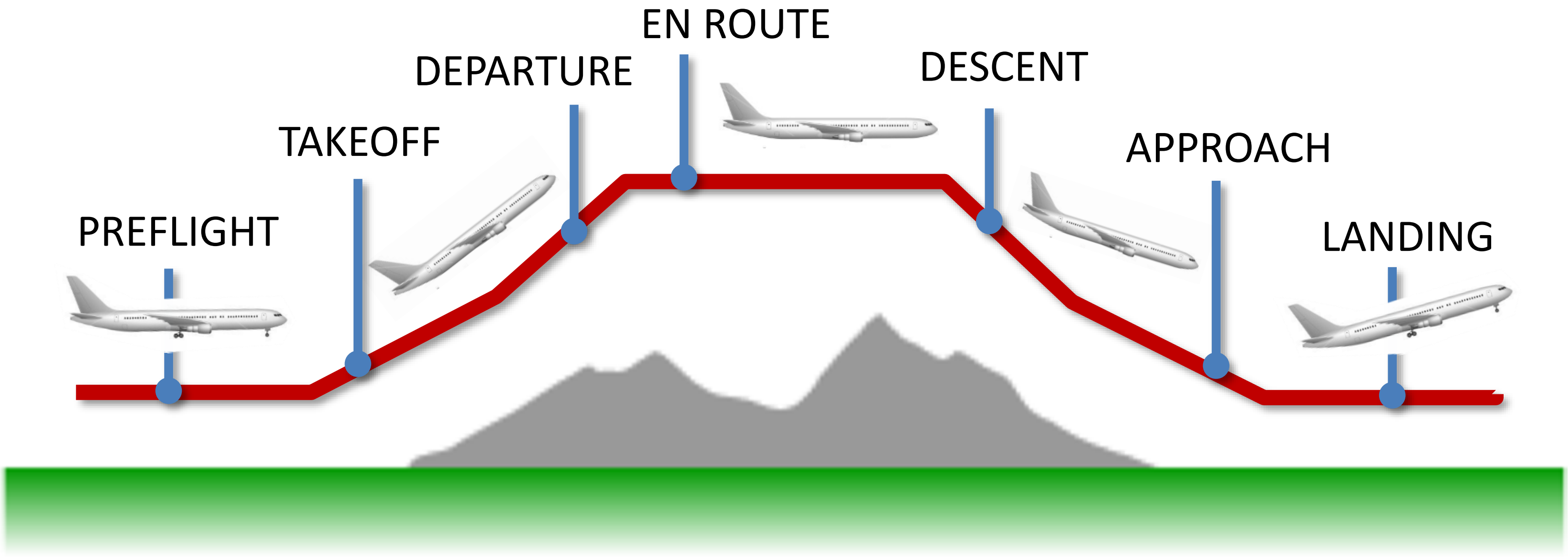
- Air Bags
- Traction Control
- Automatic Parking
- Anti-lock Brake
- Navigation Systems
- Emission Control
- Tire Pressure Monitor
- Climate Control



- Adaptive Cruise
- Night Vision
- Drive by wire
- Back-up collision sensor
- Rain-sensing Wipers

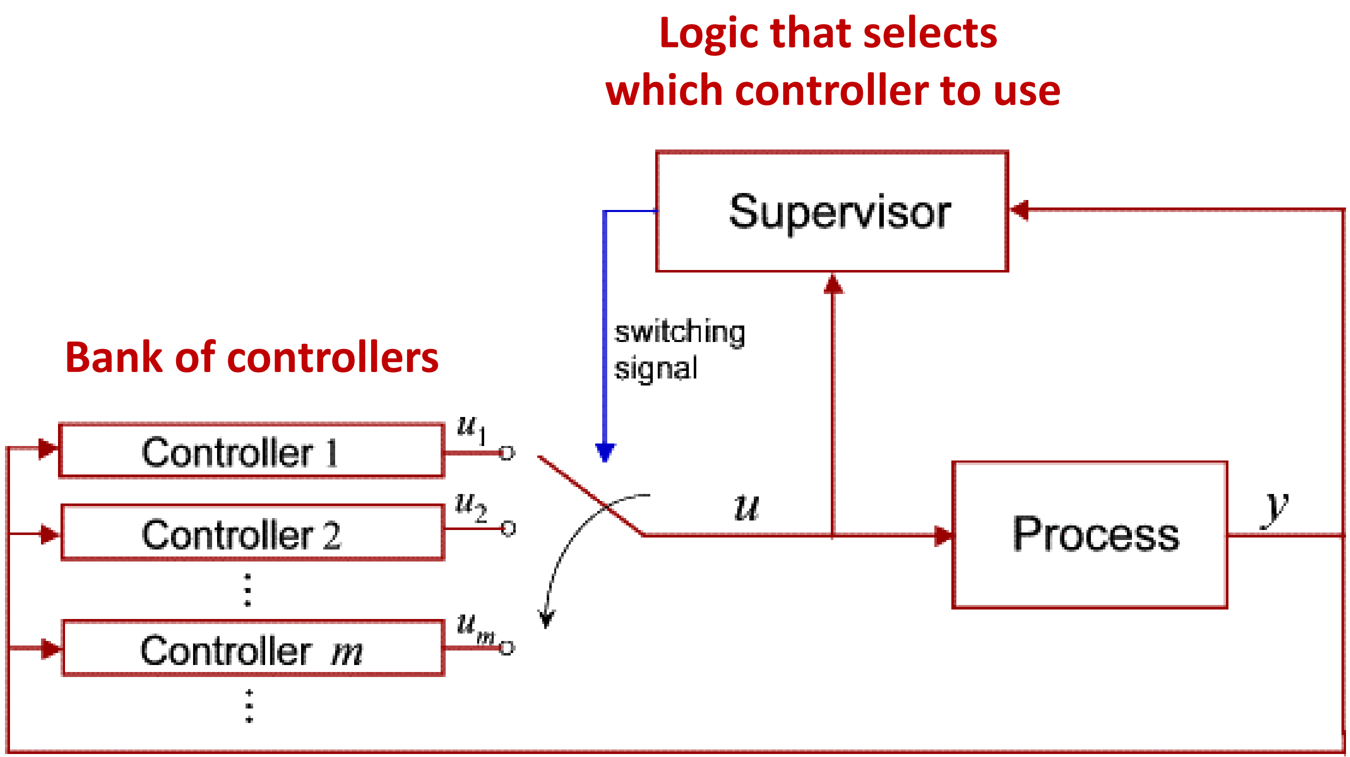
Continuous systems controlled by discrete logic *Switching control*

Control of a complex plant:
different controllers designed for different control modes (different models), switching rule coded by a DES (e.g. flight control)



Continuous systems controlled by discrete logic

Switching control



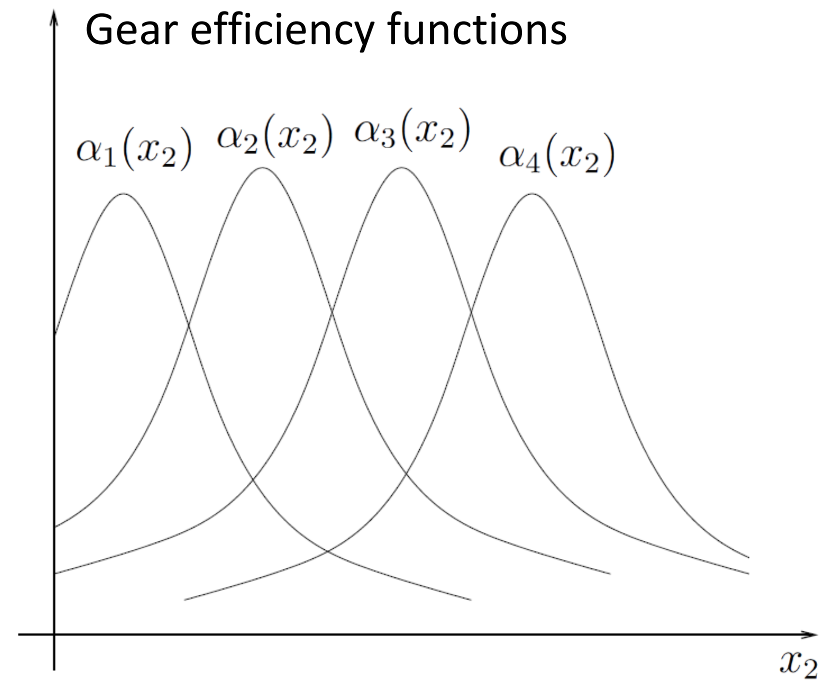
Gear shift control

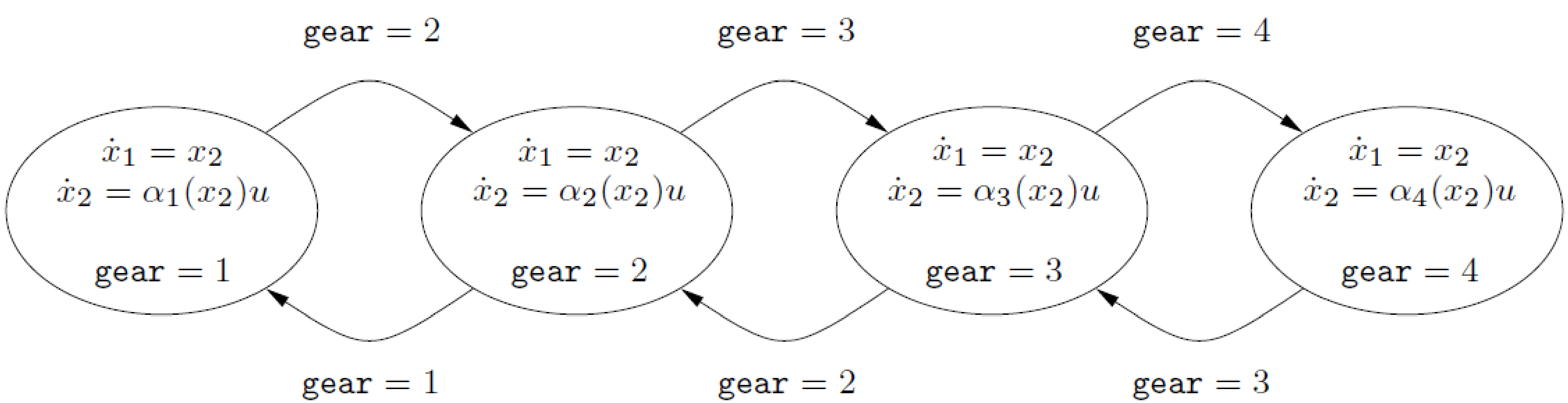
Continuous control: throttle position

$$u(t)$$

Discrete control: gear shift position

$$\texttt{gear} \in \{1, 2, 3, 4\}$$





Coordination of multi-agent systems

Air traffic control

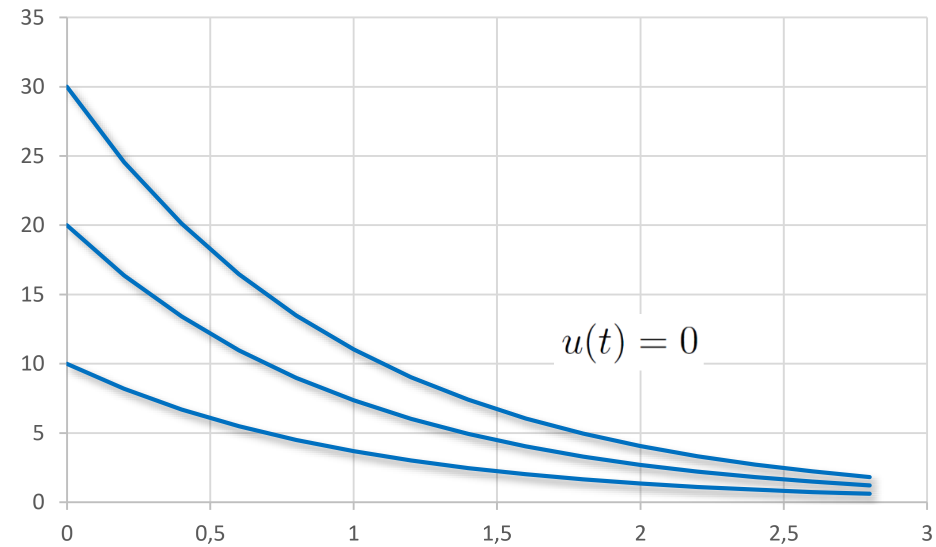
• predefined set of commands (speed change, short cut, detour...) issued by air traffic controllers to pilots to avoid conflicts

• sequencing of aircraft landing to some airport (distance to avoid turbulence caused by preceding aircraft, timing constraints)

Continuous systems controlled by discrete logic

Thermostat

Temperature in a room controlled by a thermostat switching a heater on and off

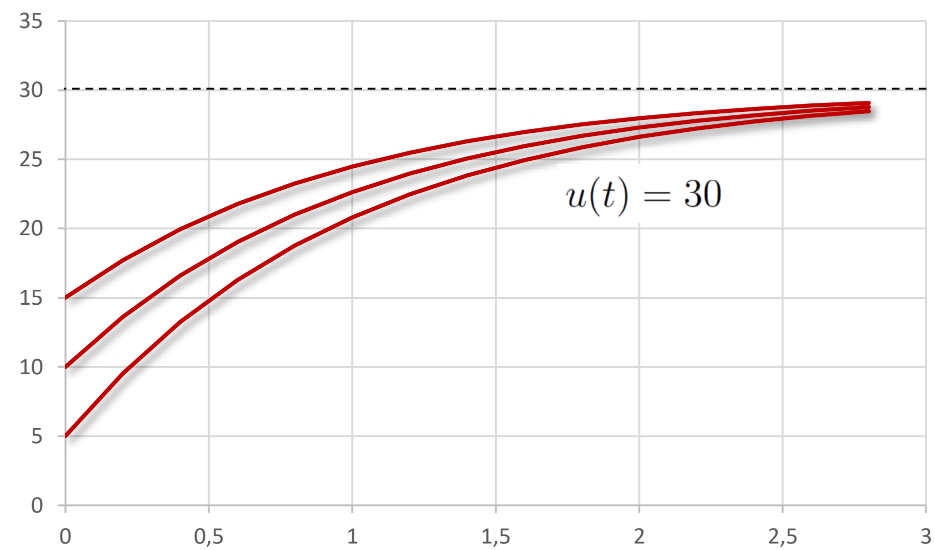


Dynamics of the temperature (in °C):

$$\dot{x}(t) = -x(t) + u(t)$$

heater on: $\dot{x}(t) = -x(t) + 30$

heater off: $\dot{x}(t) = -x(t)$

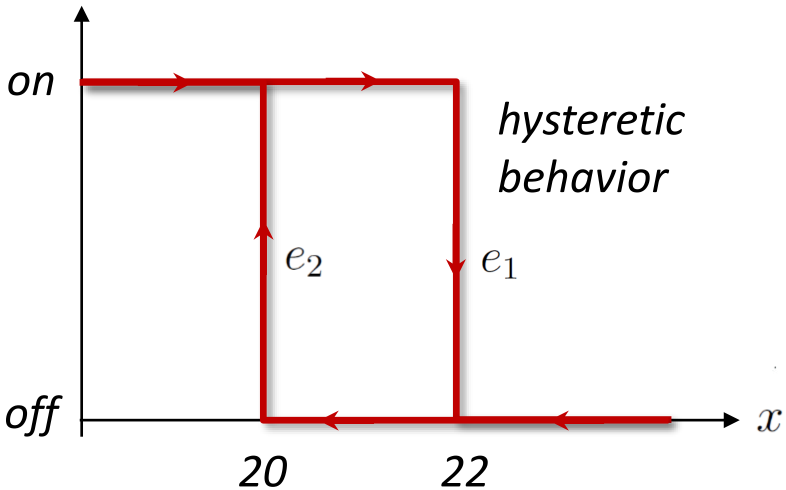


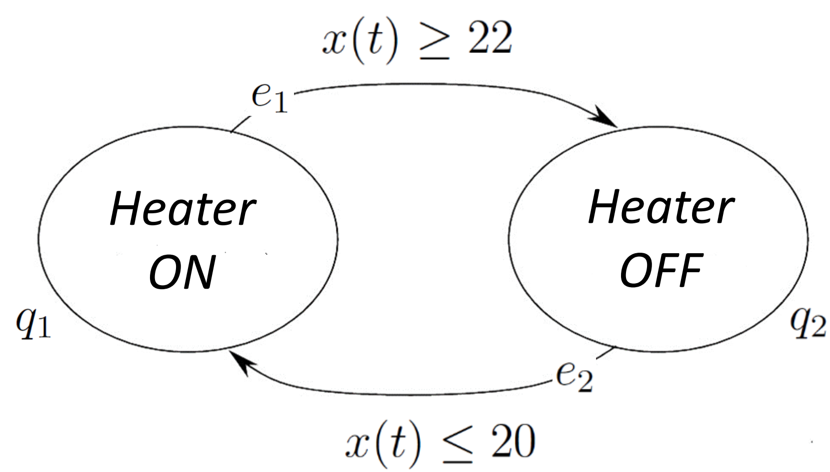
Goal:

regulate the temperature around 21°C

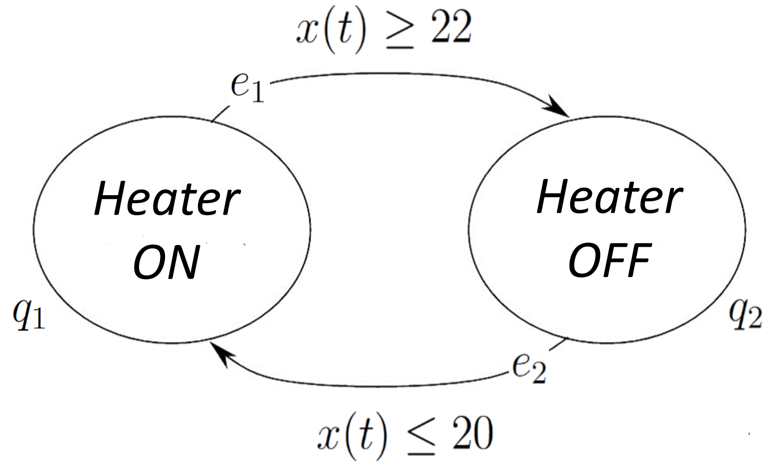
Strategy: turn the heater from OFF to ON if $x(t) \leq 20$

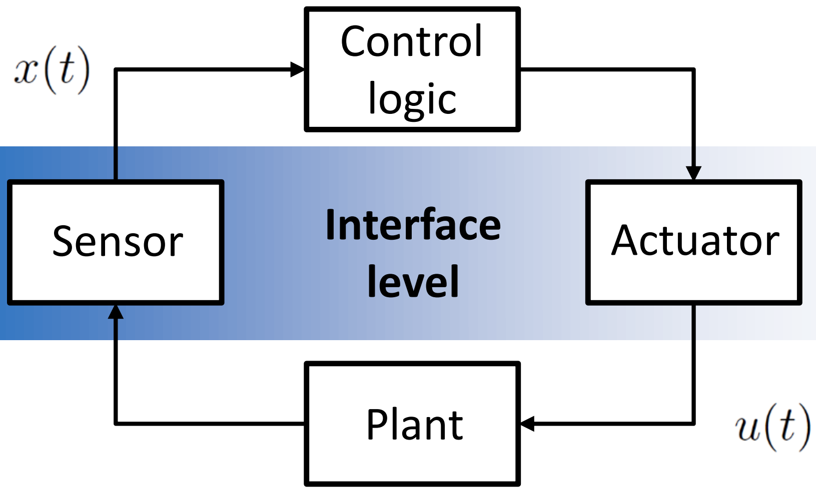
turn the heater from ON to OFF if $x(t) \geq 22$





Continuous systems controlled by a discrete logic





$u(t) = 0$ *if heater OFF*

$u(t) = 30$ *if heater ON*

$$\dot{x}(t) = -x(t) + u(t)$$

$\dot{x}(t) = -x(t)$ if heater OFF

$\dot{x}(t) = -x(t) + 30$ if heater ON

turn the heater from OFF to ON if $x(t) \leq 20$

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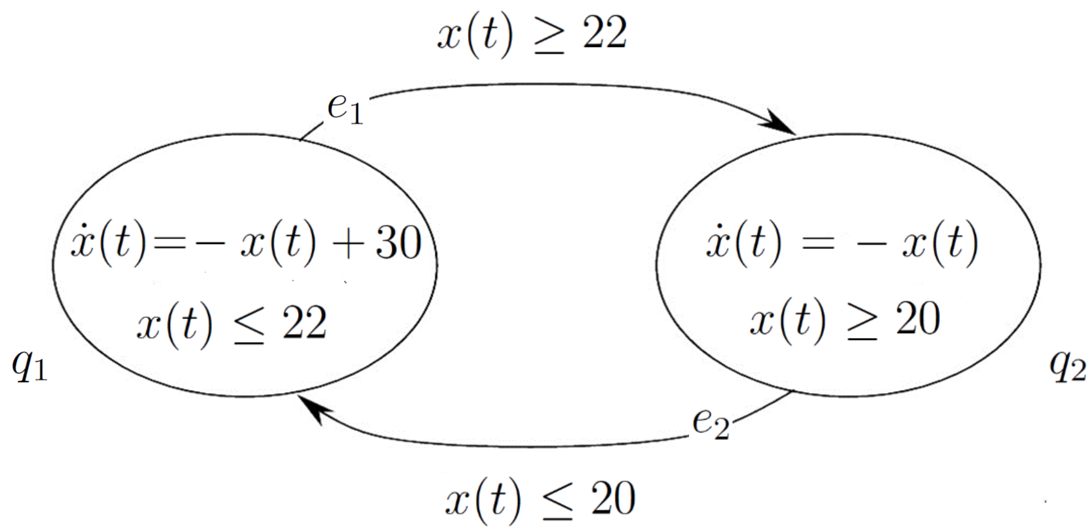

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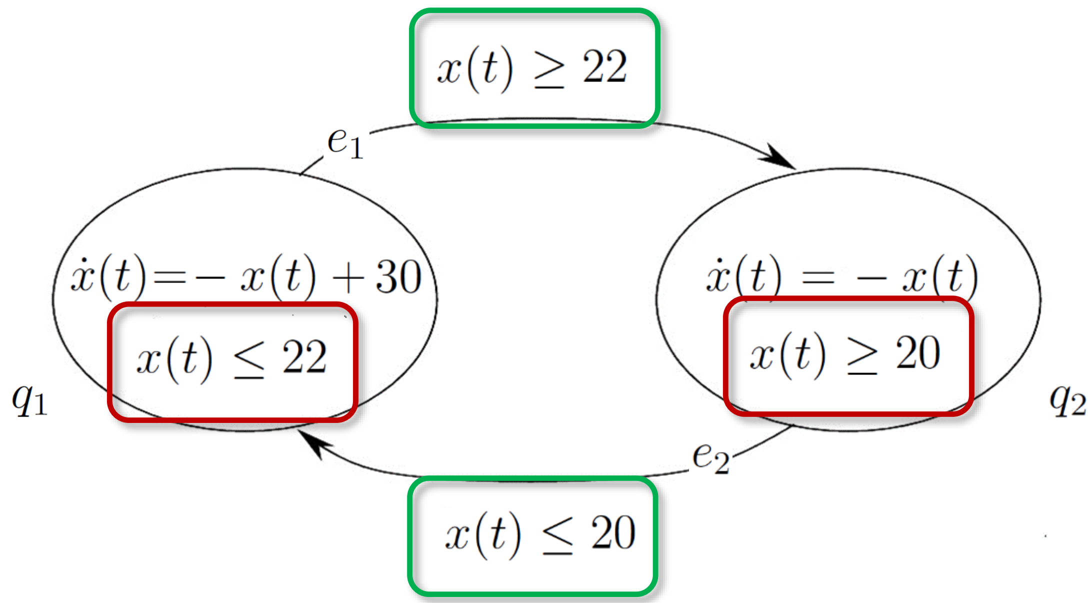

If the state exits the domain (or "invariant set"), then a discrete transition must occur

Guard conditions enable the discrete transition

$\dot{x}(t) = -\,x(t)$ if heater OFF

$\dot{x}(t) = -\,x(t) + 30$ if heater ON

turn the heater from OFF to ON if $x(t) \leq 20$

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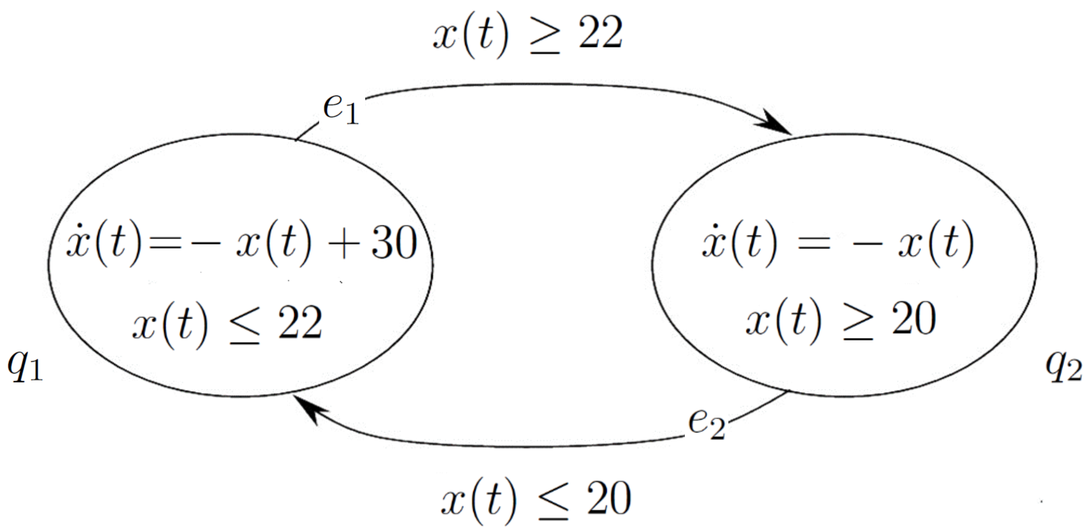

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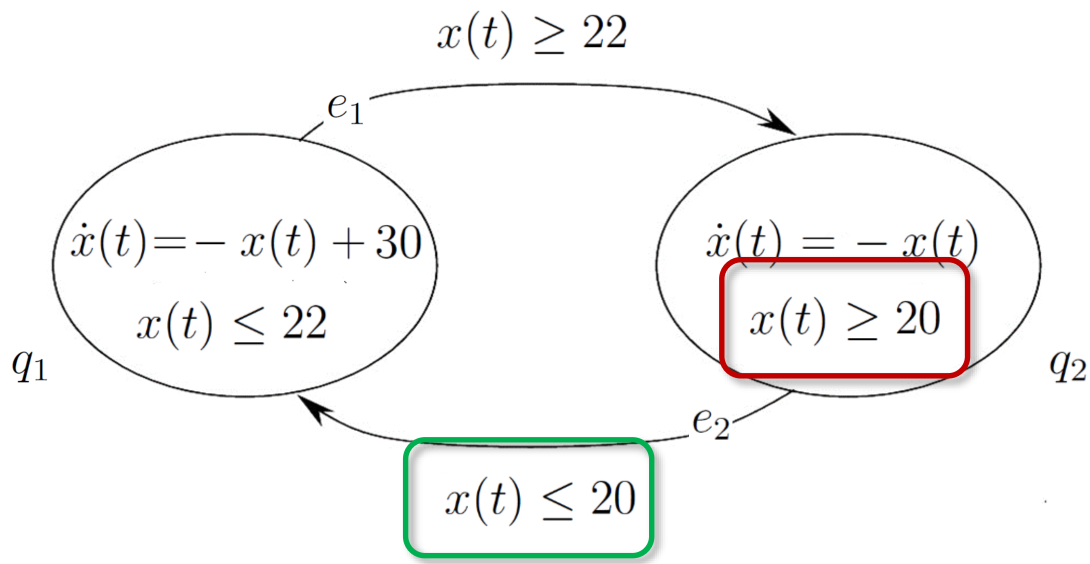

transition from OFF to ON occurs when $x(t) = 20$

$\dot{x}(t) = -x(t)$ if heater OFF

$\dot{x}(t) = -x(t) + 30$ if heater ON

turn the heater from OFF to ON if $x(t) \leq 20$

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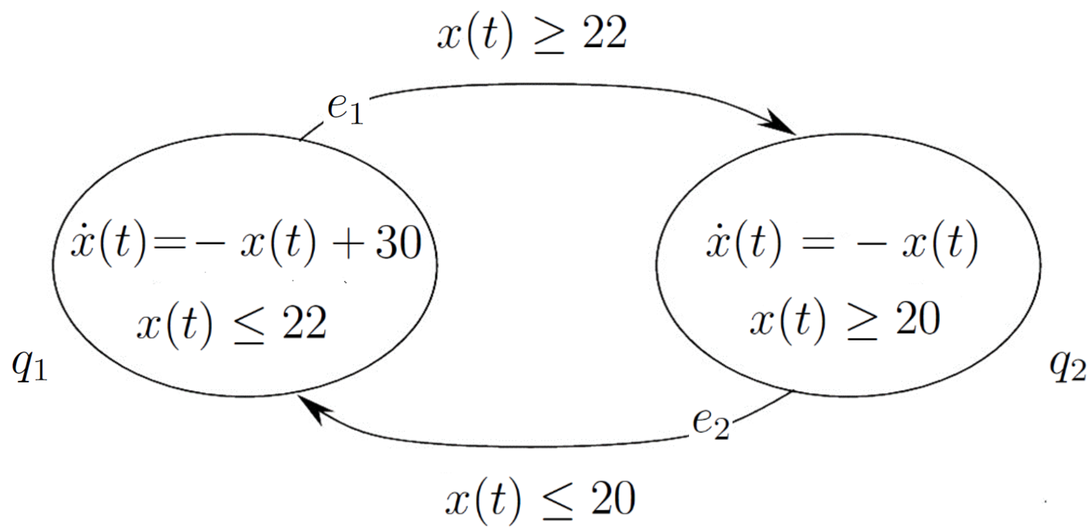

$\dot{x}(t) = -x(t)$ if heater OFF

$\dot{x}(t) = -x(t) + 30$ if heater ON

turn the heater from OFF to ON if $x(t) \leq 20$

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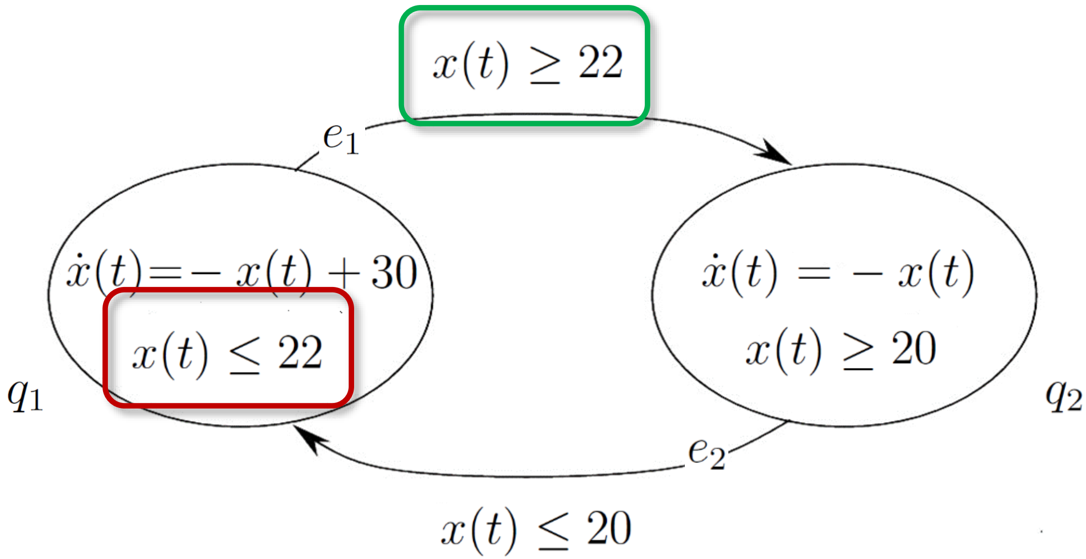

transition from ON to OFF occurs when $x(t) = 22$

$x(t) \leq 22$ $x(t) \geq 22$

$\dot{x}(t) = -x(t)$ if heater OFF

$\dot{x}(t) = -x(t) + 30$ if heater ON

turn the heater from OFF to ON if $x(t) \leq 20$

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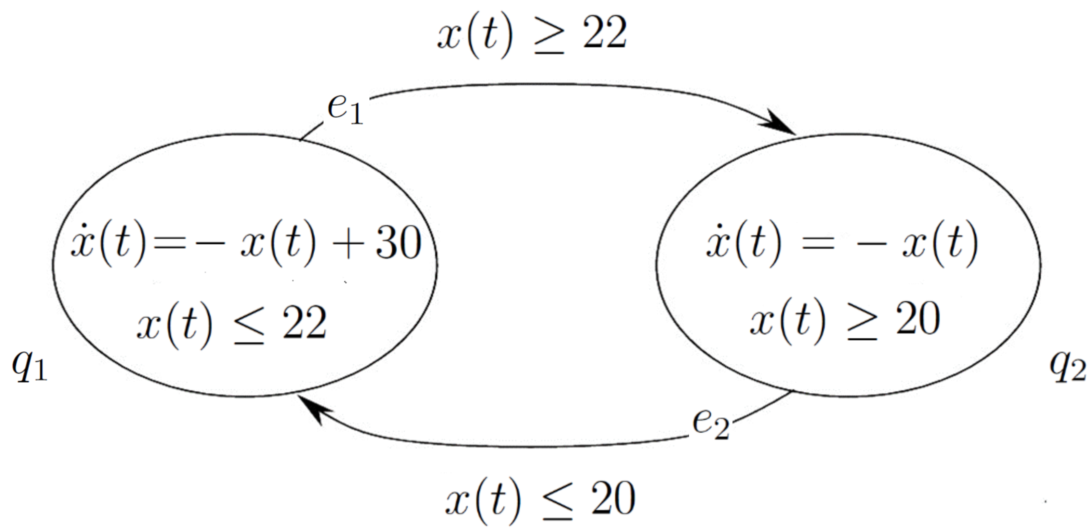

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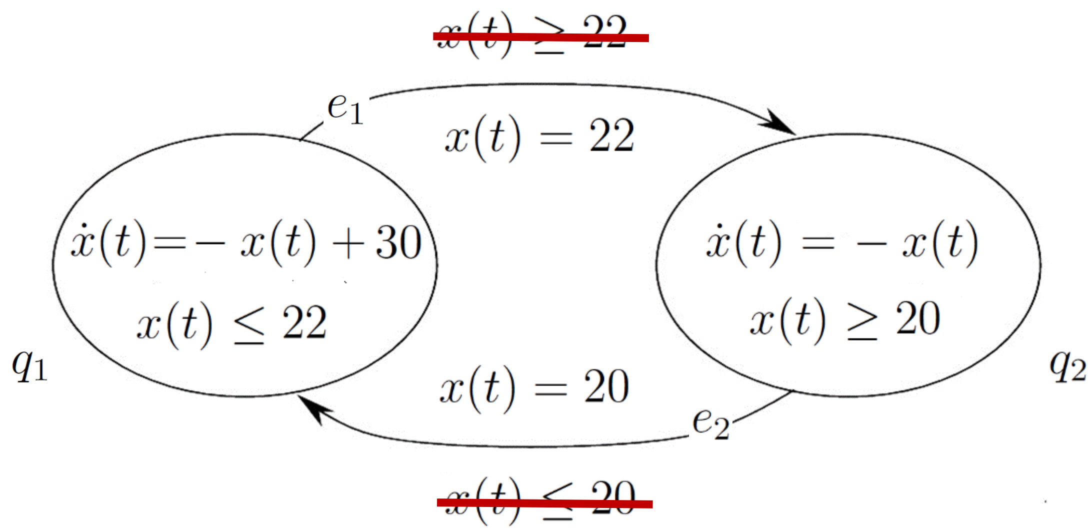
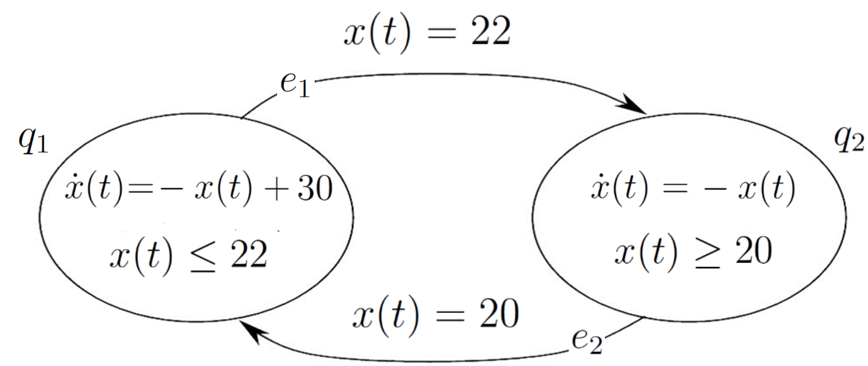


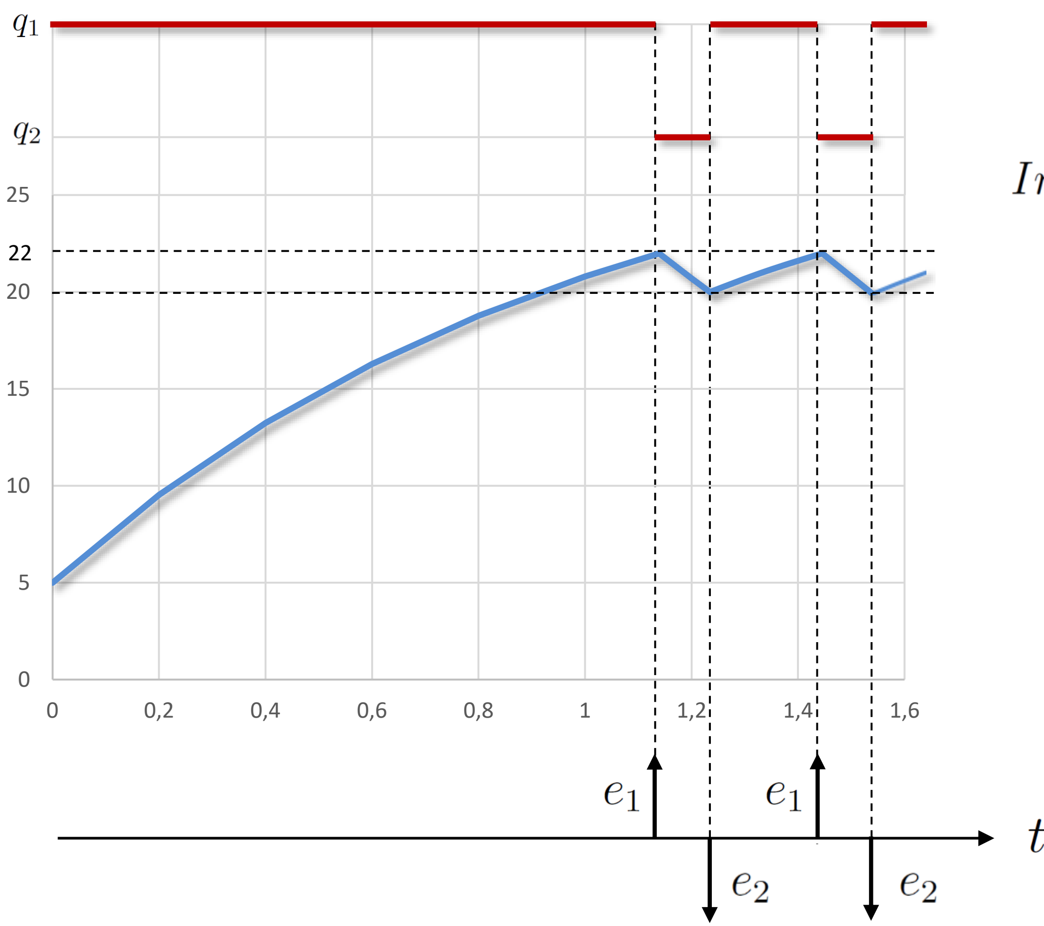
Initialization of the system

$$Init = q_1 \times \{x \leq 22\} \bigcup q_2 \times \{x \geq 20\}$$

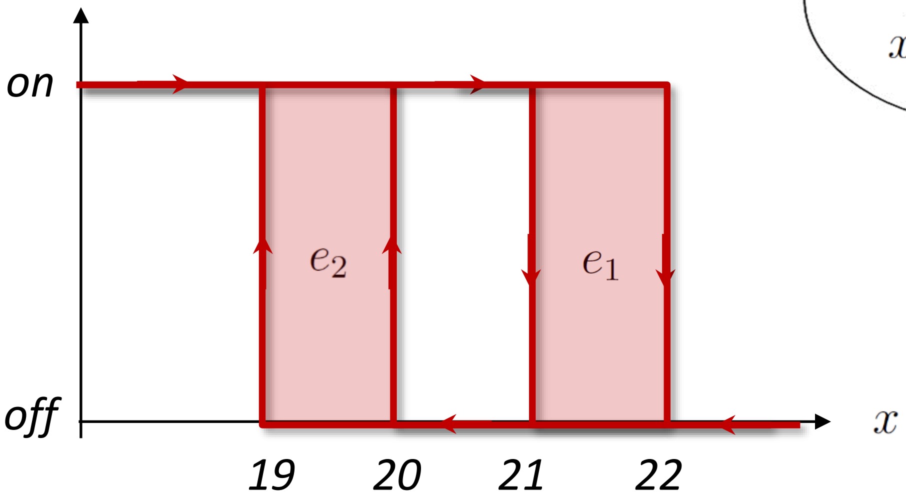
Thermostat

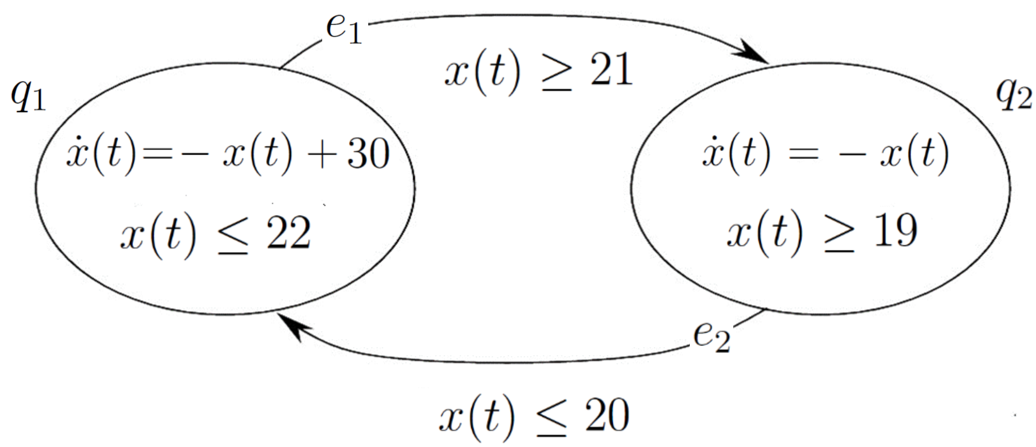


$$Init = q_1 \times \{x \leq 22\} \bigcup q_2 \times \{x \geq 20\}$$



Thermostat control





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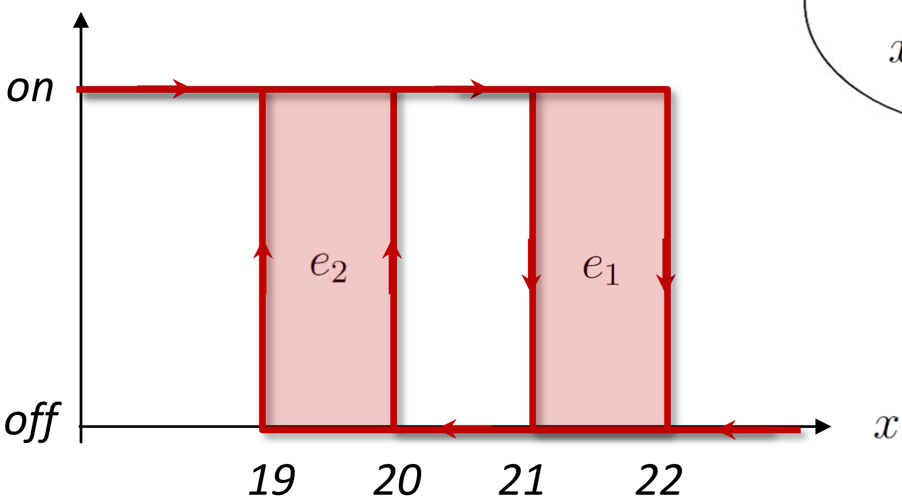
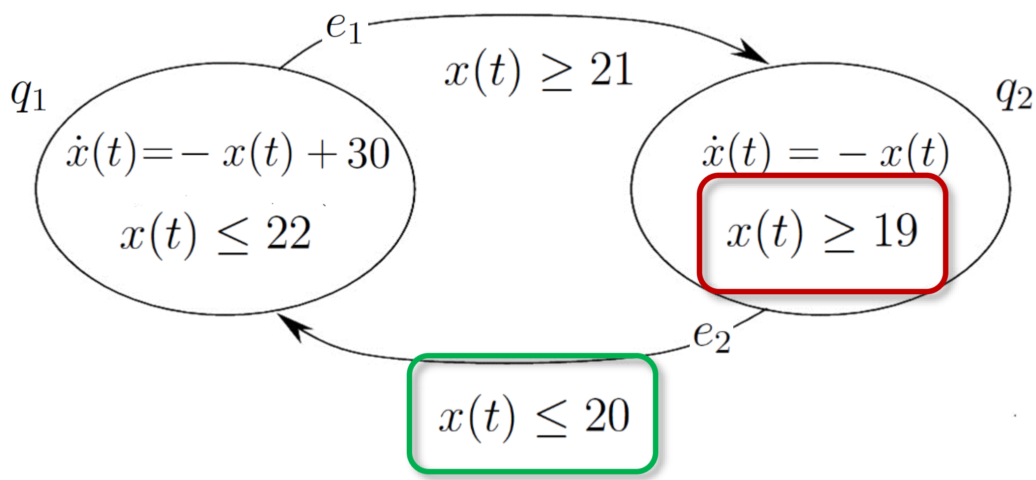
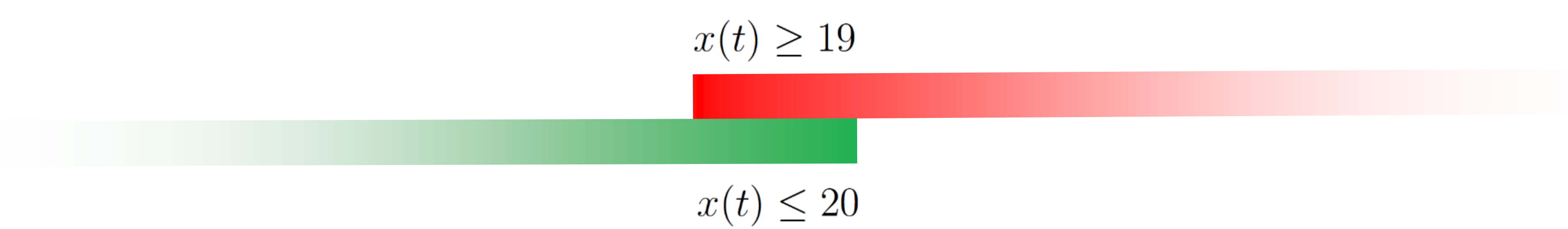




transition from OFF to ON occurs when $19 \leq x(t) \leq 20$



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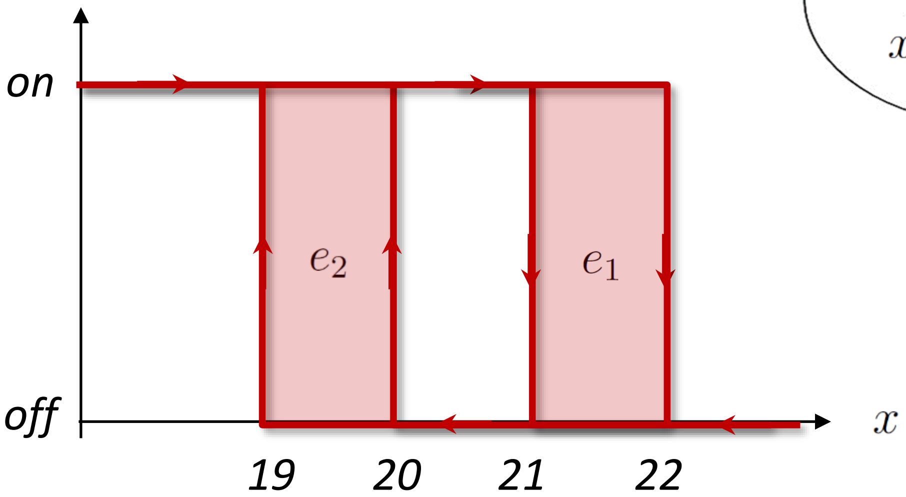
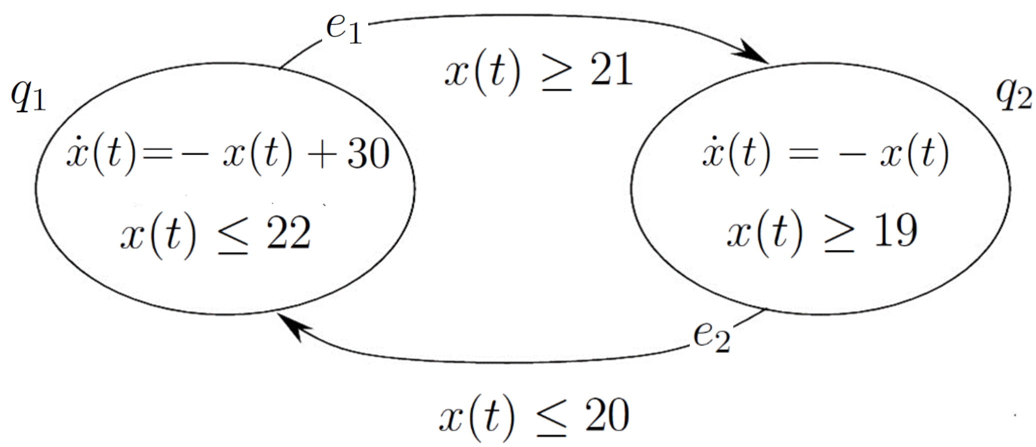
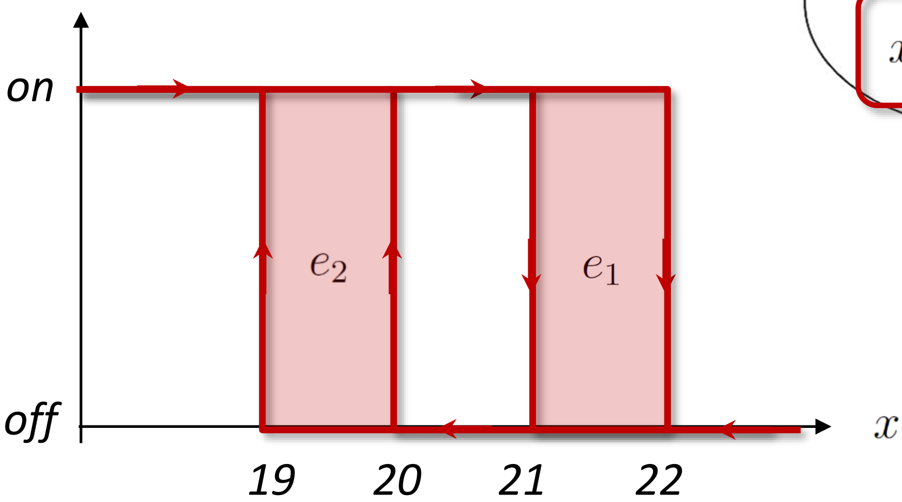
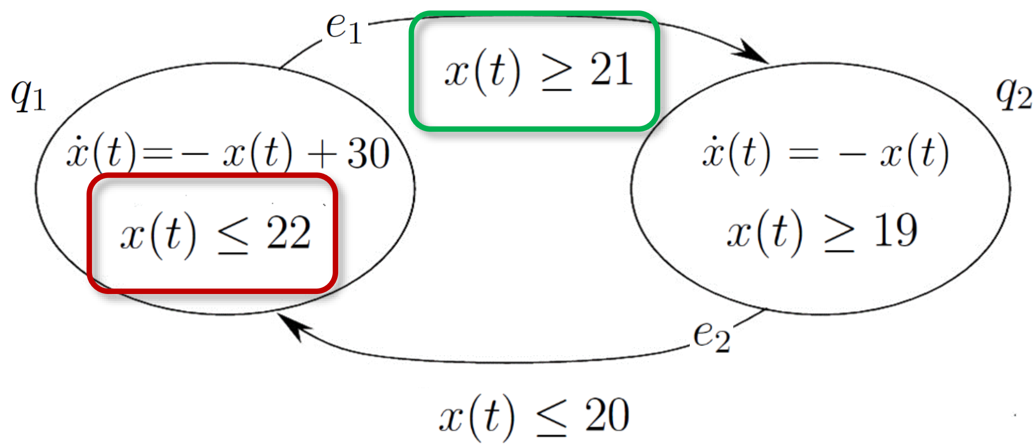


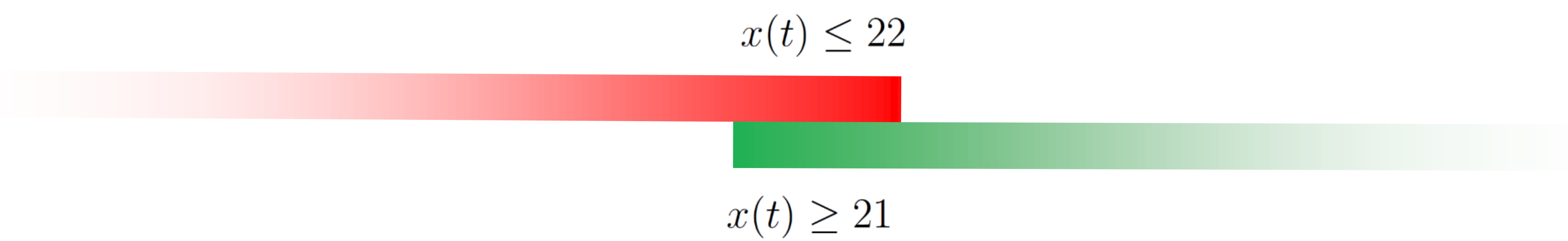


Thermostat control





transition from ON to OFF occurs when $21 \leq x(t) \leq 22$



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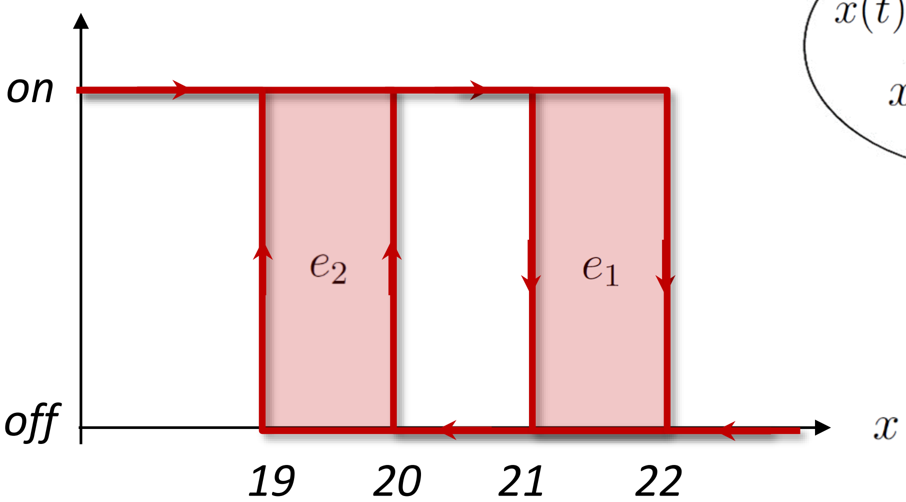
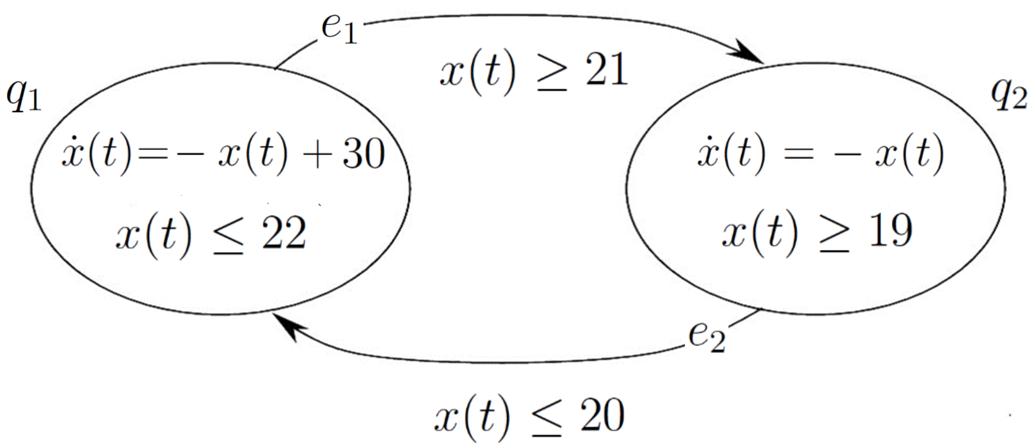
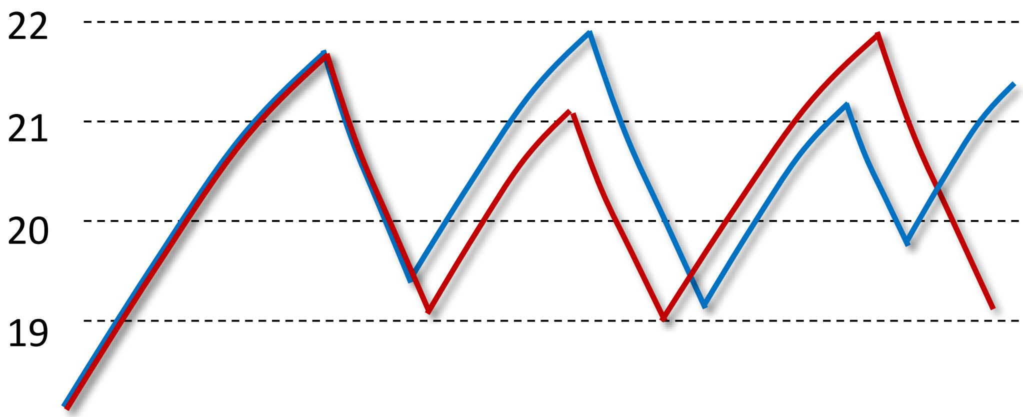




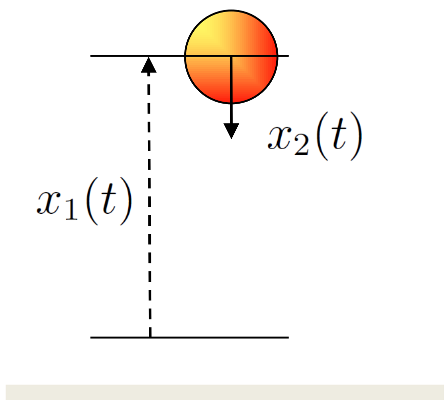


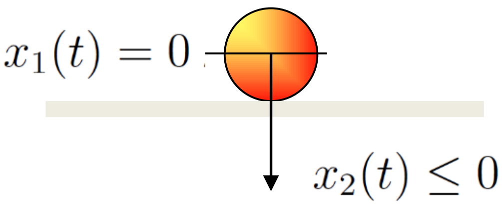
The bouncing ball

2 situations:

a) ball flying in the air

b) ball hitting the ground



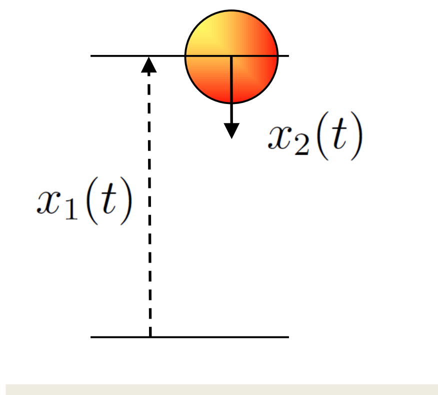


The state of the system is given by position $x_1(t)$ and velocity $x_2(t)$

The bouncing ball

First situation:

a) ball flying in the air



Conditions:

$$x_1(t) > 0 \quad \text{or}$$

$$x_1(t) = 0 \wedge x_2(t) \geq 0$$

Time-driven dynamics:

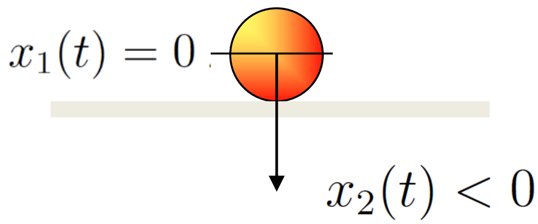
$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -g$$

The bouncing ball

Second situation:

b) ball hitting the ground



Conditions:

$$x_1(t) = 0 \wedge x_2(t) < 0$$

Event-driven dynamics:

$$x_1(t^+) = x_1(t^-) = 0$$

$$x_2(t^+) = -c x_2(t^-)$$

2 situations:

a) ball flying in the air

$x_1(t) > 0$ or

$x_1(t) = 0 \wedge x_2(t) \geq 0$

$\dot{x}_1(t) = x_2(t)$

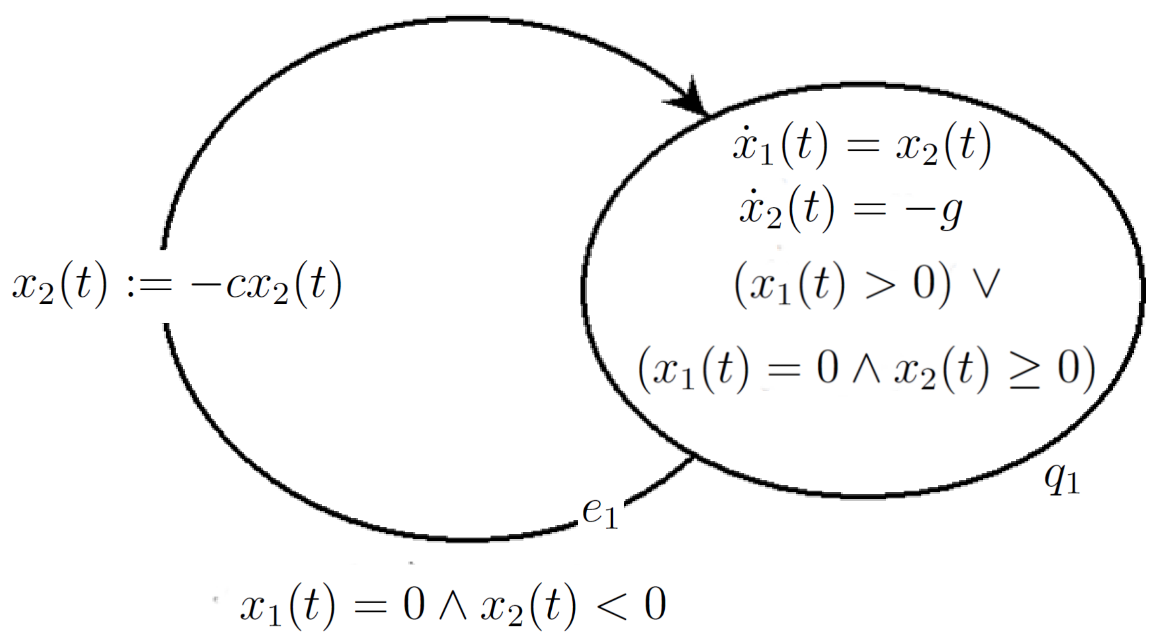
$\dot{x}_2(t) = -g$

b) ball hitting the ground

$x_1(t) = 0 \wedge x_2(t) < 0$

$x_1(t^+) = x_1(t^-) = 0$

$x_2(t^+) = -cx_2(t^-)$



2 situations:

a) ball flying in the air

$x_1(t) > 0$ or

$x_1(t) = 0 \wedge x_2(t) \geq 0$

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -g$$

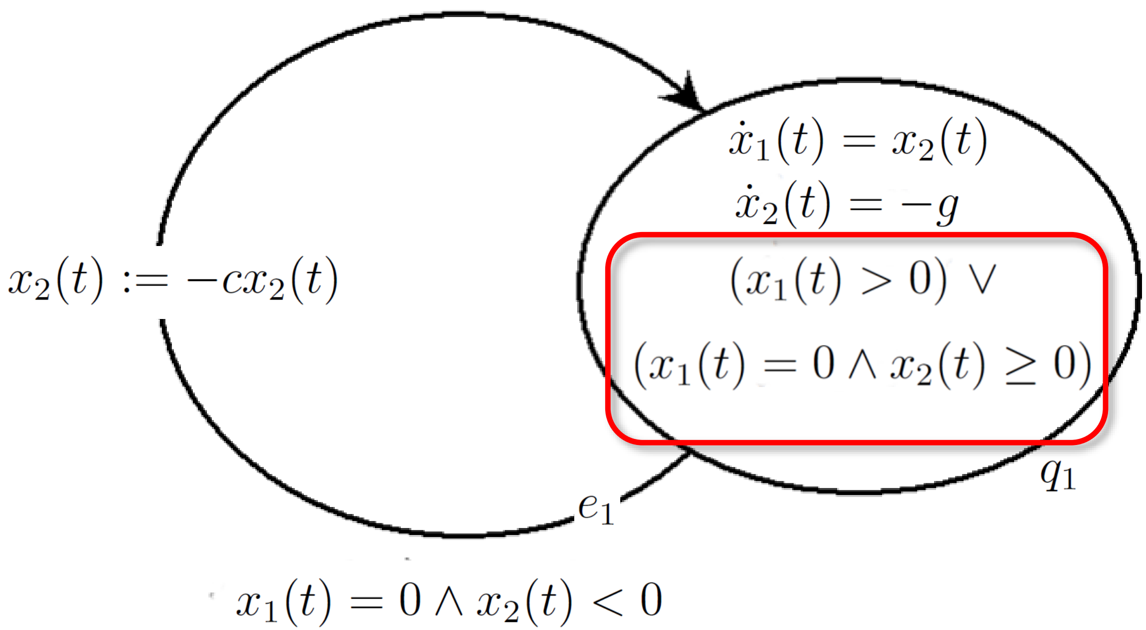
b) ball hitting the ground

$x_1(t) = 0 \wedge x_2(t) < 0$

$$x_1(t^+) = x_1(t^-) = 0$$

$$x_2(t^+) = -cx_2(t^-)$$

If the state exits the domain (or "invariant set"), then a discrete transition must occur



2 situations:

a) ball flying in the air

$x_1(t) > 0$ or

$x_1(t) = 0 \wedge x_2(t) \geq 0$

$\dot{x}_1(t) = x_2(t)$

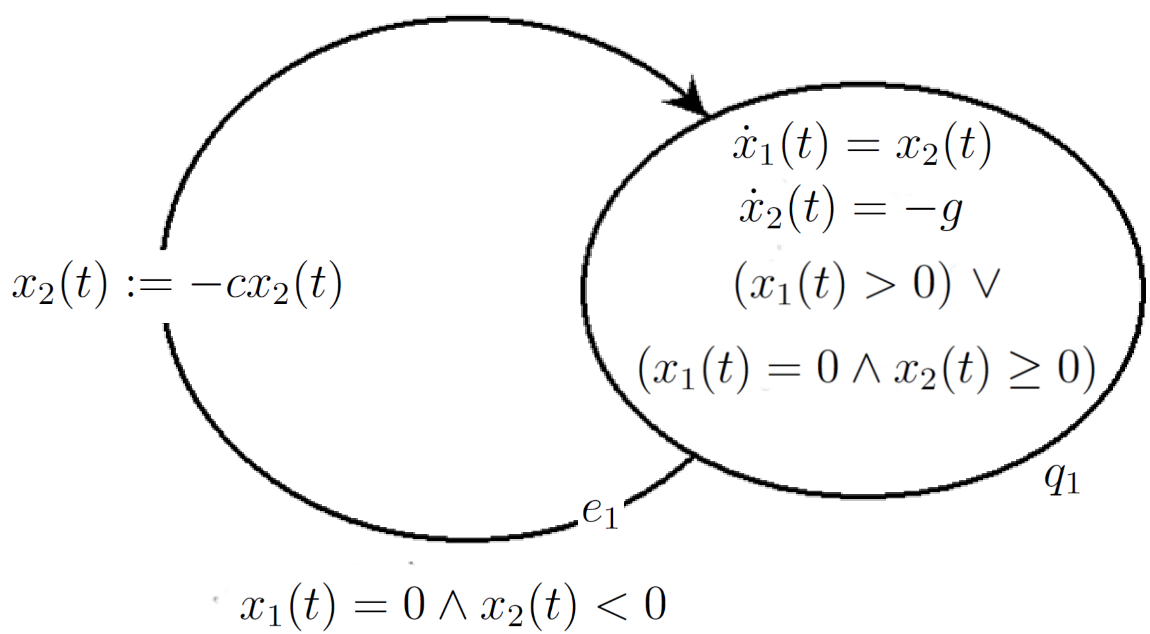
$\dot{x}_2(t) = -g$

b) ball hitting the ground

$x_1(t) = 0 \wedge x_2(t) < 0$

$x_1(t^+) = x_1(t^-) = 0$

$x_2(t^+) = -cx_2(t^-)$



2 situations:

a) ball flying in the air

$x_1(t) > 0$ or

$x_1(t) = 0 \wedge x_2(t) \geq 0$

$$\dot{x}_1(t) = x_2(t)$$

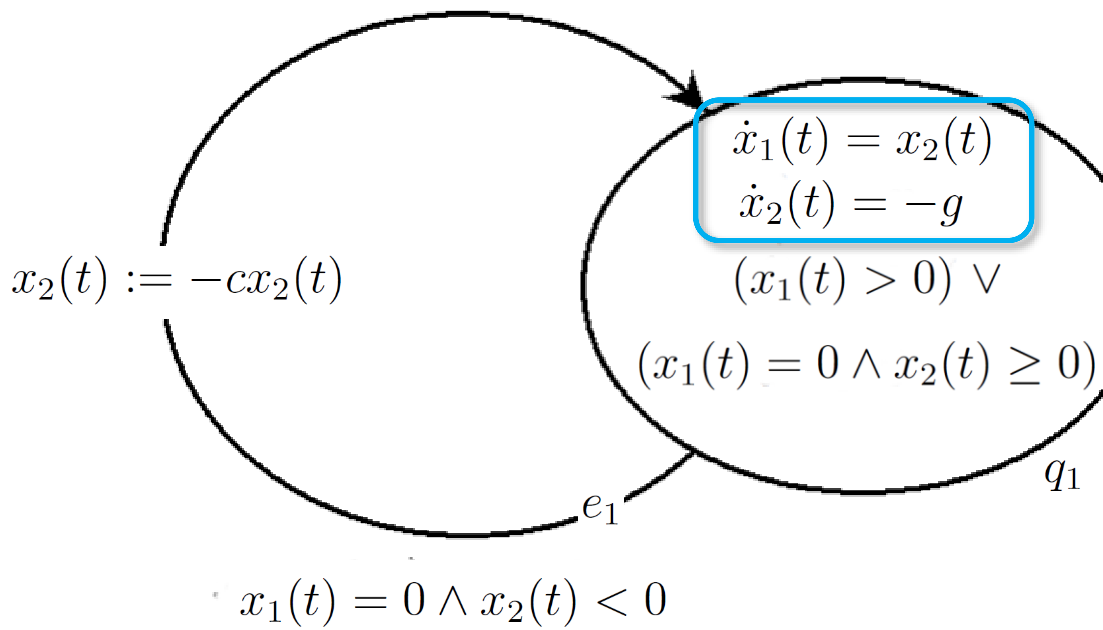
$$\dot{x}_2(t) = -g$$

b) ball hitting the ground

$x_1(t) = 0 \wedge x_2(t) < 0$

$$x_1(t^+) = x_1(t^-) = 0$$

$$x_2(t^+) = -cx_2(t^-)$$



2 situations:

a) ball flying in the air

$x_1(t) > 0$ or

$x_1(t) = 0 \wedge x_2(t) \geq 0$

$\dot{x}_1(t) = x_2(t)$

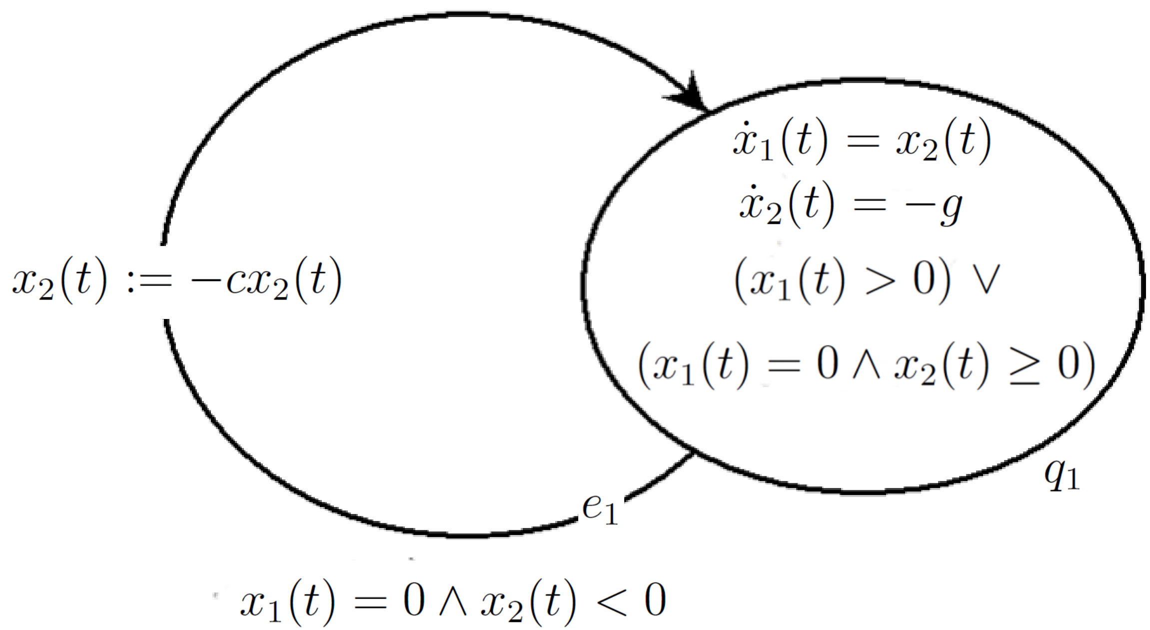
$\dot{x}_2(t) = -g$

b) ball hitting the ground

$x_1(t) = 0 \wedge x_2(t) < 0$

$x_1(t^+) = x_1(t^-) = 0$

$x_2(t^+) = -cx_2(t^-)$



2 situations:

a) ball flying in the air

$$x_1(t) > 0 \quad \text{or}$$

$$x_1(t) = 0 \wedge x_2(t) \geq 0$$

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -g$$

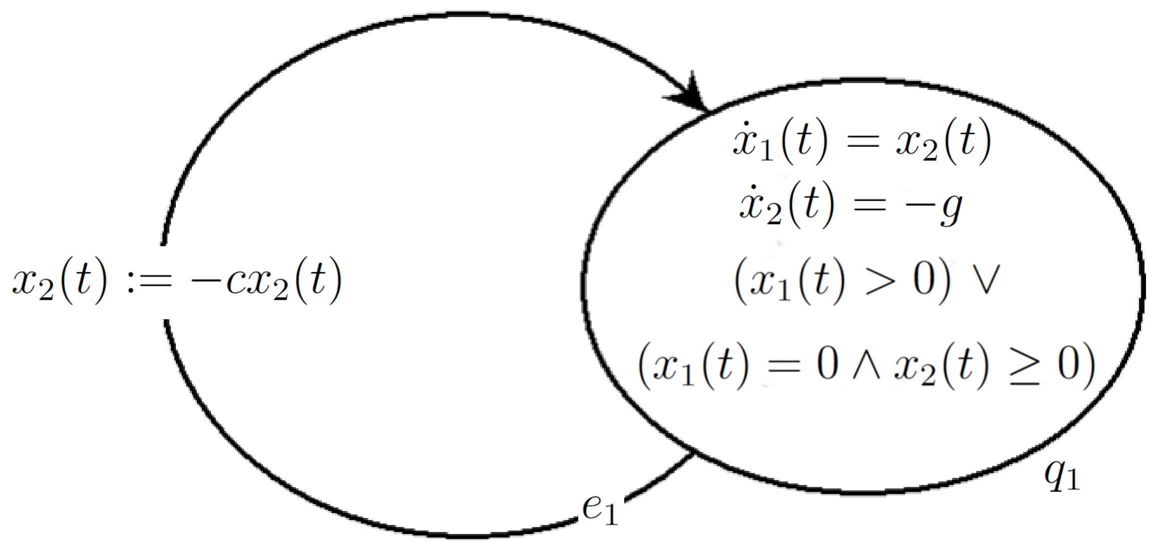
b) ball hitting the ground

$$x_1(t) = 0 \wedge x_2(t) < 0$$

$$x_1(t^+) = x_1(t^-) = 0$$

$$x_2(t^+) = -cx_2(t^-)$$

Guard conditions enable the discrete transition



$$x_1(t) = 0 \wedge x_2(t) < 0$$

2 situations:

a) ball flying in the air

$x_1(t) > 0$ or

$x_1(t) = 0 \wedge x_2(t) \geq 0$

$\dot{x}_1(t) = x_2(t)$

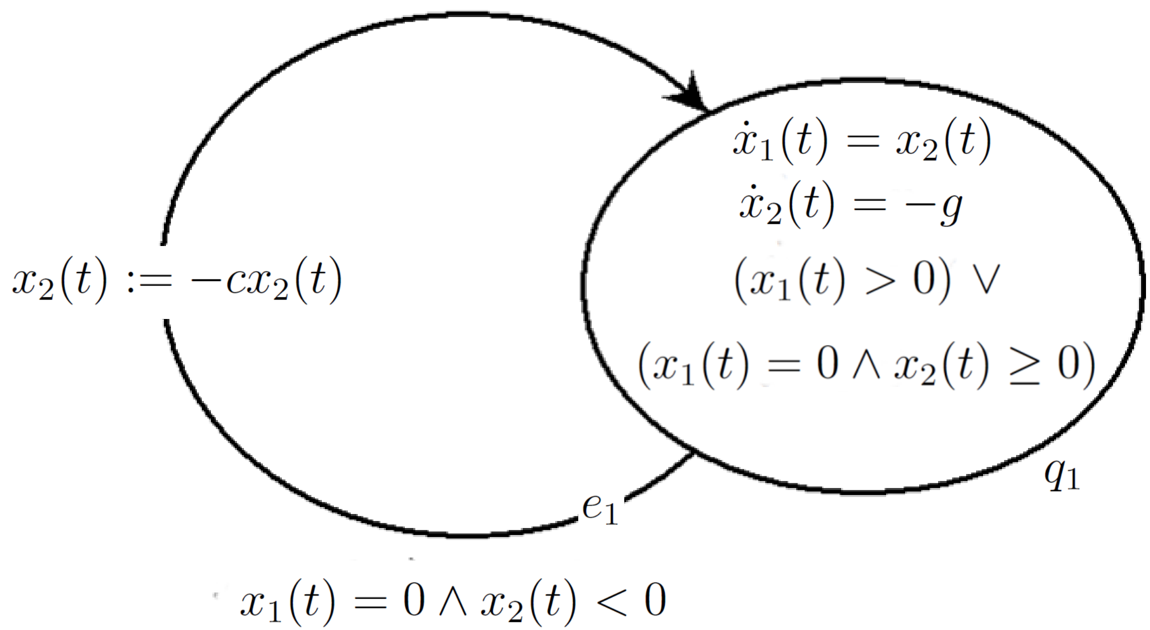
$\dot{x}_2(t) = -g$

b) ball hitting the ground

$x_1(t) = 0 \wedge x_2(t) < 0$

$x_1(t^+) = x_1(t^-) = 0$

$x_2(t^+) = -cx_2(t^-)$



2 situations:

a) ball flying in the air

$$x_1(t) > 0$$

or

$$x_1(t) = 0 \wedge x_2(t) \geq 0$$

$$\dot{x}_1(t) = x_2(t)$$

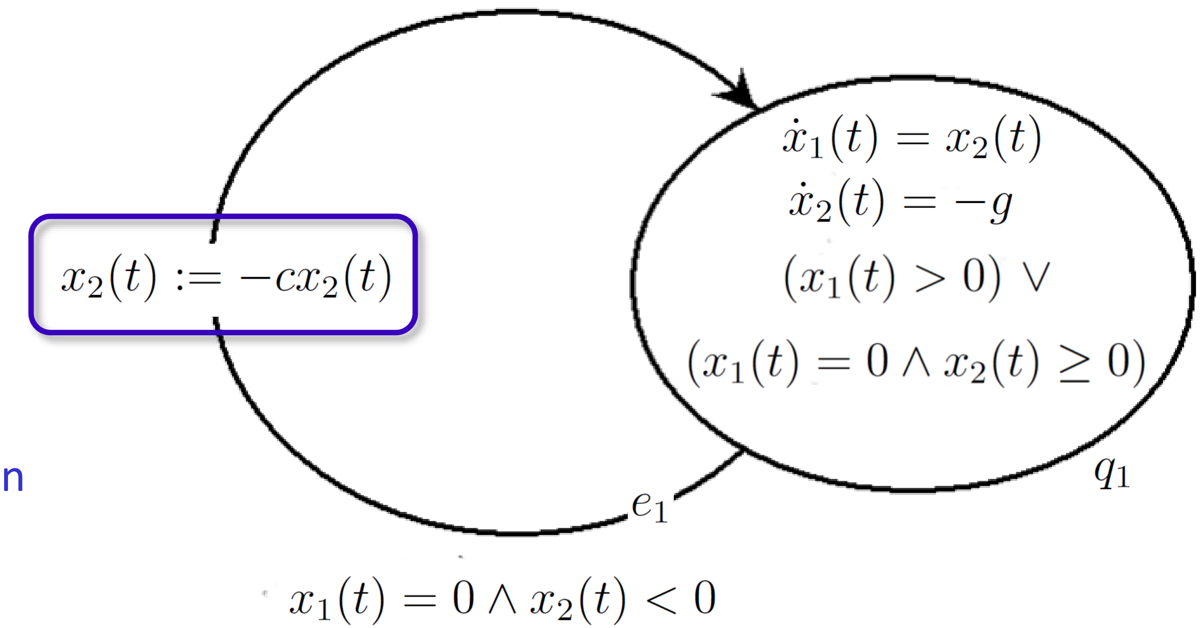
$$\dot{x}_2(t) = -g$$

b) ball hitting the ground

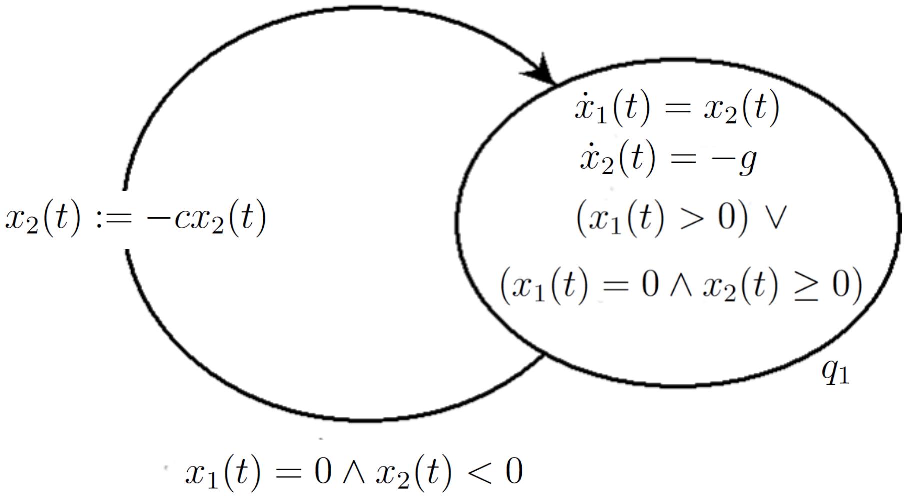
$$x_1(t) = 0 \wedge x_2(t) < 0$$

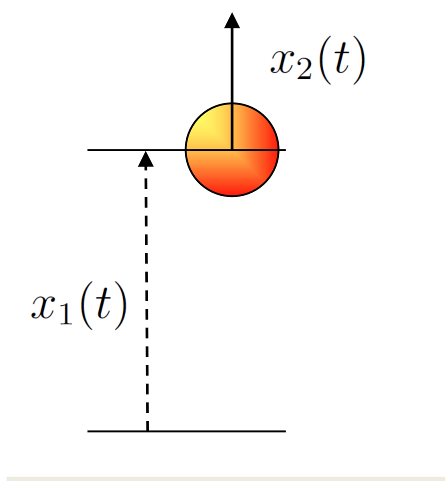
$$x_1(t^+) = x_1(t^-) = 0$$

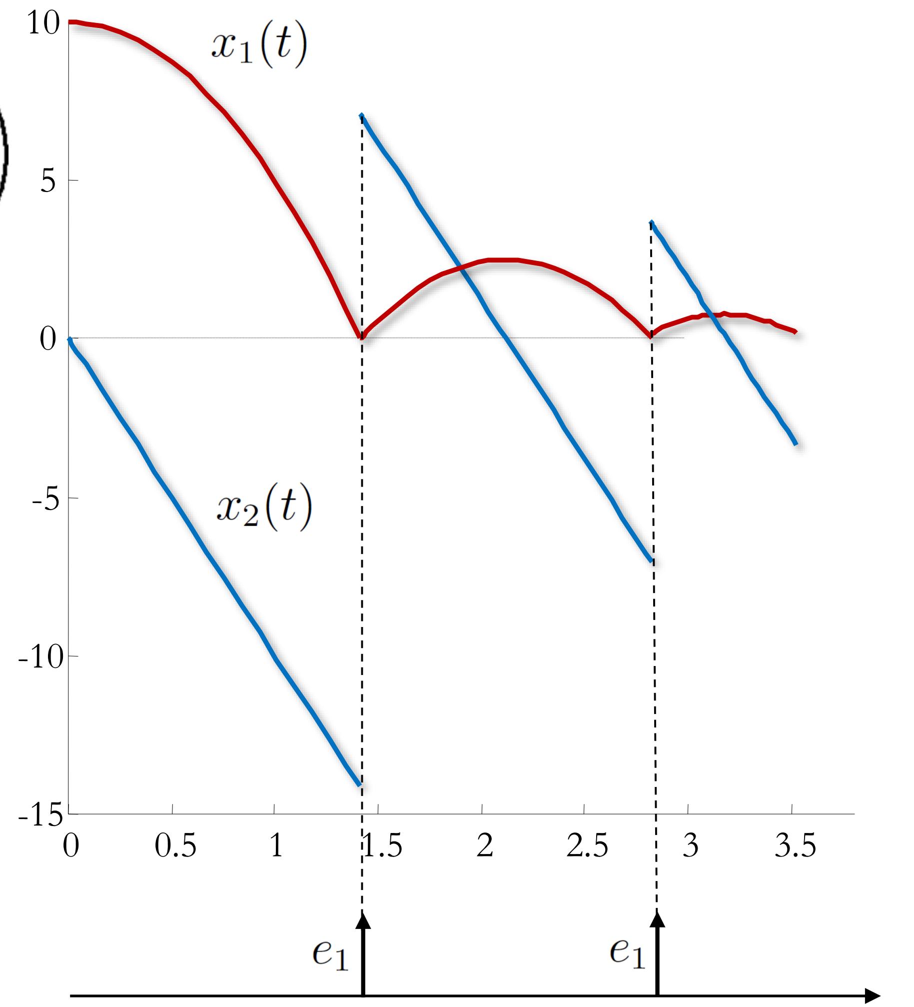
$$x_2(t^+) = -c x_2(t^-)$$



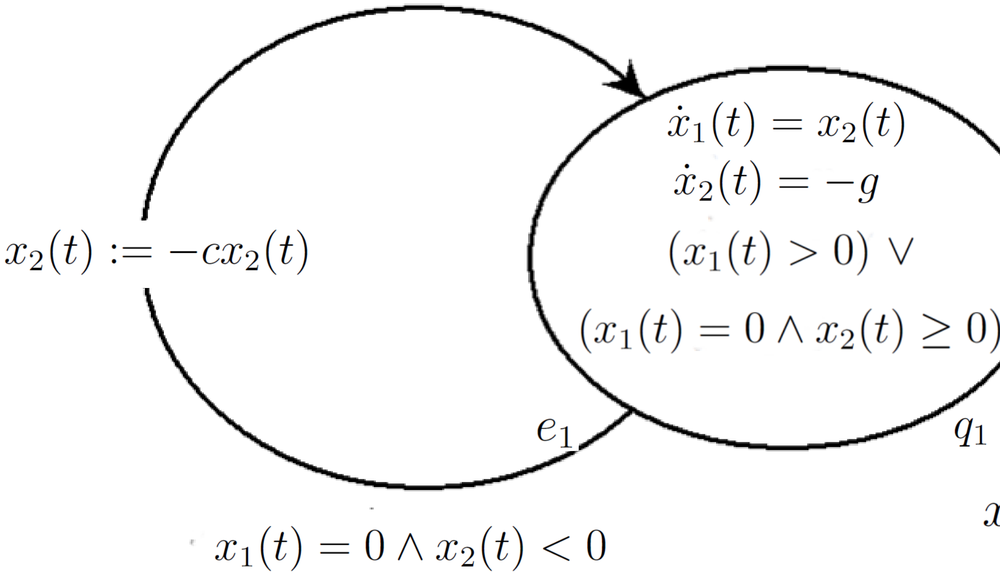
Reset of the continuous state due to the transition

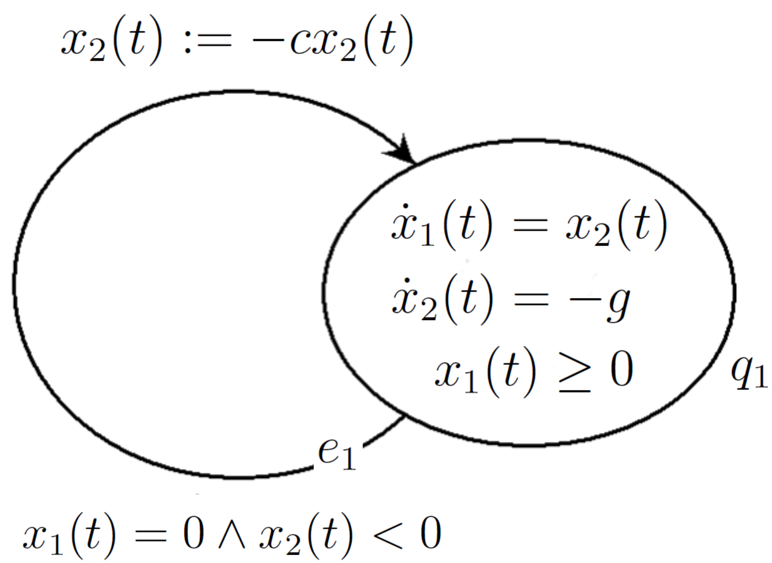

$\dot{x}_1(t) = x_2(t)$
$\dot{x}_2(t) = -g$
$(x_1(t) > 0) \vee$
$(x_1(t) = 0 \wedge x_2(t) \geq 0)$
q_1
$x_2(t) := -cx_2(t)$
$x_1(t) = 0 \wedge x_2(t) < 0$


$x_2(t)$
$x_1(t)$


$x_1(t)$
$x_2(t)$
10
5
0
-5
-10
-15
0
0.5
1
1.5
2
2.5
3
3.5
e_1
e_1
t

The bouncing ball: *simplified hybrid model*





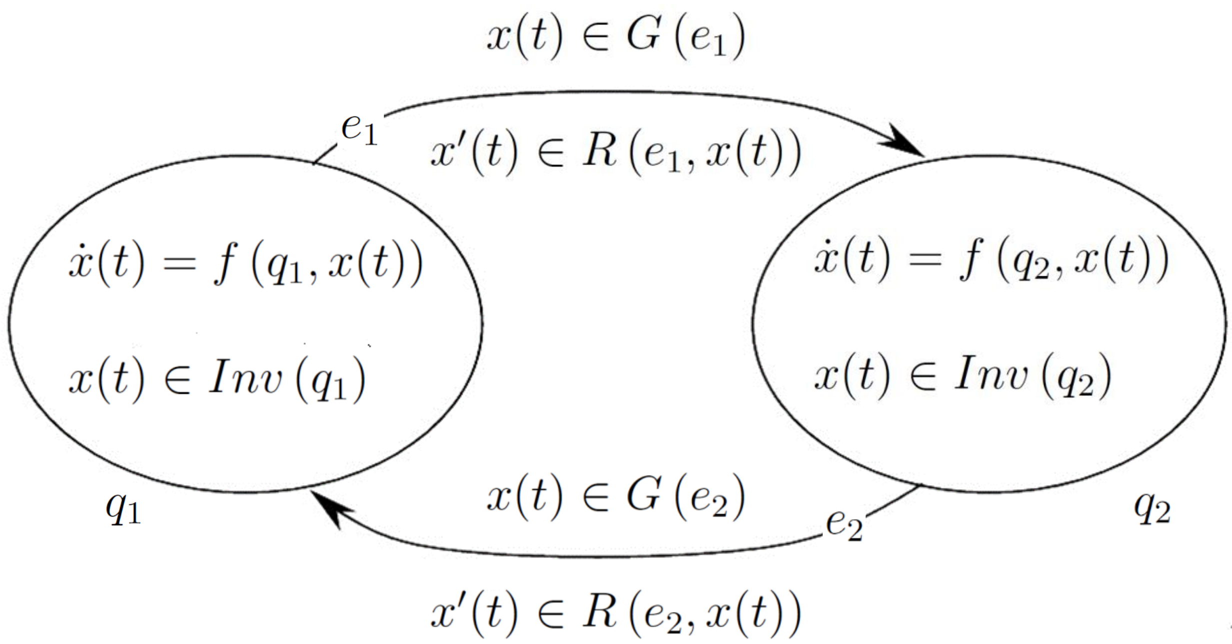
Hybrid system

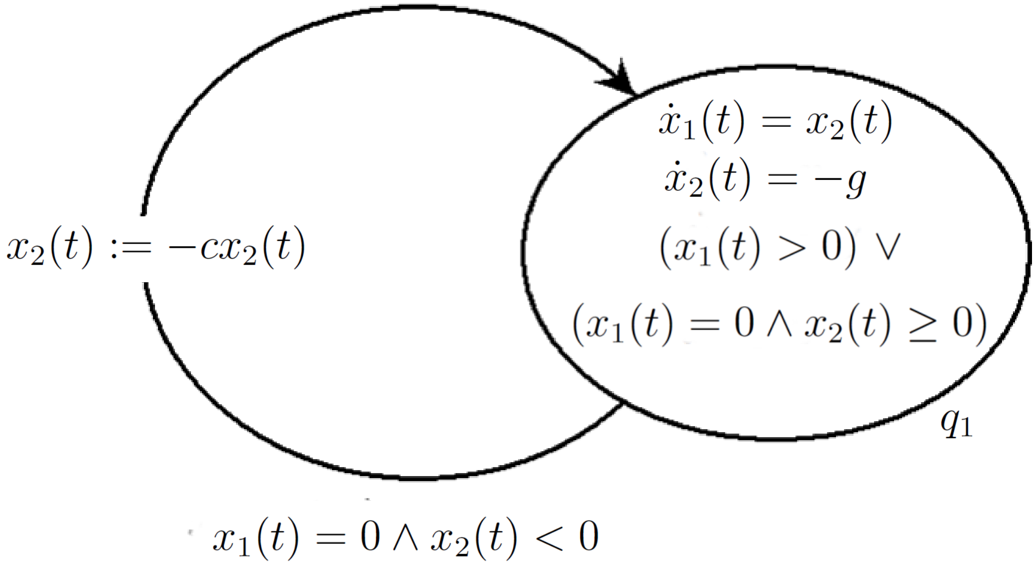
A hybrid system H is a collection

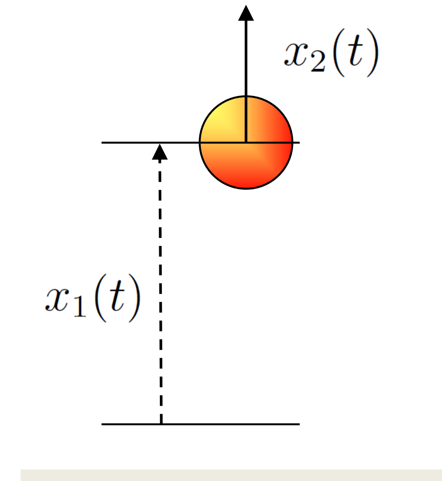
$$H = (Q, X, Init, f, Inv, E, G, R)$$

- $Q = \{q_1, q_2, \ldots\}$ is a set of **discrete states**
- $X = \mathbb{R}^n$ is a set of **continuous states**
- $Init \subseteq Q \times X$ is a set of **initial states**
- $f(\cdot, \cdot) : Q \times X \to \mathbb{R}^n$ is a **vector field**
- $Inv(\cdot) : Q \to 2^X$ is a **domain**

- $E \subseteq Q \times Q$ is a set of **edges**
- $G(\cdot) : E \rightarrow 2^X$ is a **guard condition**
- $R(\cdot, \cdot) : E \times X \rightarrow 2^X$ is a **reset map**







$$H = (Q, X, Init, f, Inv, E, G, R)$$

$$Q = \{q_1\}$$

$$X = \mathbb{R}^2$$

$$Init = q_1 \times \{x_1 \geq 0\}$$

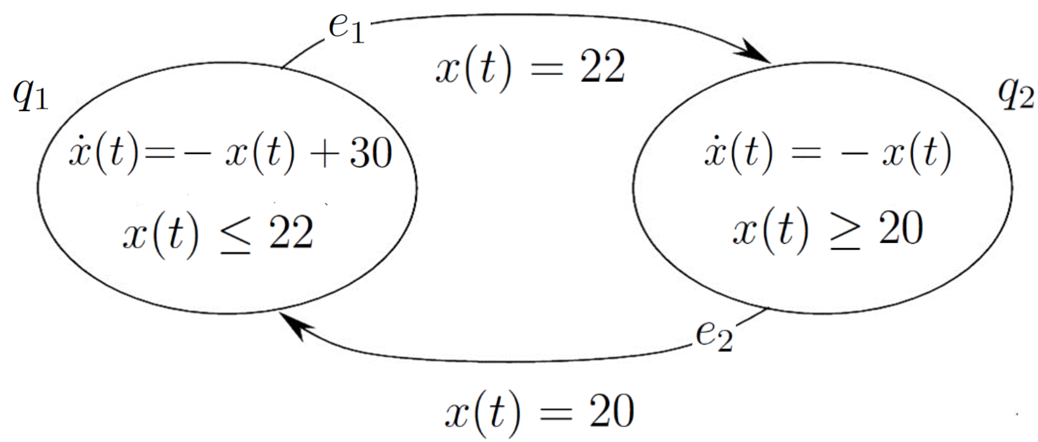
$$f(q_1, x_1, x_2) = \begin{pmatrix} x_2 \\ -g \end{pmatrix}$$

$$Inv(q_1) = \{x_1 > 0\} \bigcup \{x_1 = 0 \wedge x_2 \geq 0\}$$

$$E = \{e_1 = (q_1, q_1)\}$$

$$G(e_1) = \{x_1 = 0 \wedge x_2 < 0\}$$

$$R(e_1, x_1, x_2) = \begin{pmatrix} x_1 \\ -cx_2 \end{pmatrix}$$



$$H = (Q, X, Init, f, Inv, E, G, R)$$

$Q = \{q_1, q_2\}$

$Inv(q_1) = \{x \leq 22\},\ Inv(q_2) = \{x \geq 20\}$

$X = \mathbb{R}$

$E = \{e_1 = (q_1, q_2),\ e_2 = (q_2, q_1)\}$

$Init = q_1 \times \{x \leq 22\} \bigcup q_2 \times \{x \geq 20\}$

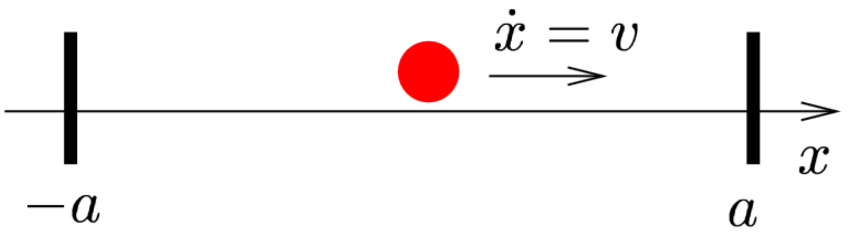
$G(e_1) = \{x = 22\},\ G(e_2) = \{x = 20\}$

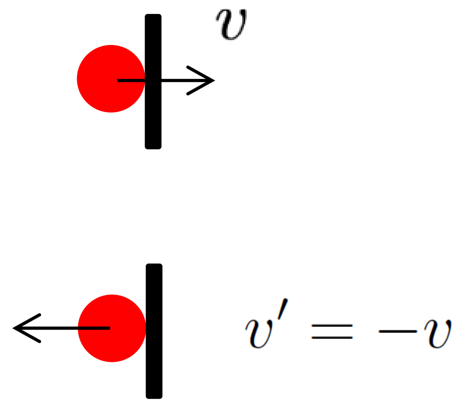
$f(q_1, x) = -x + 30,\ f(q_2, x) = -x$

$R(e_1, x) = R(e_2, x) = x$

Time to exercise

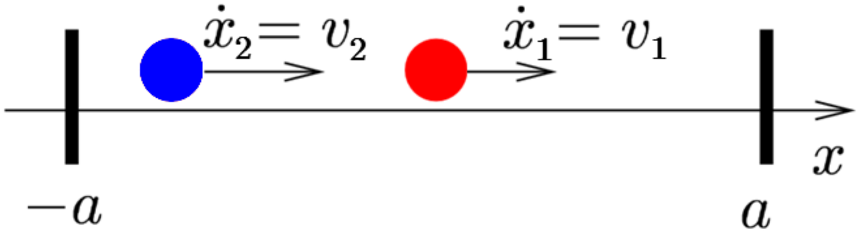
Frictionless movement of a particle in a bounded interval subject to elastic collisions at the end points of the interval itself



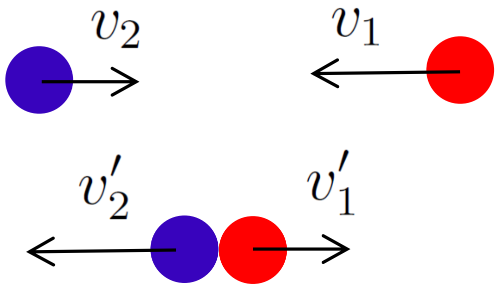


Time to exercise

Frictionless movement of two particles in a bounded interval subject to elastic collisions between them and at the end points of the interval itself

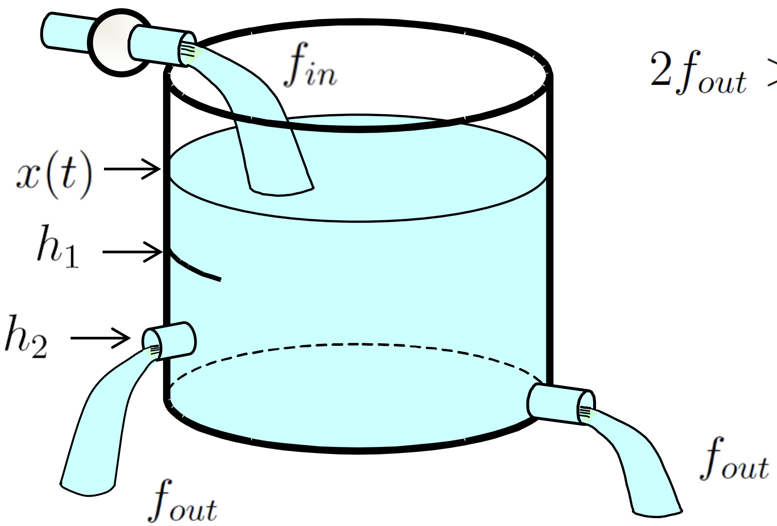


$$v_1' = v_2$$
$$v_2' = v_1$$



Time to exercise

Define the hybrid system for the following phenomenon:



$$2f_{out} > f_{in} > f_{out}$$

$x(t)$ level of water in the tank

f_{in} input flow (constant)

f_{out} output flow (constant) from each output pipe

h_2 height of the second output pipe

h_1 threshold value for the water level

Dynamics of the level

$\dot{x}(t) =$ input flow - output flow

Control logic

Input valve open if $x(t) < h_1$

Input valve closed if $x(t) \geq h_1$

Hybrid time set

A hybrid time set τ is a sequence (finite or infinite) of intervals

$$\tau = \{I_0, I_1, \ldots, I_N\}$$

such that

τ_k represent times of discrete transitions

- $I_k = [\tau_k, \tau'_k]$ for all $k = 0, 1, \ldots, N-1$

consecutive intervals, without gaps

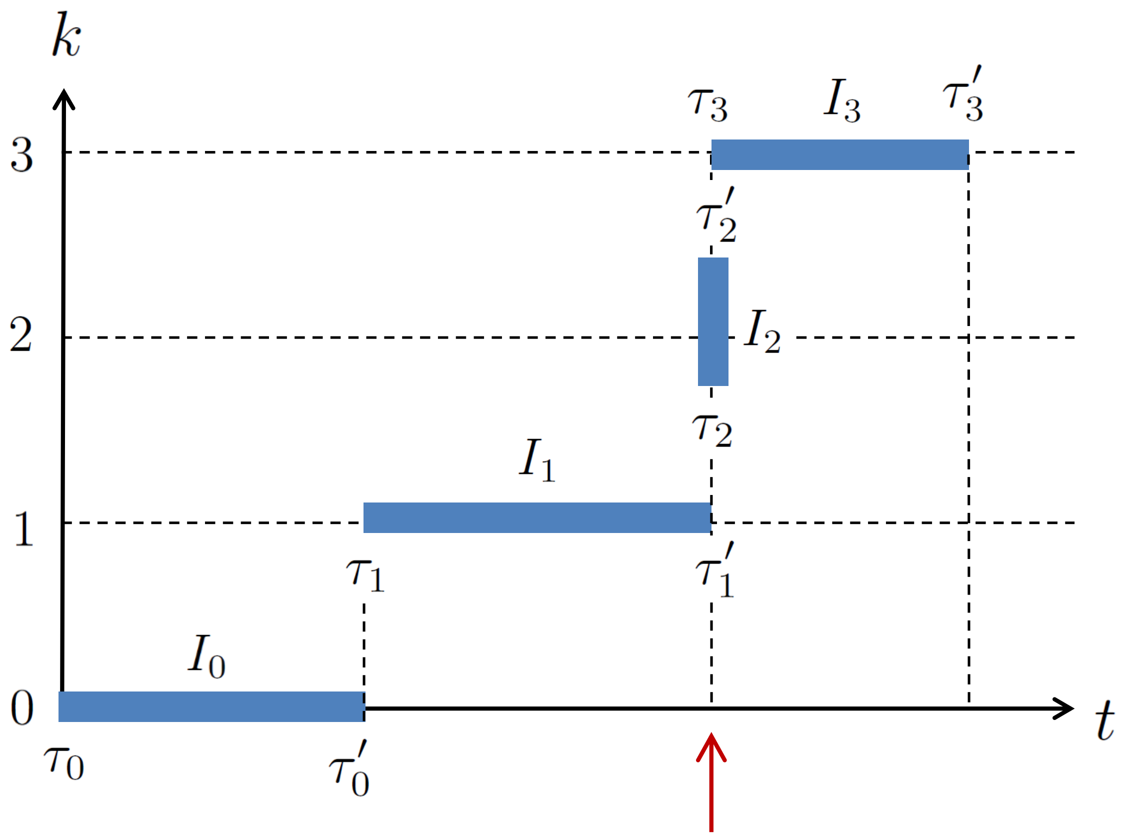
- $\tau_k \leq \tau'_k = \tau_{k+1}$ for all k

intervals can be degenerate to represent multiple transitions at the same time

- if $N < \infty$ then either $I_N = [\tau_N, \tau'_N]$ or $I_N = [\tau_N, \tau'_N)$

Hybrid time set

$\tau = \{I_0, I_1, \ldots, I_N\}$ ***LENGTH***



multiple transitions at this time

- Discrete extent:

$$N + 1$$

number of discrete transitions

- Continuous extent:

$$\sum_{k=0}^{N} (\tau_k' - \tau_k)$$

total duration of intervals in t

Hybrid trajectory

A hybrid trajectory is a triple $\chi = (\tau, q, x)$ where

- τ is a hybrid time set $\tau = \{I_0, I_1, \ldots, I_N\}$
- q is a sequence of functions $q_0(\cdot), q_1(\cdot), \ldots, q_N(\cdot)$

$$q_k(\cdot) : I_k \to Q$$

- x is a sequence of functions $x_0(\cdot), x_1(\cdot), \ldots, x_N(\cdot)$

$$x_k(\cdot) : I_k \to \mathbb{R}^n$$

Hybrid execution

A hybrid execution of a hybrid system

$$H = (Q, X, Init, f, Inv, E, G, R)$$

is a hybrid trajectory $\chi = (\tau, q, x)$ such that

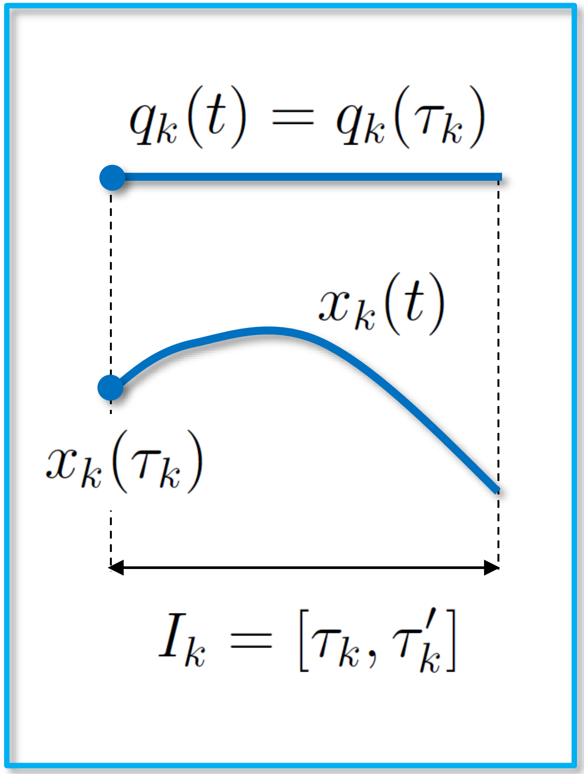
- **Initial condition**

$$(q_0(0), x_0(0)) \in Init$$

- **Continuous evolutions**
  - $q_k(\cdot) : I_k \to Q$ is constant over $t \in I_k$
  - $x_k(\cdot) : I_k \to \mathbb{R}^n$ is the solution of the

    diff. equation $\dot{x}_k(t) = f(q_k(t), x_k(t))$

    starting at $x_k(\tau_k)$
  - $x_k(t) \in Inv(q_k(t))$



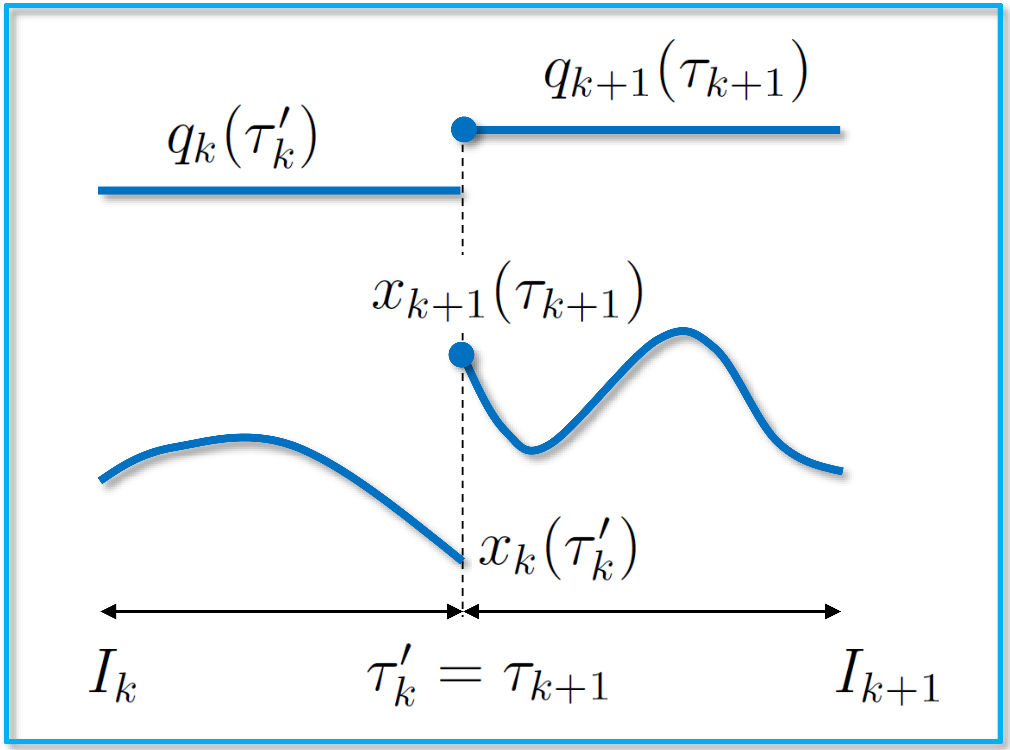
Event-driven transitions

$$H = (Q, X, Init, f, Inv, E, G, R)$$

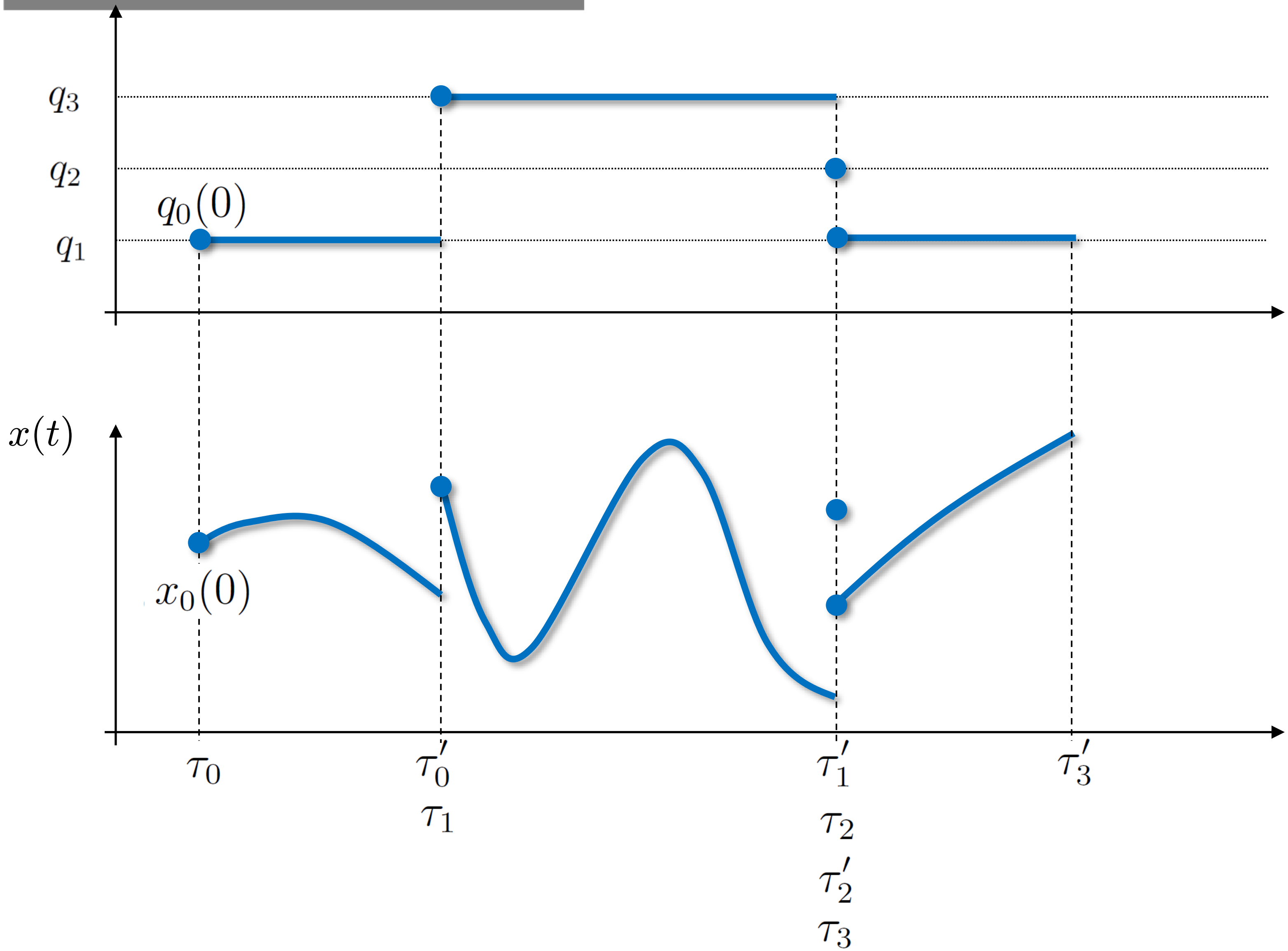
$$(q_k(\tau_k'), q_{k+1}(\tau_{k+1})) \in E$$

$$x_k(\tau_k') \in G\left(q_k(\tau_k'), q_{k+1}(\tau_{k+1})\right)$$

$$x_{k+1}(\tau_{k+1}) \in R\left(q_k(\tau_k'), q_{k+1}(\tau_{k+1}), x_k(\tau_k')\right)$$



Hybrid execution



Execution time: $\tau(\chi) = \sum_{k=0}^{N} (\tau'_k - \tau_k)$

An execution is called

Finite if τ is a finite sequence and the last interval is closed

$$N < \infty \quad \textbf{\textit{and}} \quad I_N = [\tau_N, \tau'_N]$$

Infinite if τ is an infinite sequence, or if the execution time is infinite

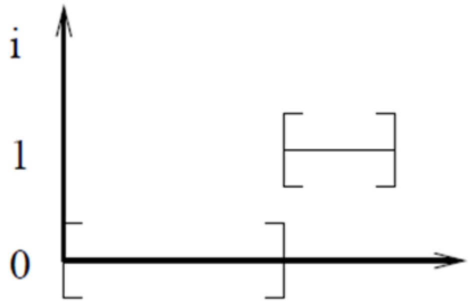
$$N = \infty \quad \textbf{\textit{or}} \quad \tau(\chi) = \infty$$

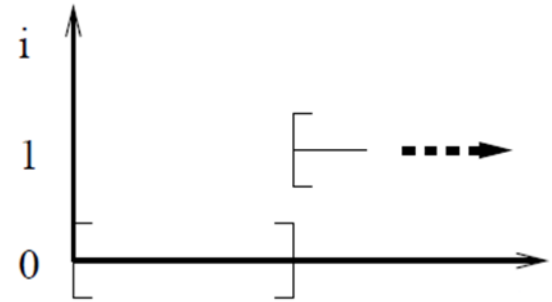
Zeno if τ is an infinite sequence but the execution time is finite

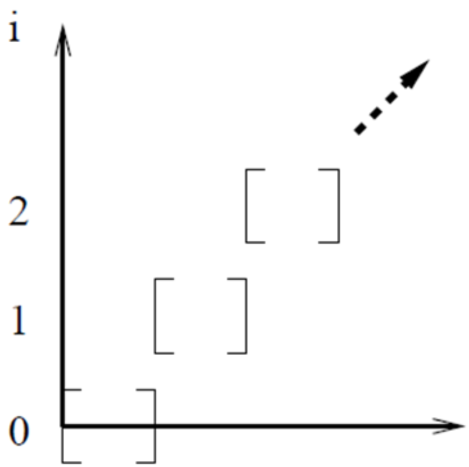
$$N = \infty \quad \textbf{\textit{and}} \quad \tau(\chi) < \infty$$

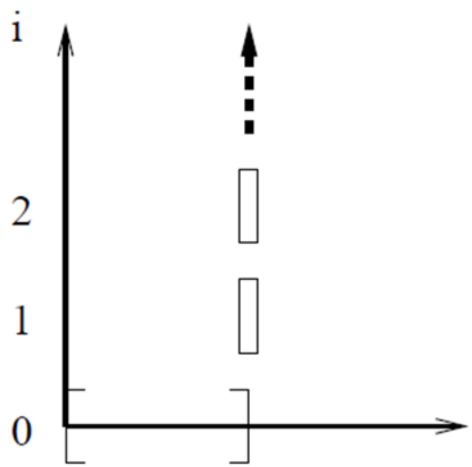
Time to exercise

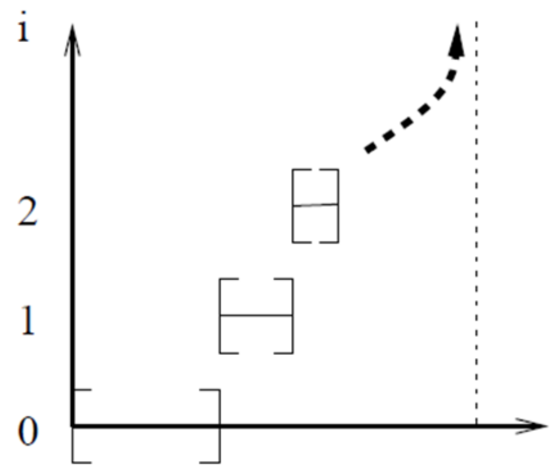
For each execution, determine if it is finite, infinite or Zeno





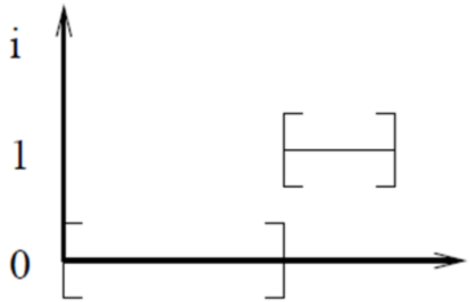




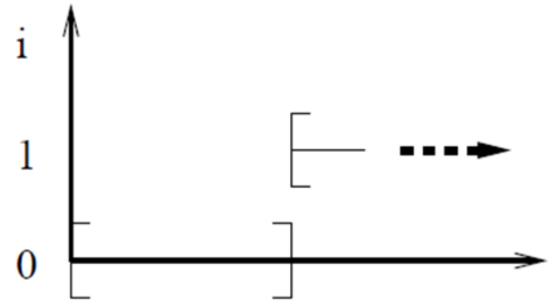


Time to exercise

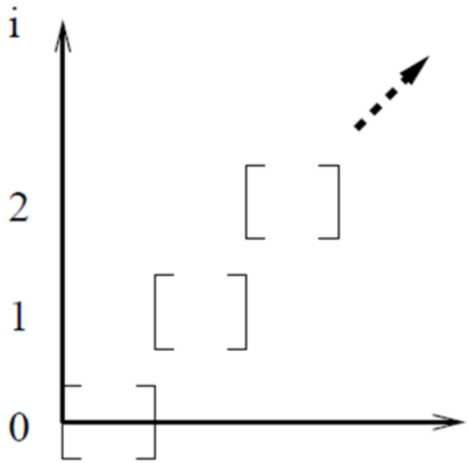
For each execution, determine if it is finite, infinite or Zeno



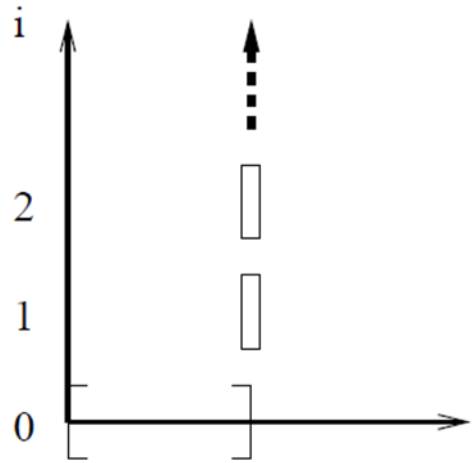
Finite



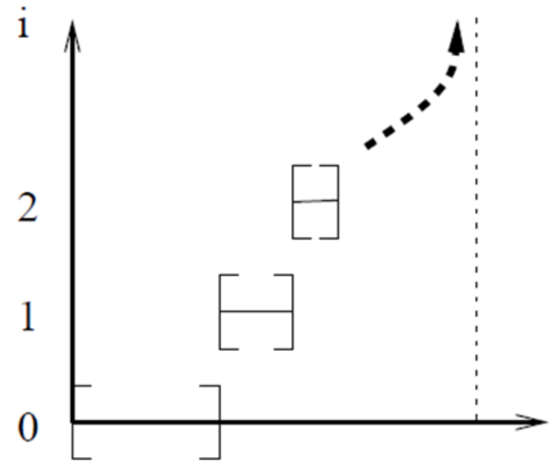
Infinite



Infinite



Zeno



Zeno

Reachable and outside states

A state $(\bar{q}, \bar{x}) \in Q \times X$ of a hybrid system H is reachable if there exists a finite execution ending in $(\bar{q}, \bar{x})$

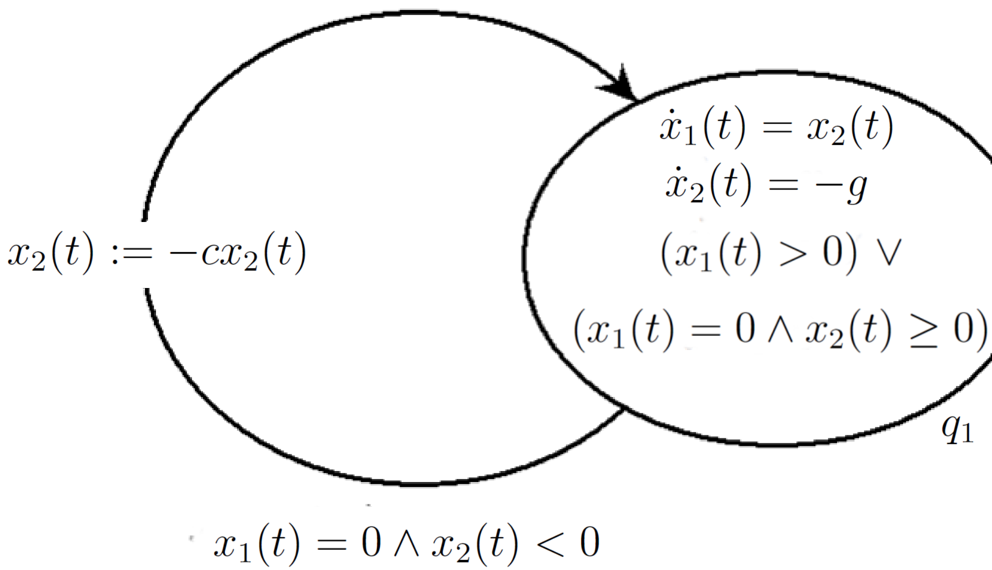
$$Init \subseteq Reach$$

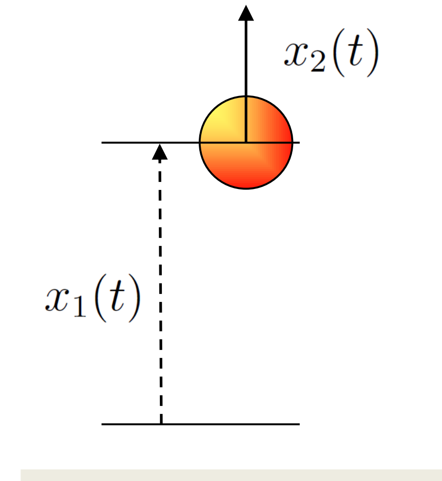
A state $(\bar{q}, \bar{x}) \in Q \times X$ of a hybrid system H is an outside state if continuous evolution from that state forces the system to exit the domain instantaneously.

$$Out = \{(q, x) \in Q \times X | \forall \epsilon > 0,\ \exists t \in [0, \epsilon)\ such\ that\ (q, x(t)) \notin Inv(q)\}$$

$$\bigcup_{q \in Q} \{q\} \times \overline{Inv(q)} \subseteq Out$$

states outside $Inv(q)$





$$Reach = Init = q_1 \times \{x_1 \geq 0\}$$

$$Out = \{q_1 \times \{x_1 < 0\}\} \cup \{q_1 \times \{x_1 = 0, x_2 < 0\}\}$$

Existence of executions

A hybrid system H is called ***non-blocking*** if there exists an infinite execution starting at each initial state $(q, x) \in Init$

A hybrid system H is called ***deterministic*** if at each initial state $(q, x) \in Init$ corresponds a unique execution

A hybrid system H is called ***Zenonian*** if there exist an initial state $(q, x) \in Init$ to which corresponds a zeno execution

$$N = \infty \quad \textbf{\textit{and}} \quad \tau(\chi) < \infty$$

A hybrid system H is ***non-blocking*** if

(A deterministic hybrid system H is ***non-blocking*** if and only if)

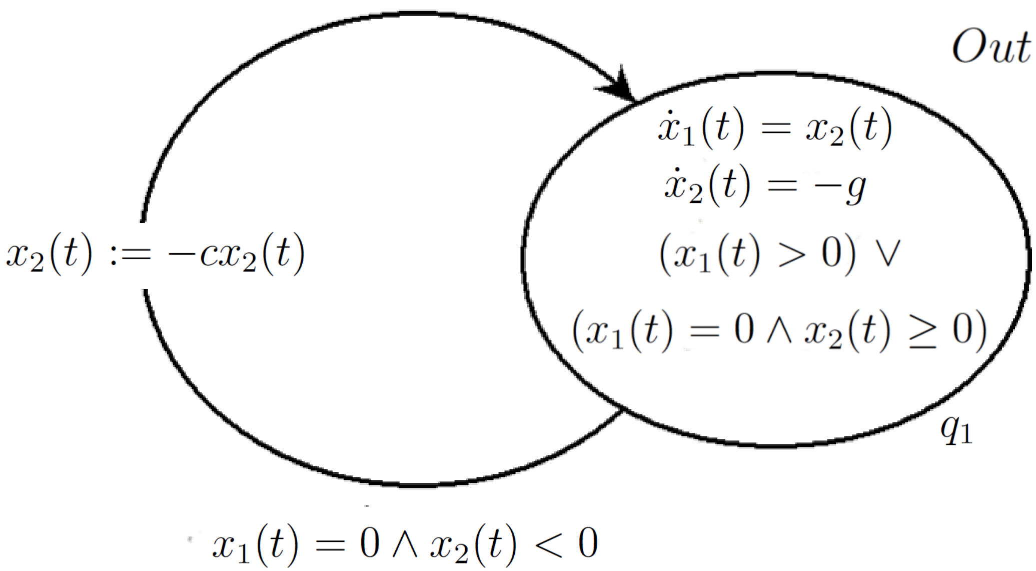
1) $f(q,\cdot)$ is Lipschitz continuous for each $q \in Q$

2) $\forall (q,x) \in Out \cap Reach,\ \exists (q,q') \in E : x \in G(q,q')$

$$Out = \{q_1 \times \{x_1 < 0\}\} \cup \{q_1 \times \{x_1 = 0, x_2 < 0\}\}$$

$$Reach = Init = q_1 \times \{x_1 \geq 0\}$$

$$Out \cap Reach = \{q_1 \times \{x_1 = 0, x_2 < 0\}\}$$



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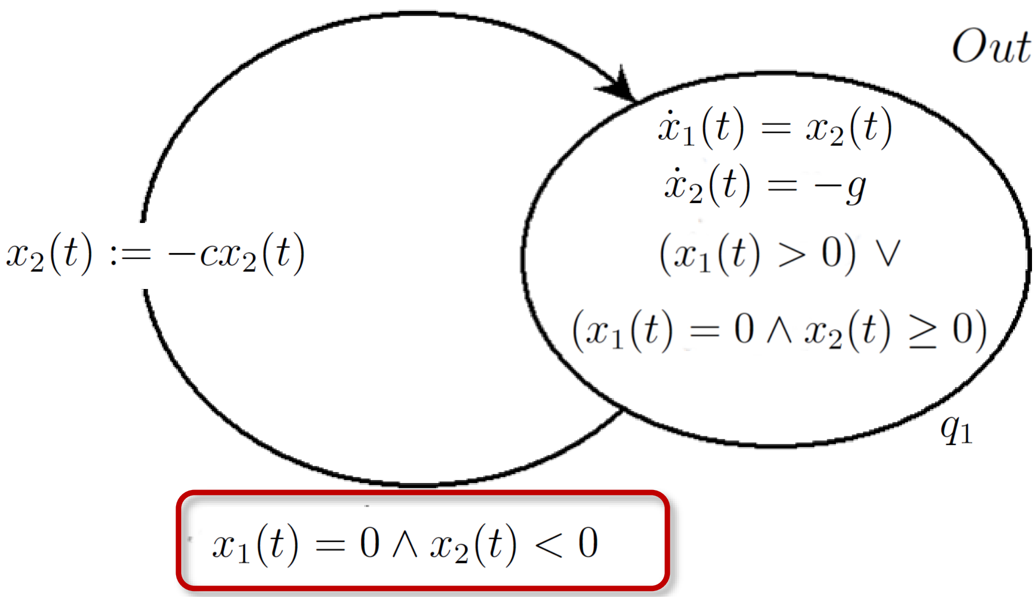
1) $f(q,\cdot)$ is Lipschitz continuous for each $q \in Q$

2) $\forall (q,x) \in Out \cap Reach,\ \exists (q,q') \in E : x \in G(q,q')$

$$Out = \{q_1 \times \{x_1 < 0\}\} \cup \{q_1 \times \{x_1 = 0, x_2 < 0\}\}$$

$$Reach = Init = q_1 \times \{x_1 \geq 0\}$$

$$Out \cap Reach = \{q_1 \times \{x_1 = 0, x_2 < 0\}\}$$

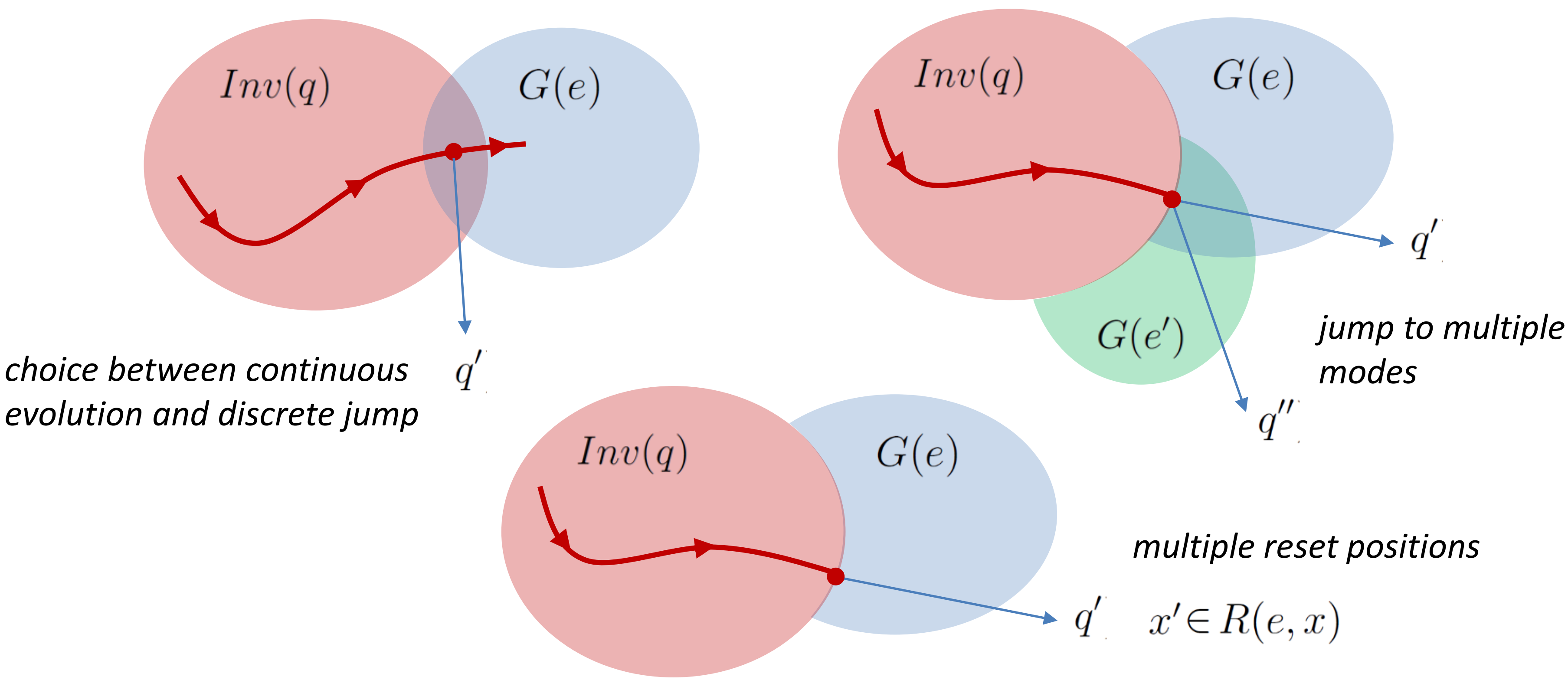


A hybrid system H is ***deterministic*** if and only if $\forall (q, x) \in Reach$

1) If $x \in G(e)$ for some $e = (q, q') \in E$, then $(q, x) \in Out$

2) If $e = (q, q') \in E$ and $e' = (q, q'') \in E$, then $x \notin G(e) \cap G(e')$

3) If $e = (q, q') \in E$ and $x \in G(e)$, then $R(e, x)$ contains a single element



Existence and uniqueness of executions

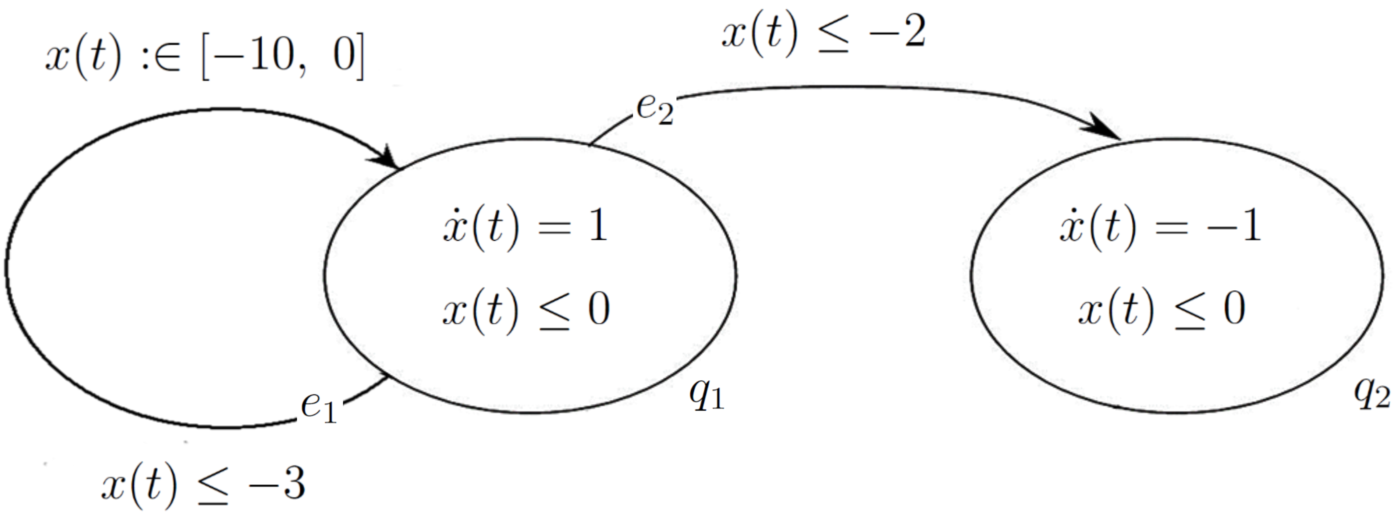
A hybrid automaton has a unique infinite execution for each initial state if it is non-blocking and deterministic.

- $f(q, \cdot)$ is Lipschitz continuous for each $q \in Q$
- $\forall (q, x) \in Out \cap Reach,\ \exists (q, q') \in E : x \in G(q, q')$

This concerns with global existence of a solution, not uniqueness!

- If $x \in G(e)$ for some $e = (q, q') \in E$, then $(q, x) \in Out$
- If $e = (q, q') \in E$ and $e' = (q, q'') \in E$, then $x \notin G(e) \cap G(e')$
- If $e = (q, q') \in E$ and $x \in G(e)$, then $R(e, x)$ contains a single element

Time to exercise



$Init = q_1 \times \{x \in [-10,\ 0]\}$

Is the hybrid system ***deterministic*** ?

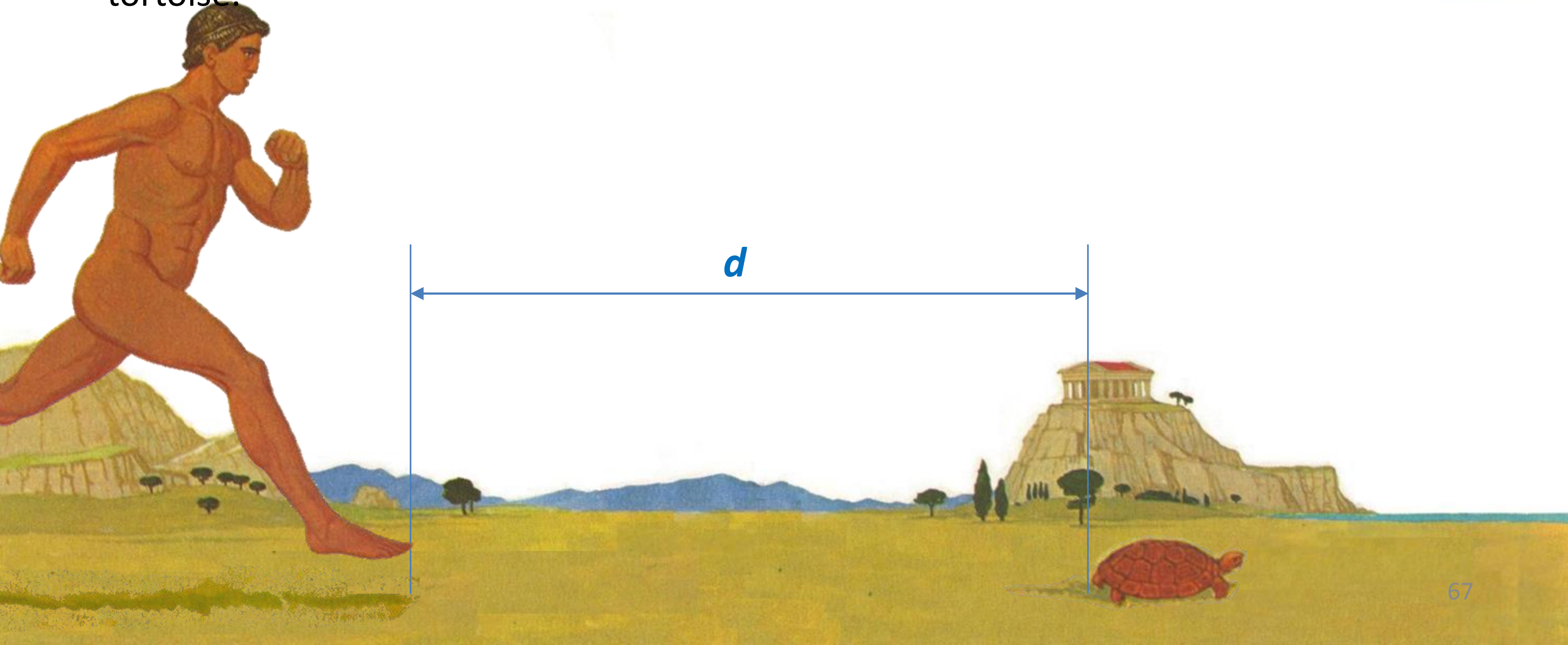
Is the hybrid system ***non-blocking*** ?

Compute the sets Out and $Reach$

A hybrid system is ***non-blocking*** if $\forall (q, x) \in Out \cap Reach,\ \exists (q, q') \in E : x \in G(q, q')$

Zenonian Hybrid models

Achilles is in a footrace with the tortoise and allows the tortoise a head start.
After some finite time, Achilles reaches the tortoise's starting point.
During this time, the tortoise has run a much shorter distance.
It will then take Achilles some further time to run that distance, by which time the tortoise will have advanced farther... Thus, whenever Achilles arrives somewhere the tortoise has been, he still has some distance to go before he can even reach the tortoise.



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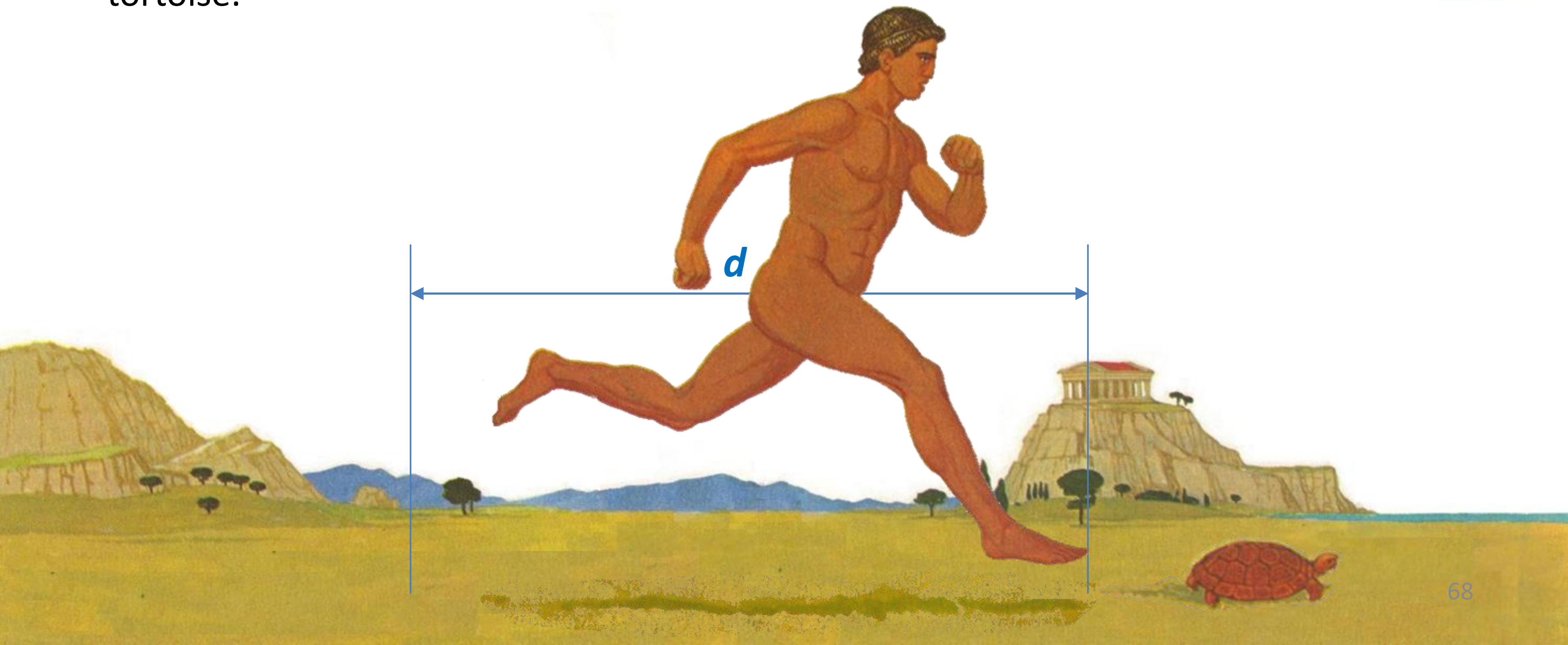
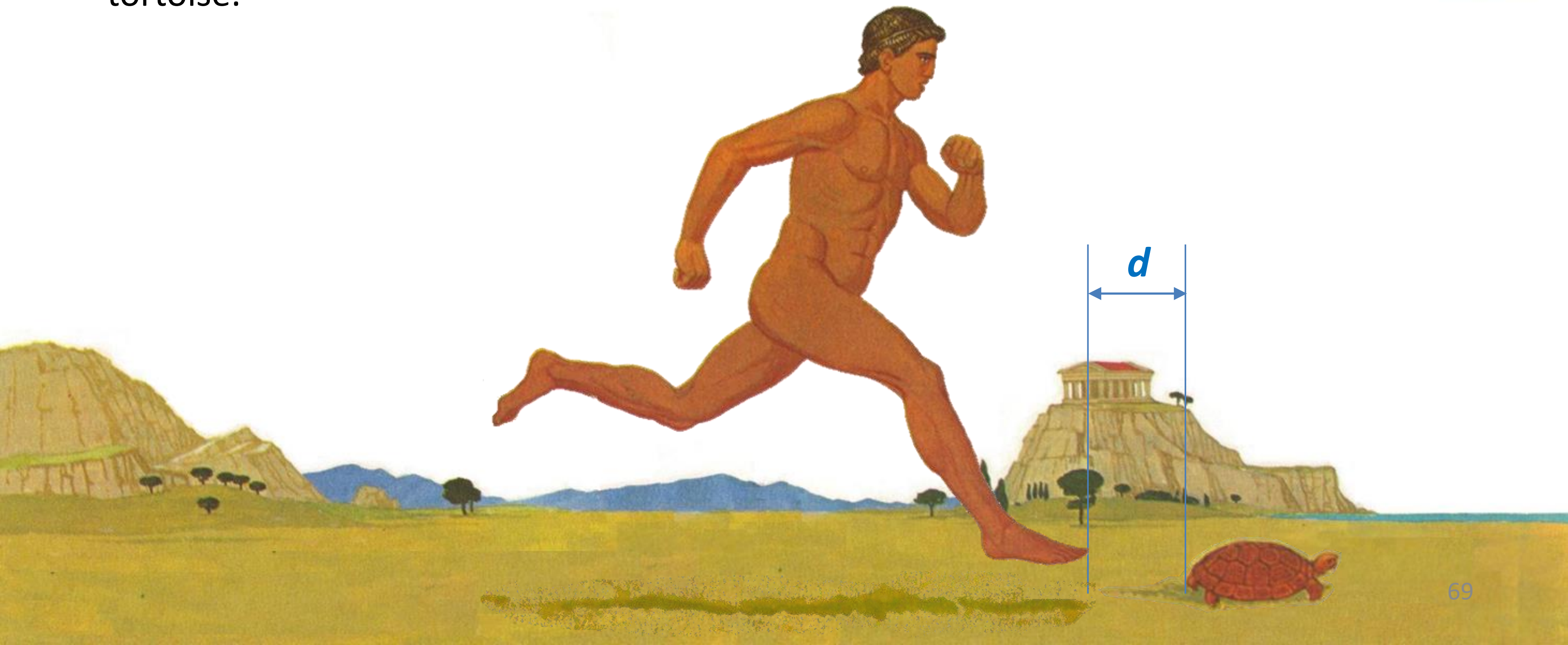


Zenonian Hybrid models

Achilles is in a footrace with the tortoise and allows the tortoise a head start.
After some finite time, Achilles reaches the tortoise's starting point.
During this time, the tortoise has run a much shorter distance.
It will then take Achilles some further time to run that distance, by which time the tortoise will have advanced farther... Thus, whenever Achilles arrives somewhere the tortoise has been, he still has some distance to go before he can even reach the tortoise.

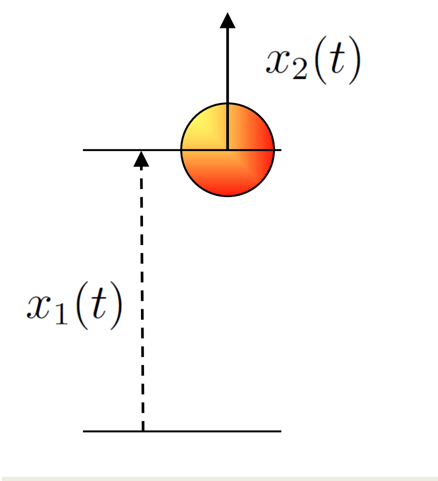


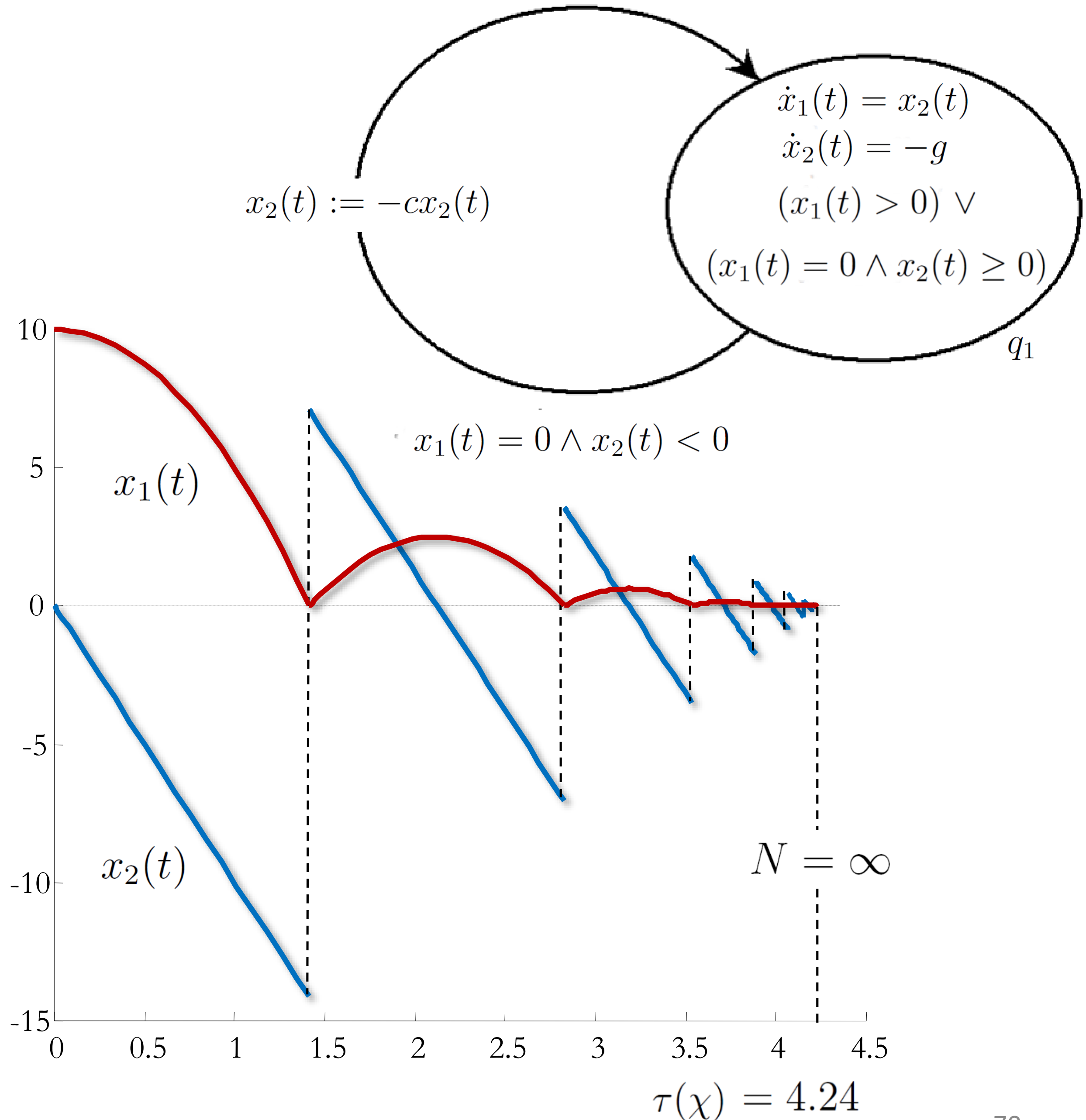
$$c = 1/2$$

$$x_1(0) = 10$$

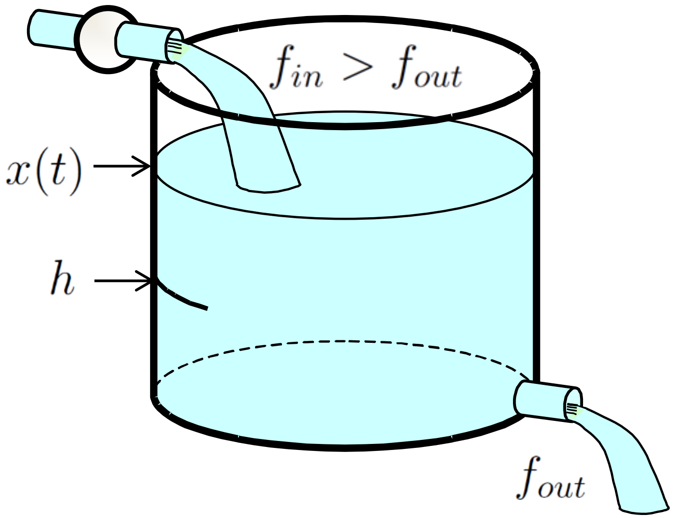
$$x_2(0) = 0$$

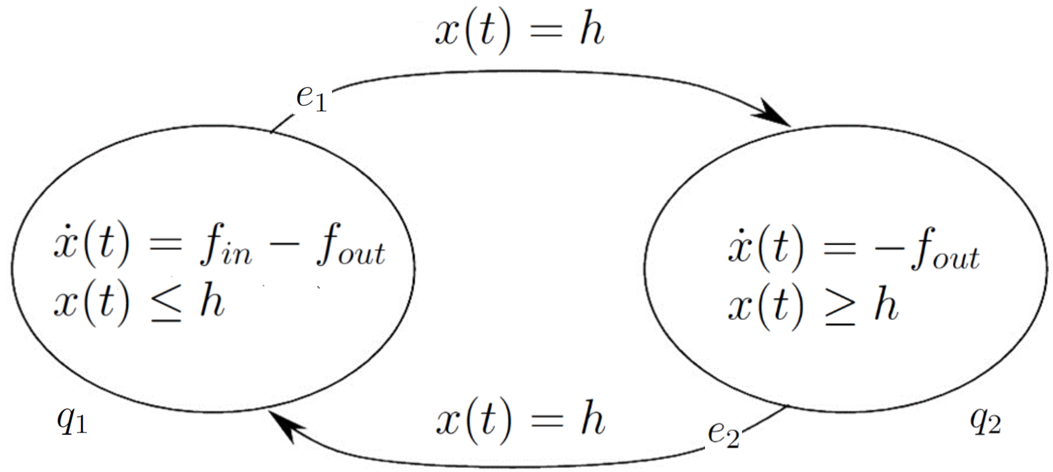
$$g = 10$$

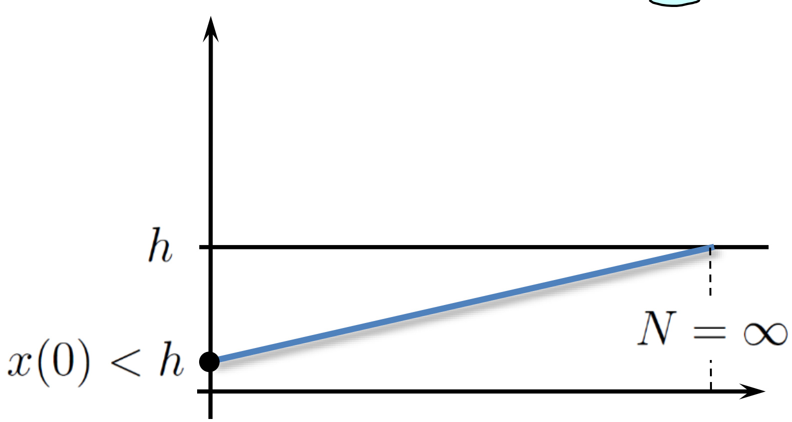


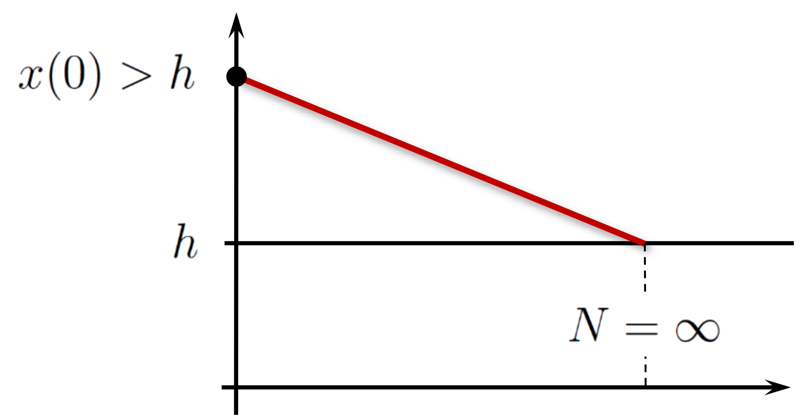


The valve is opened if the level is below the assigned threshold and is closed if the level is above the assigned threshold





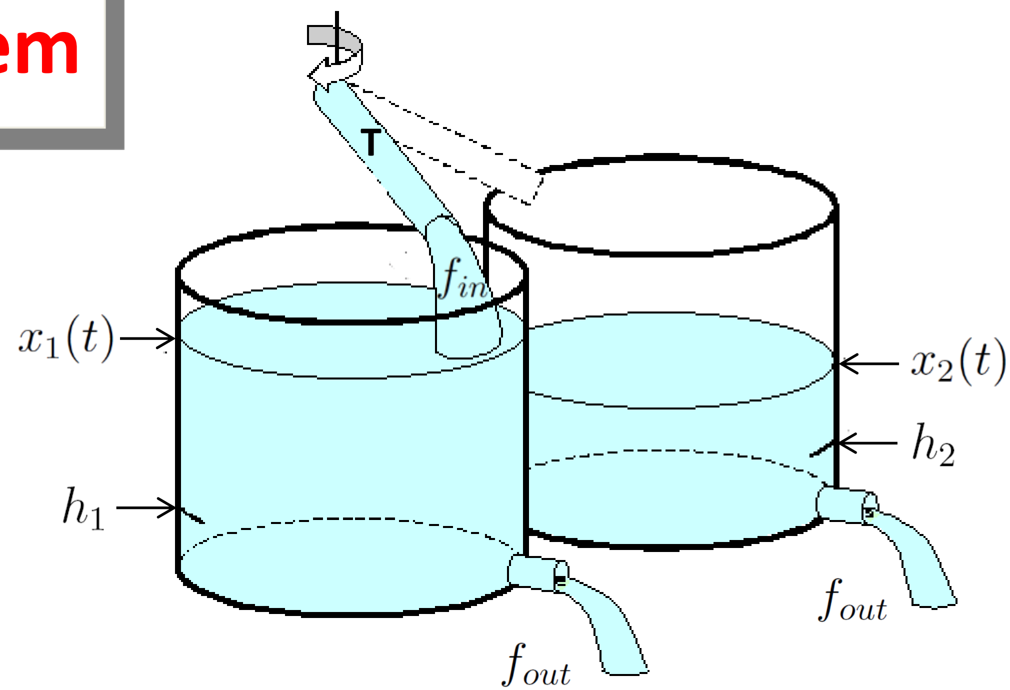




$$\tau(\chi) = x(0)/\left(f_{in} - f_{out}\right)$$

$$\tau(\chi) = x(0)/f_{out}$$

Switched Flow System



Instantaneously move the input to any tank in which the level falls below the assigned threshold

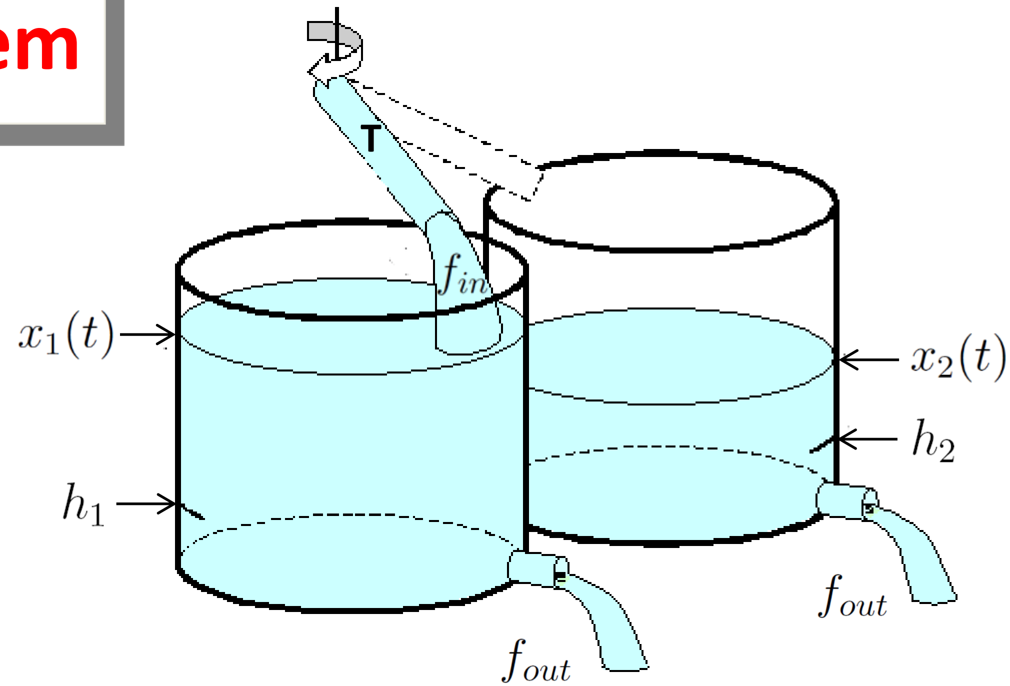
$$x_1(0) > h_1$$

$$x_2(0) > h_2$$

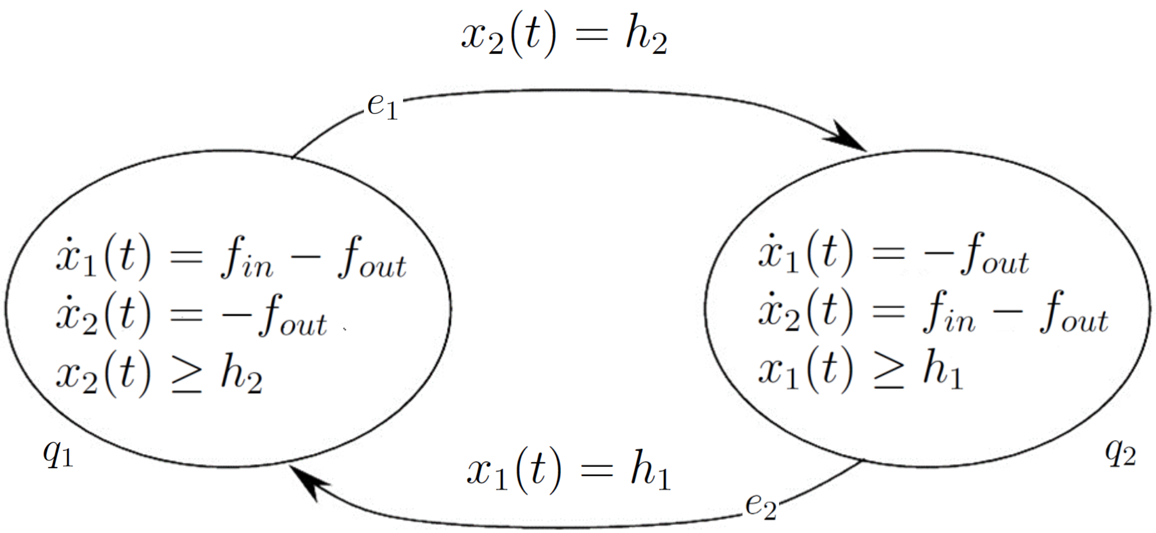
$$2f_{out} > f_{in} > f_{out}$$

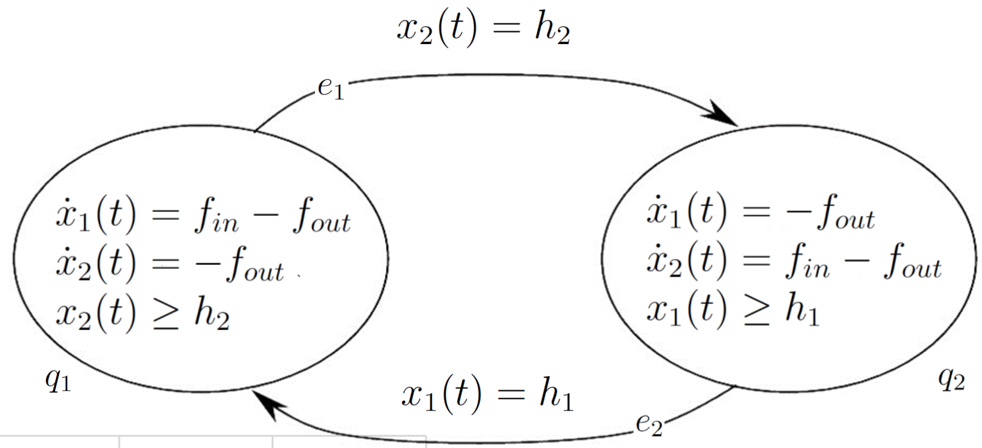
$x_i(t)$	water level of the *i*-th tank
f_{out}	output flow of the *i*-th tank
h_i	threshold assigned to tank *i*
f_{in}	input flow

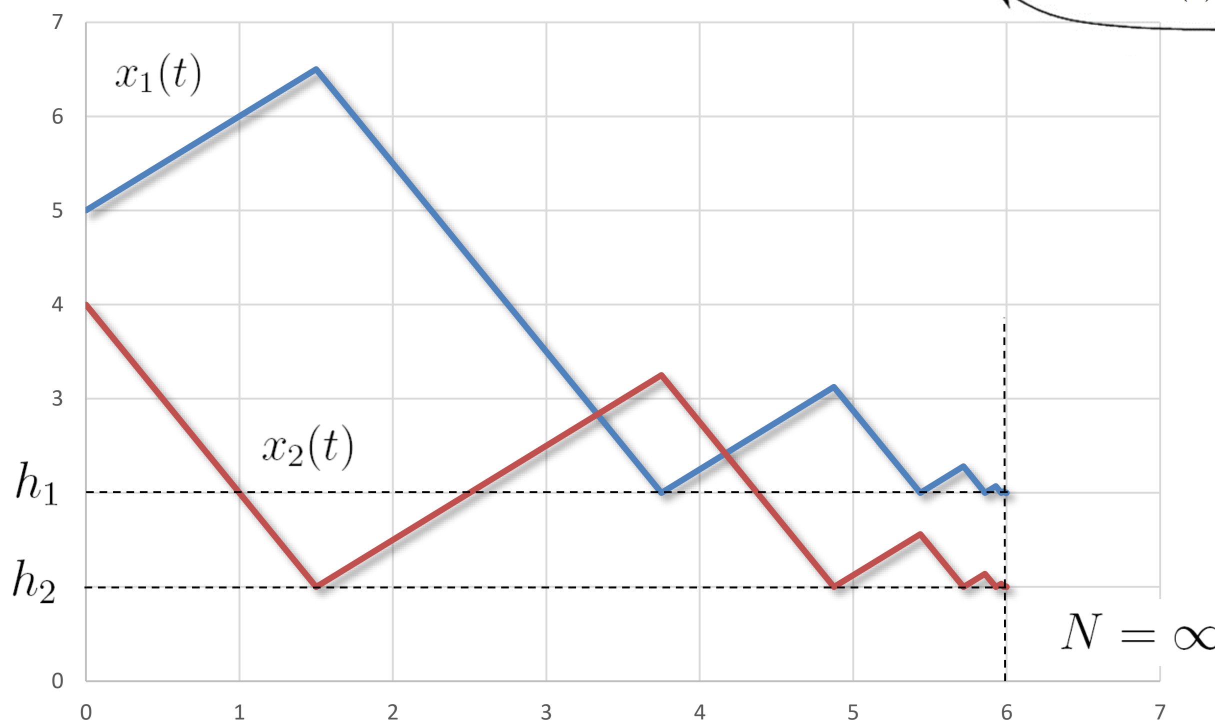
Switched Flow System



Hybrid model







$x_1(0) = 5$

$x_2(0) = 4$

$h_1 = 2$

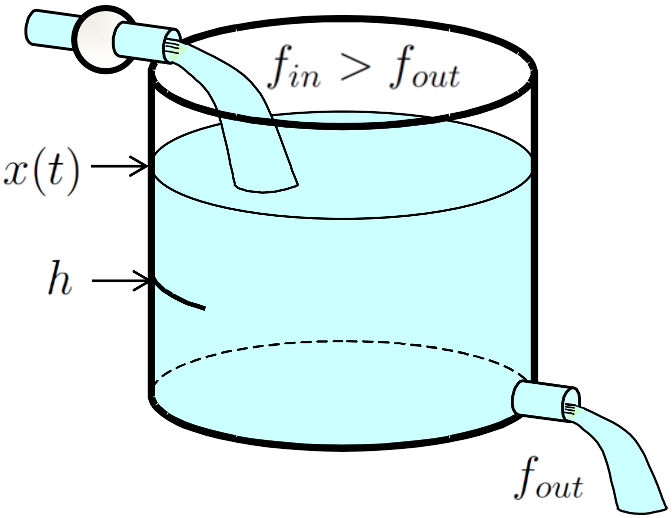
$h_2 = 1$

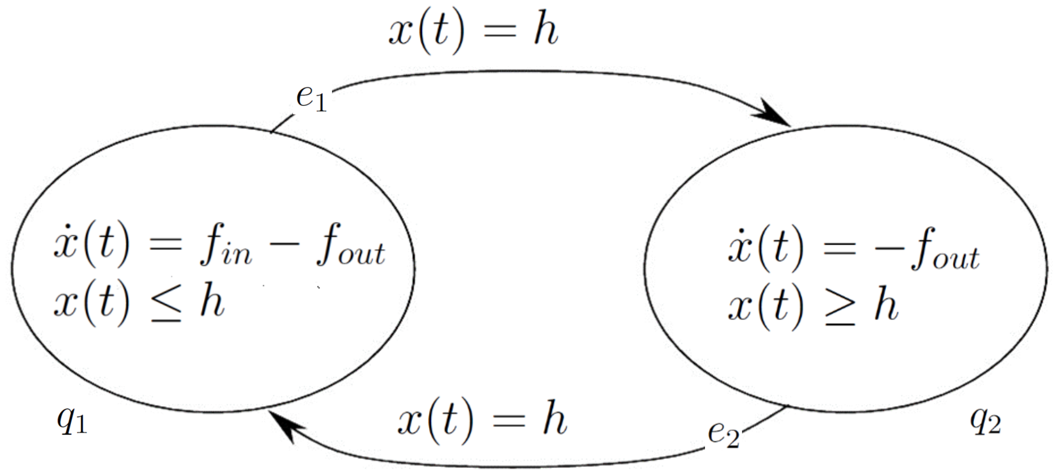
$f_{in} = 3$

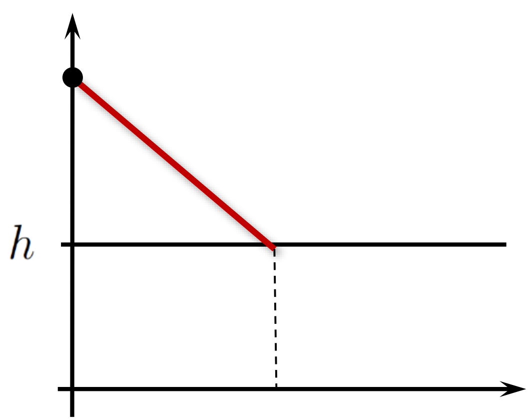
$f_{out} = 2$

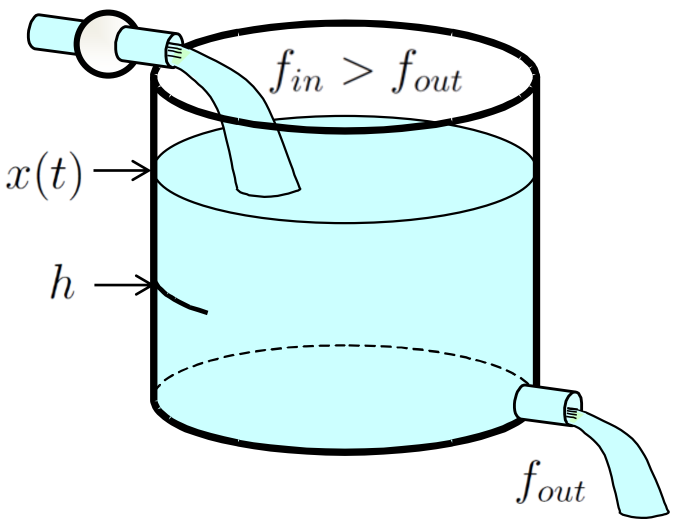
$$\tau(\chi) = \frac{x_1(0) + x_2(0) - h_1 - h_2}{2 f_{out} - f_{in}}$$

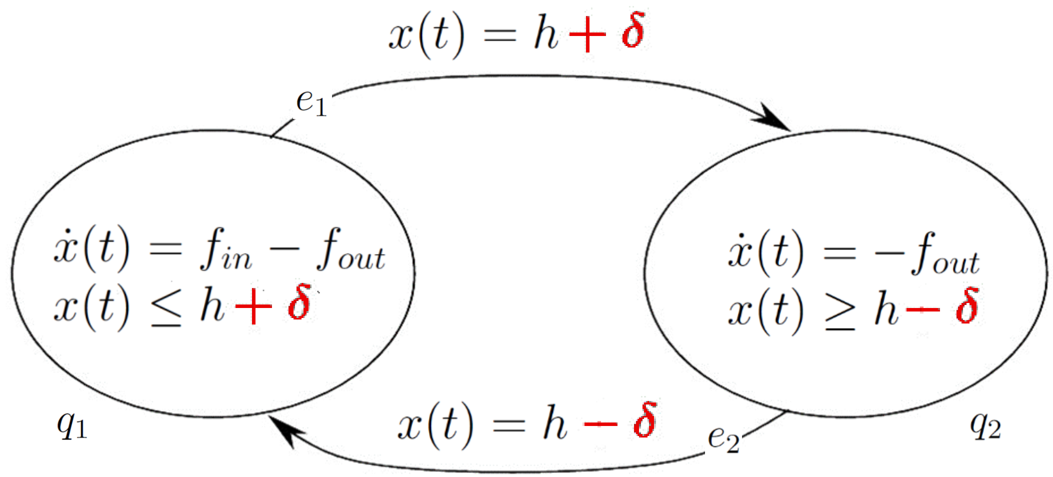
Regularization of Zeno hybrid automata

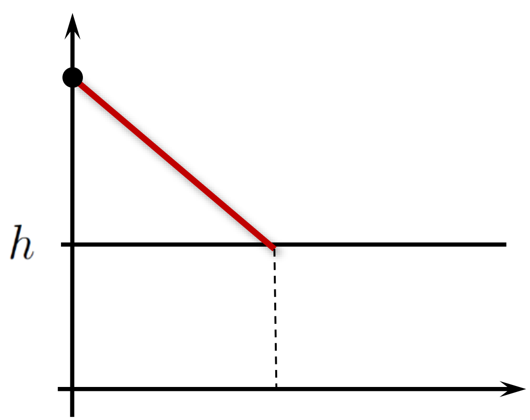

$f_{in} > f_{out}$
$x(t)$
h
f_{out}

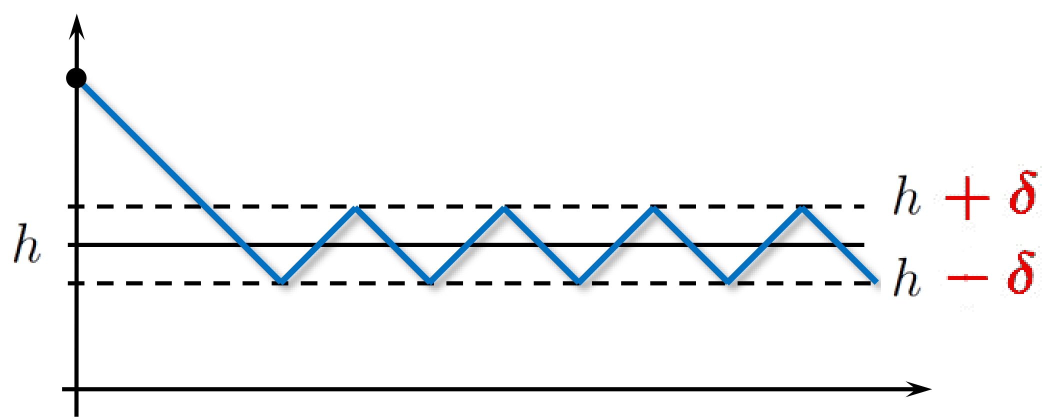

$x(t) = h$
e_1
$\dot{x}(t) = f_{in} - f_{out}$
$x(t) \leq h$
$\dot{x}(t) = -f_{out}$
$x(t) \geq h$
q_1
$x(t) = h$
e_2
q_2

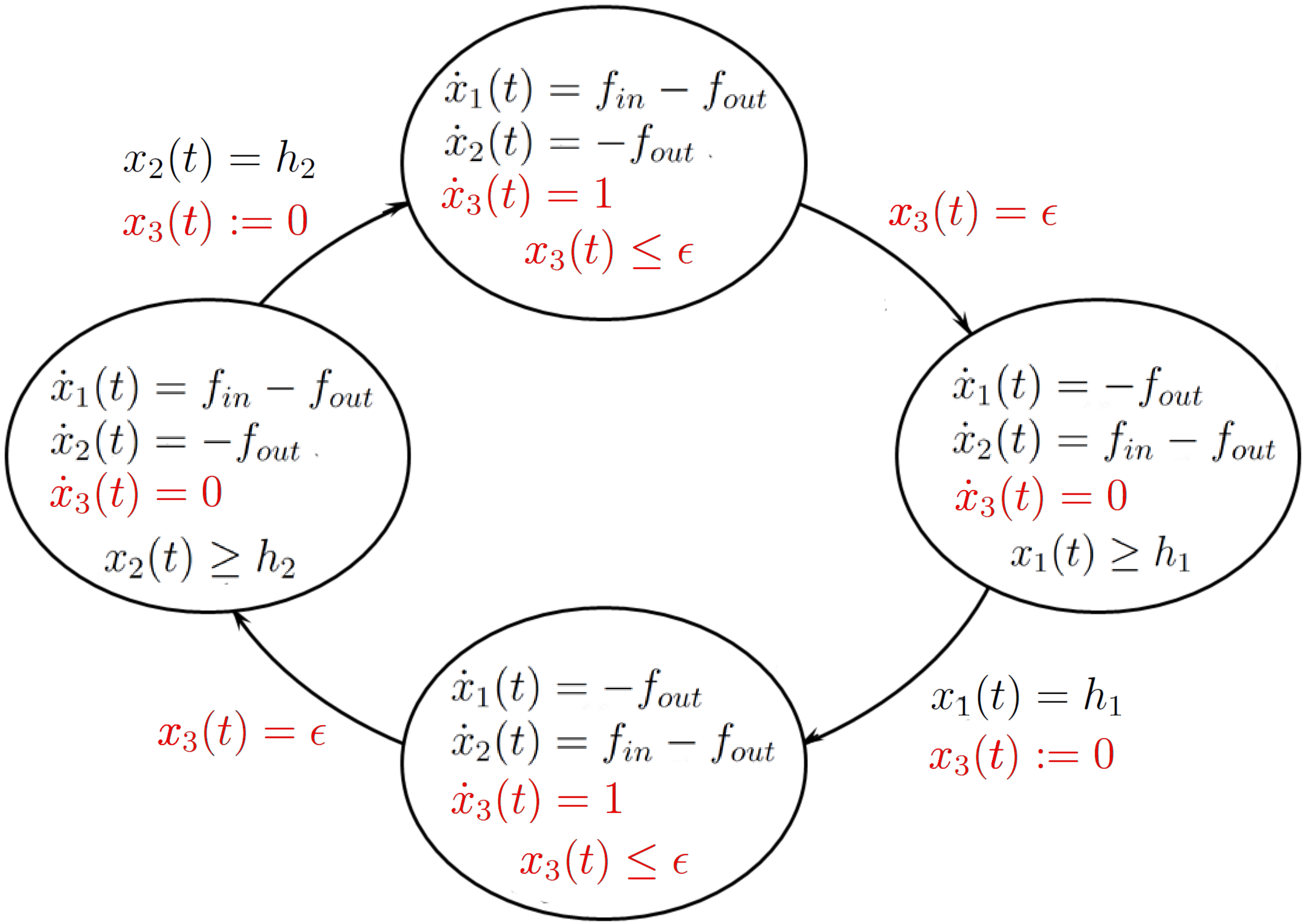

h


$f_{in} > f_{out}$
$x(t)$
h
f_{out}


$x(t) = h + \delta$
e_1
$\dot{x}(t) = f_{in} - f_{out}$
$x(t) \leq h + \delta$
$\dot{x}(t) = -f_{out}$
$x(t) \geq h - \delta$
q_1
$x(t) = h - \delta$
e_2
q_2


h

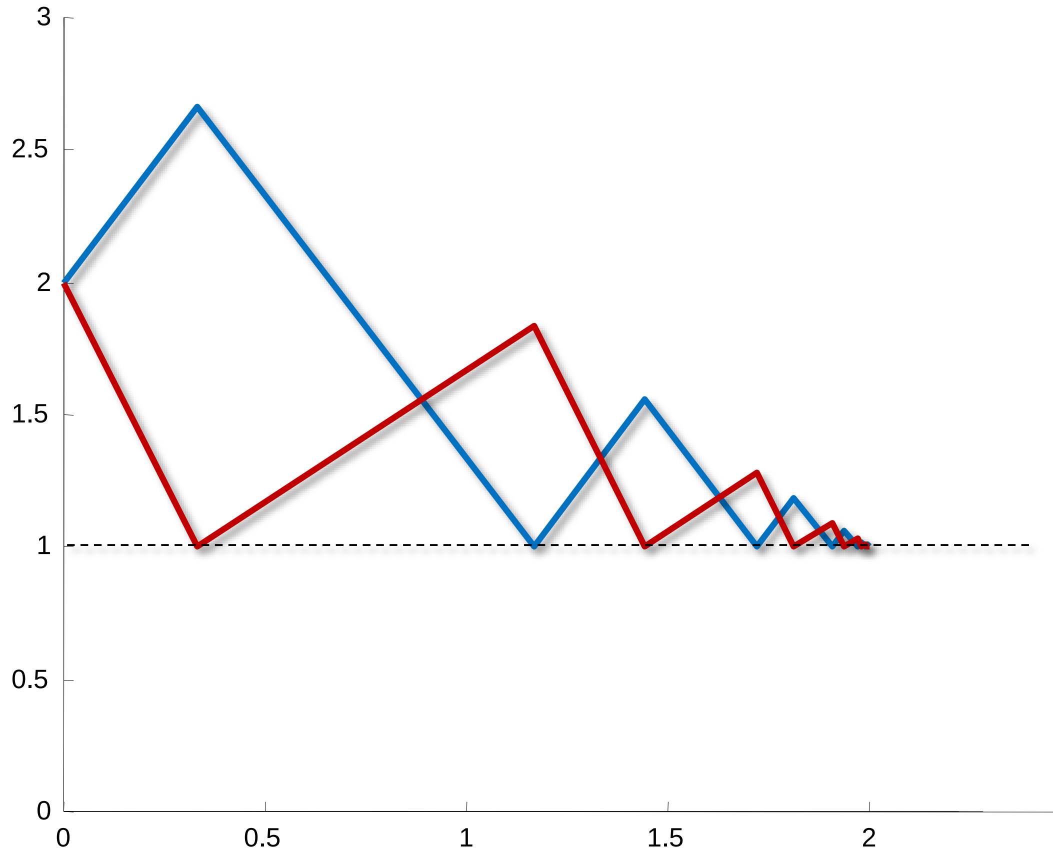

h
$h + \delta$
$h - \delta$

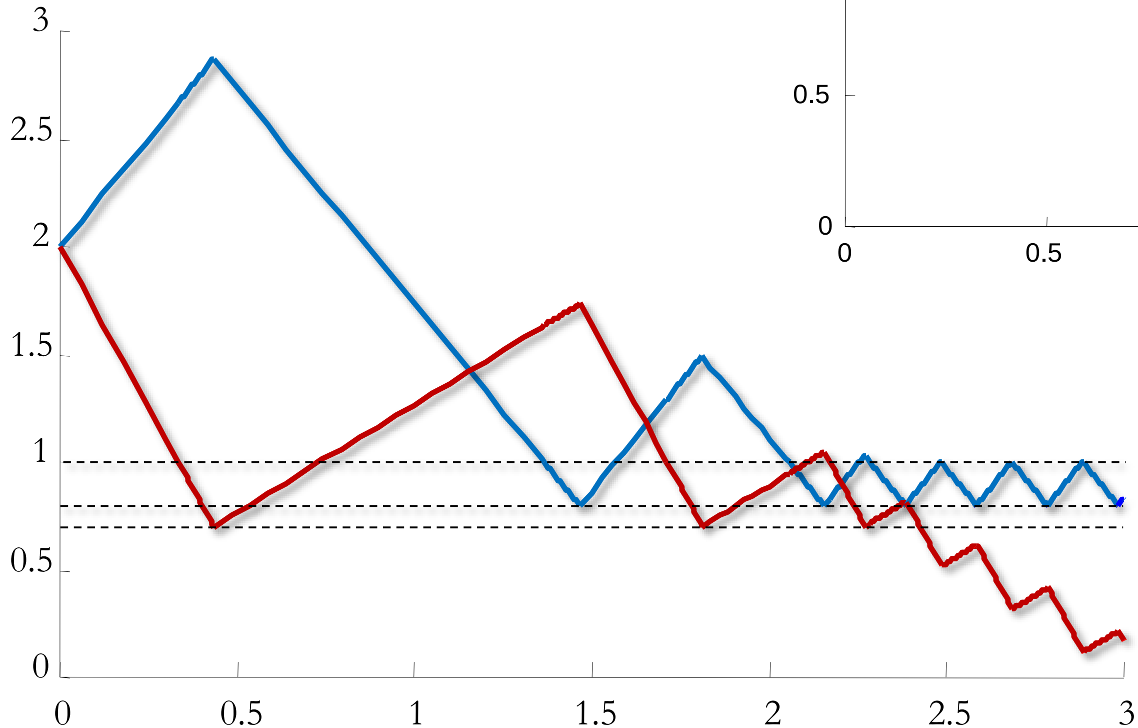

$\dot{x}_1(t) = f_{in} - f_{out}$
$\dot{x}_2(t) = -f_{out}$
$\dot{x}_3(t) = 1$
$x_3(t) \leq \epsilon$
$x_3(t) = \epsilon$
$\dot{x}_1(t) = -f_{out}$
$\dot{x}_2(t) = f_{in} - f_{out}$
$\dot{x}_3(t) = 0$
$x_1(t) \geq h_1$
$x_1(t) = h_1$
$x_3(t) := 0$
$\dot{x}_1(t) = -f_{out}$
$\dot{x}_2(t) = f_{in} - f_{out}$
$\dot{x}_3(t) = 1$
$x_3(t) \leq \epsilon$
$x_3(t) = \epsilon$
$\dot{x}_1(t) = f_{in} - f_{out}$
$\dot{x}_2(t) = -f_{out}$
$\dot{x}_3(t) = 0$
$x_2(t) \geq h_2$
$x_2(t) = h_2$
$x_3(t) := 0$

$$x_1(0) = 2 \qquad f_{in} = 4$$

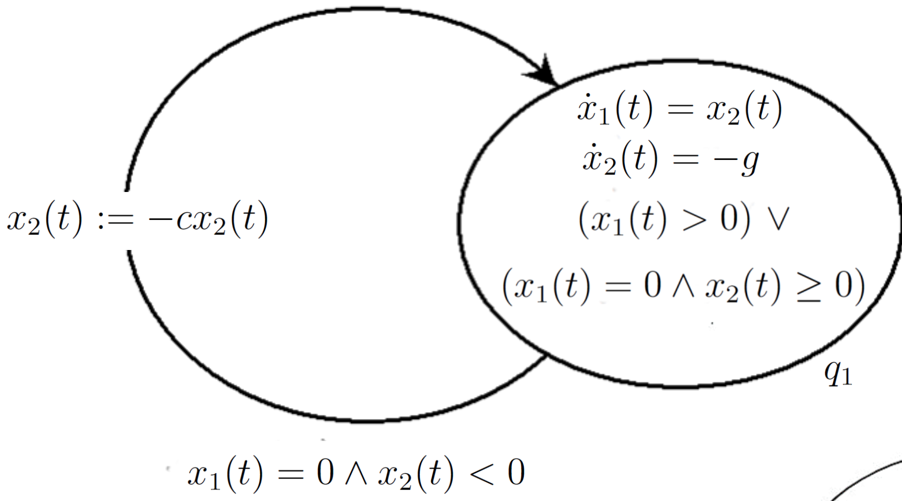
$$x_2(0) = 2 \qquad f^1_{out} = 2$$

$$h_1 = h_2 = 1 \qquad f^2_{out} = 3$$

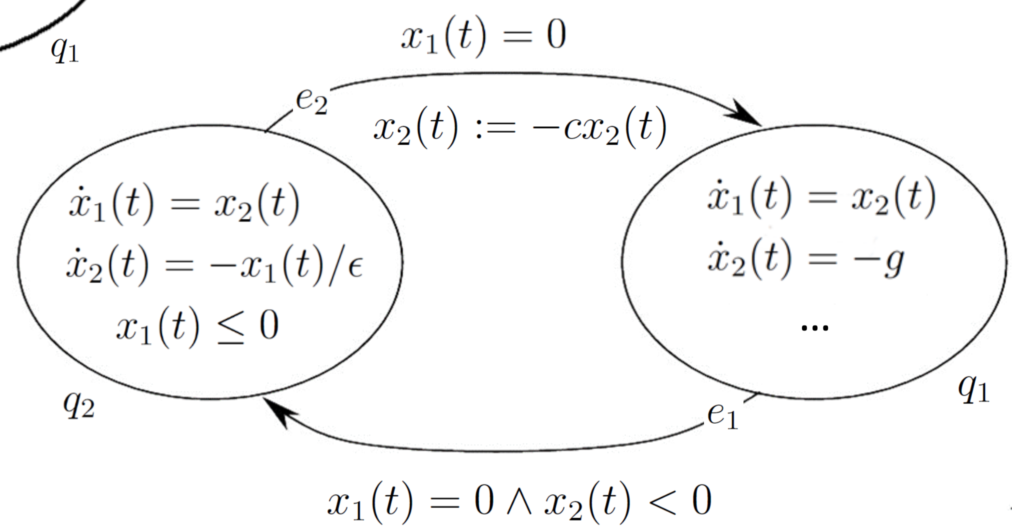




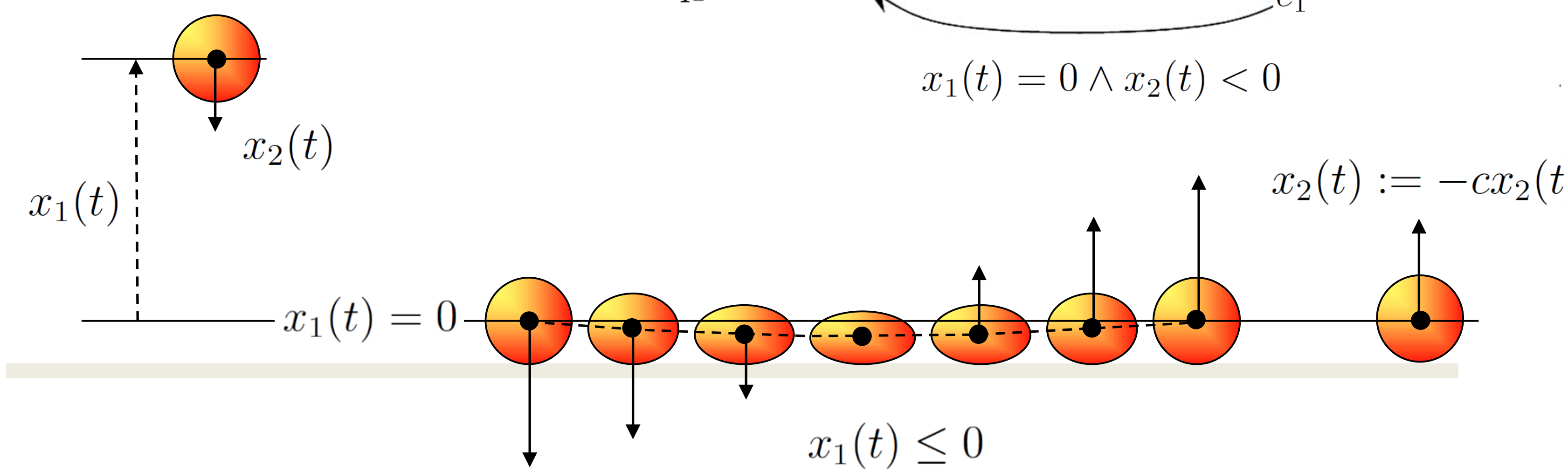
$$\epsilon = 0.1$$



$\dot{x}_1(t) = x_2(t)$
$\dot{x}_2(t) = -g$
$(x_1(t) > 0) \vee$
$(x_1(t) = 0 \wedge x_2(t) \geq 0)$
q_1
$x_2(t) := -cx_2(t)$
$x_1(t) = 0 \wedge x_2(t) < 0$



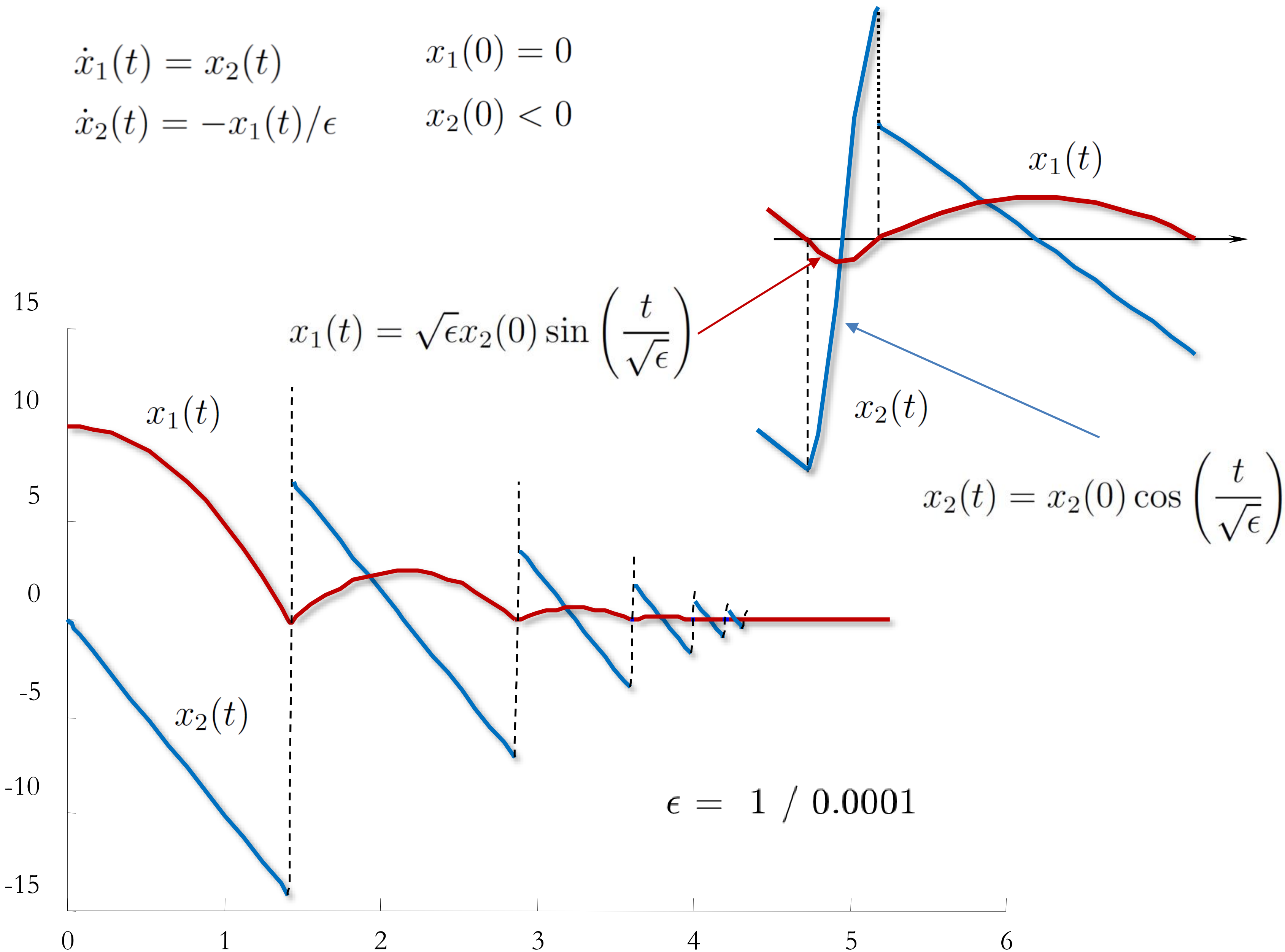
$x_1(t) = 0$
e_2
$x_2(t) := -cx_2(t)$
$\dot{x}_1(t) = x_2(t)$
$\dot{x}_2(t) = -x_1(t)/\epsilon$
$x_1(t) \leq 0$
q_2
$\dot{x}_1(t) = x_2(t)$
$\dot{x}_2(t) = -g$
...
q_1
e_1
$x_1(t) = 0 \wedge x_2(t) < 0$



$x_2(t)$
$x_1(t)$
$x_2(t) := -cx_2(t)$
$x_1(t) = 0$
$x_1(t) \leq 0$

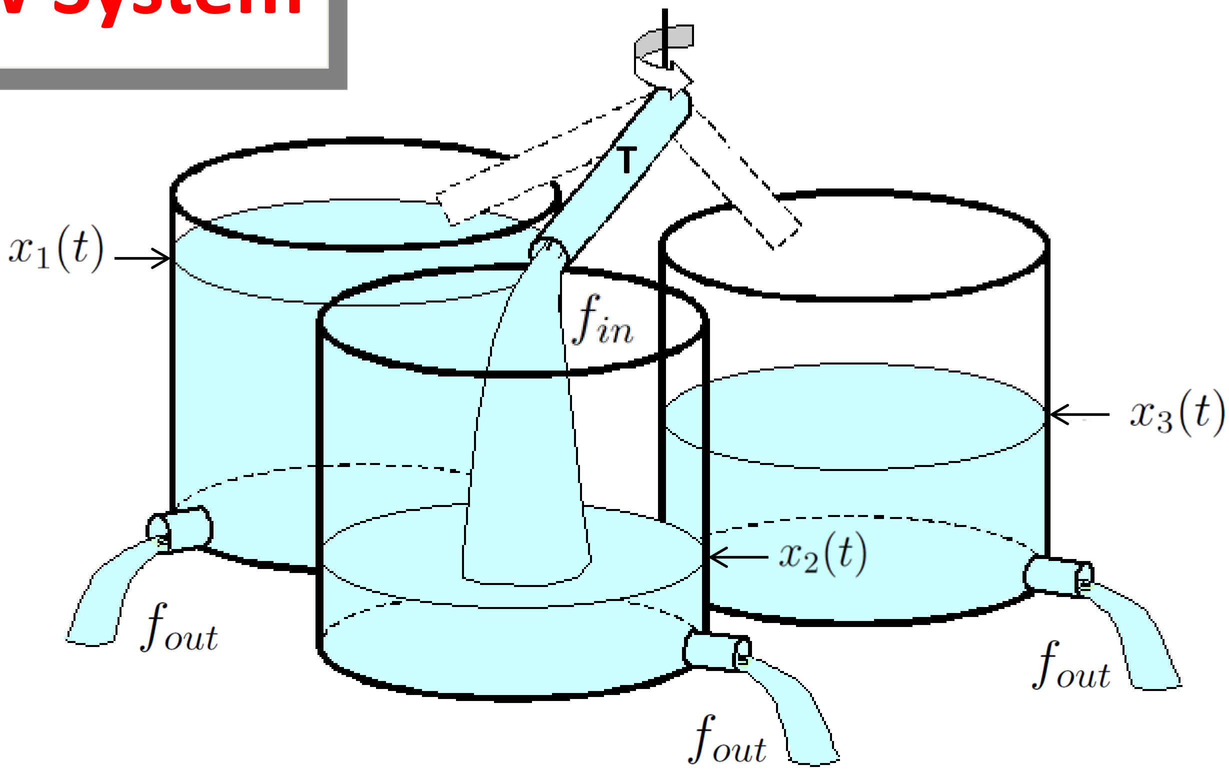
$$\dot{x}_1(t) = x_2(t) \qquad x_1(0) = 0$$

$$\dot{x}_2(t) = -x_1(t)/\epsilon \qquad x_2(0) < 0$$



Switched Flow System

Instantaneously move the input to any tank in which the level falls to zero



$$x_1(0) \geq 0$$

$$x_2(0) \geq 0$$

$$x_3(0) \geq 0$$

$$x_1(0) + x_2(0) + x_3(0) = 1$$

$$f_{in} = 3f_{out} = 1$$

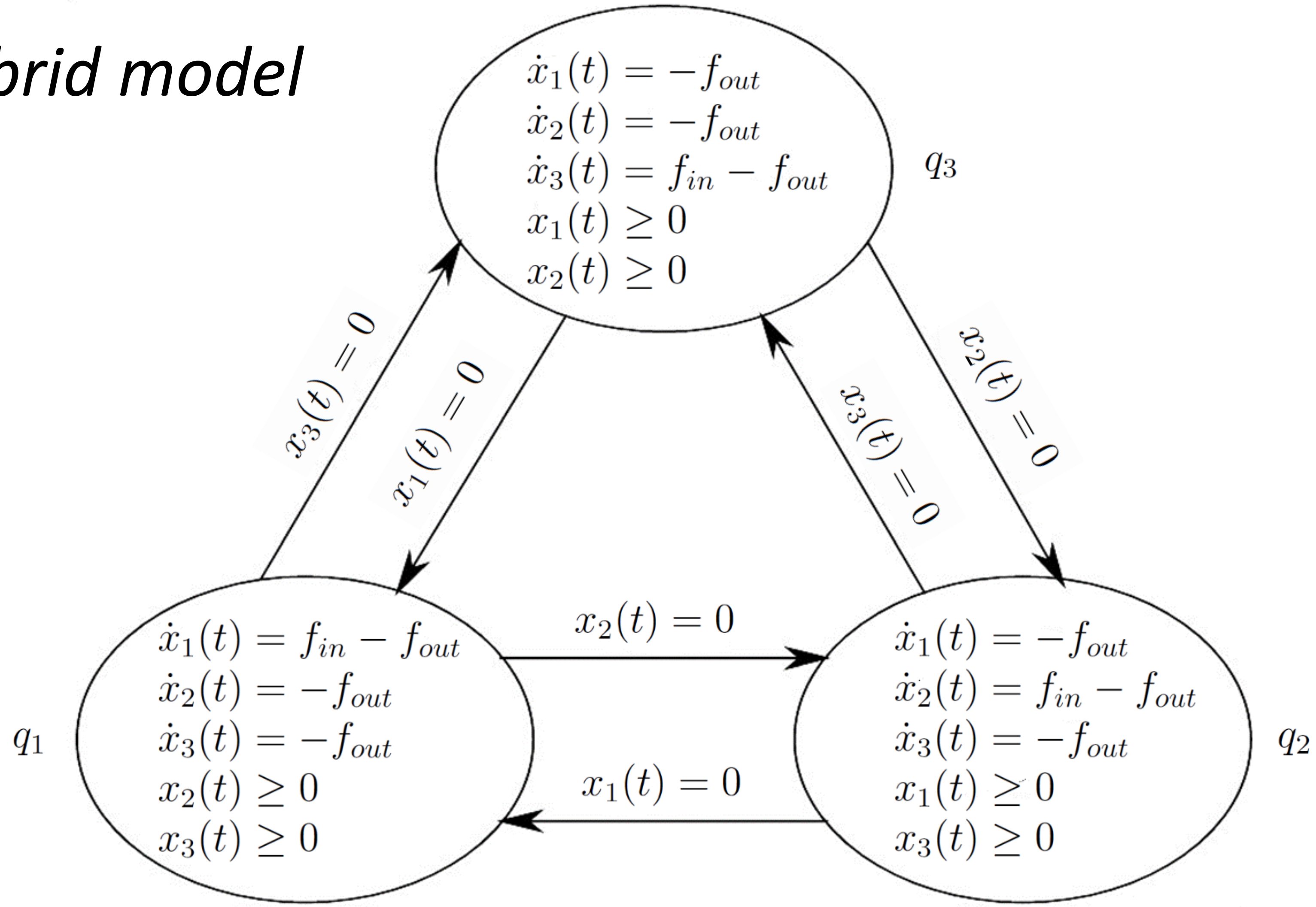
$x_i(t)$ water level of the *i*-th tank

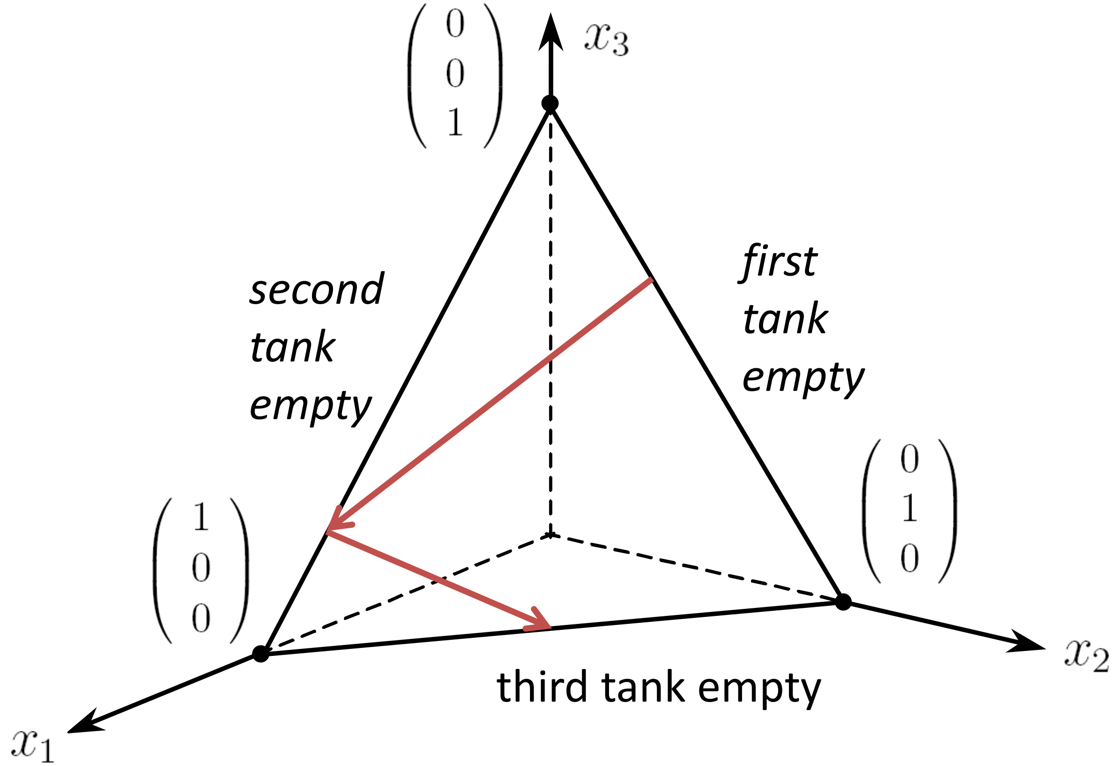
f_{out} output flow of the *i*-th tank

f_{in} input flow

Switched Flow System

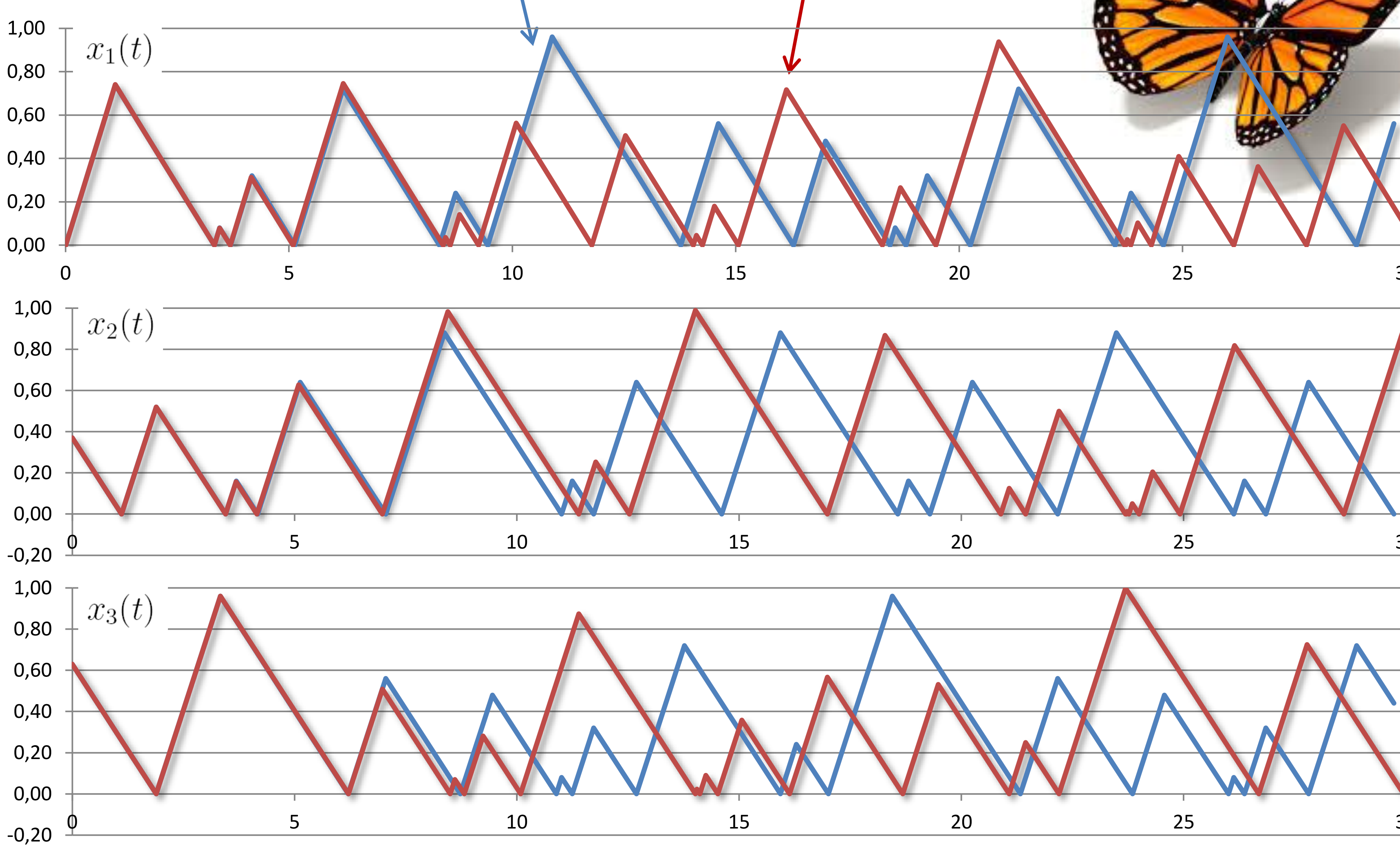
Hybrid model

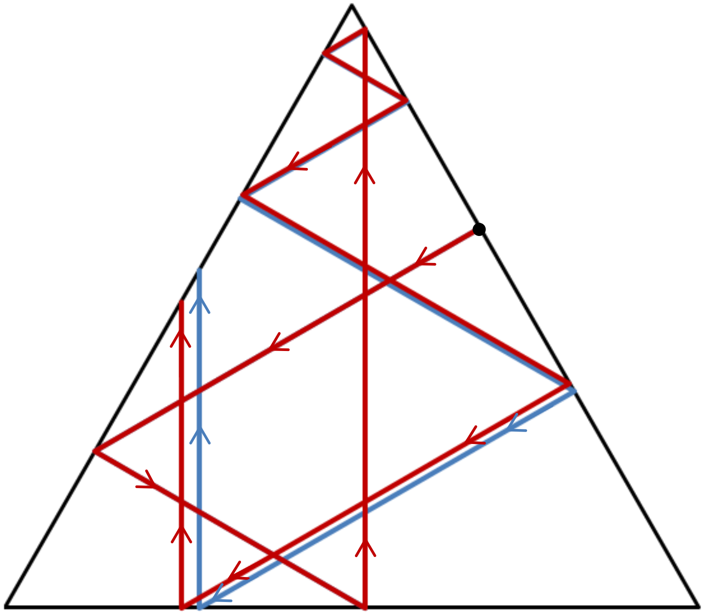


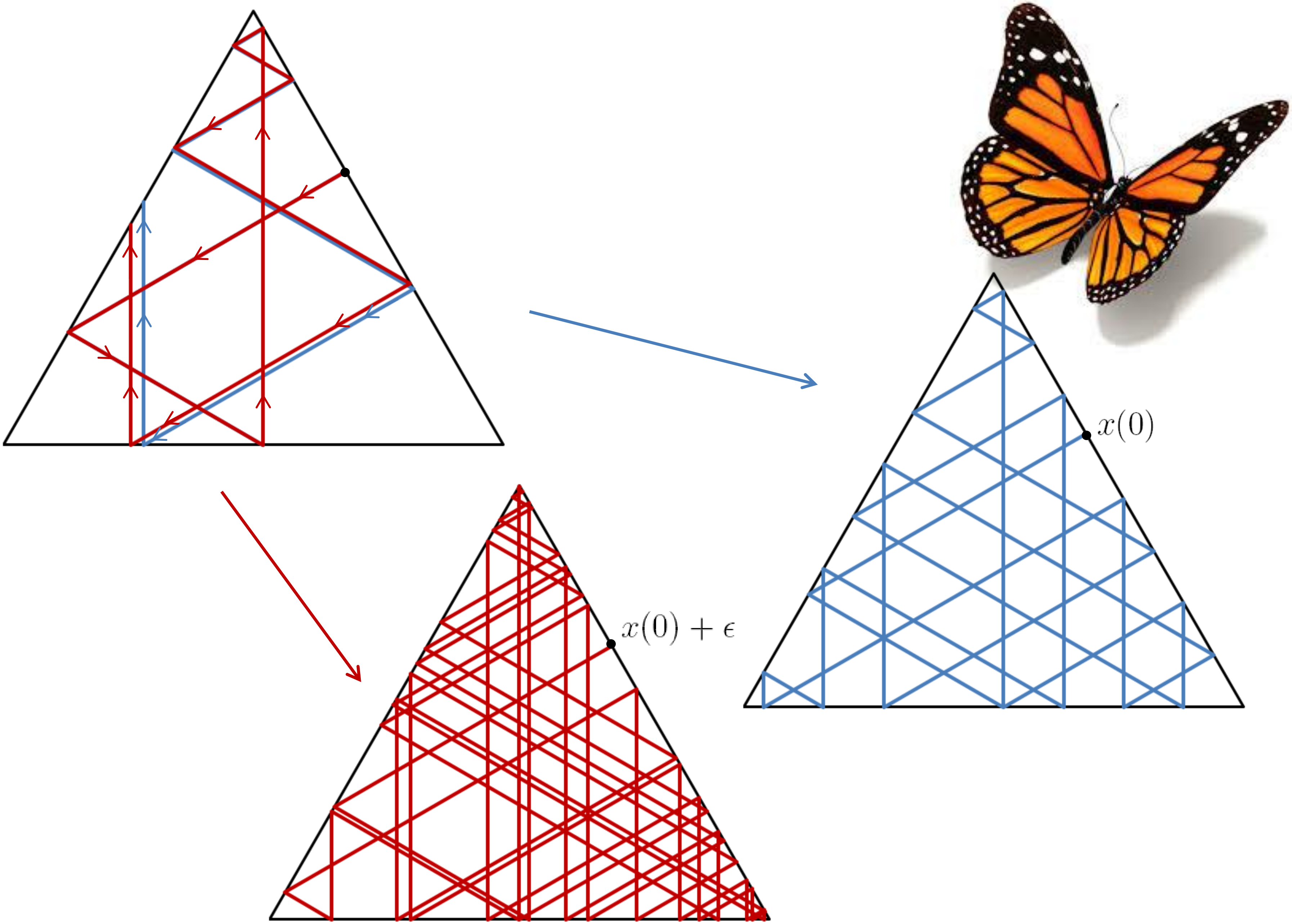


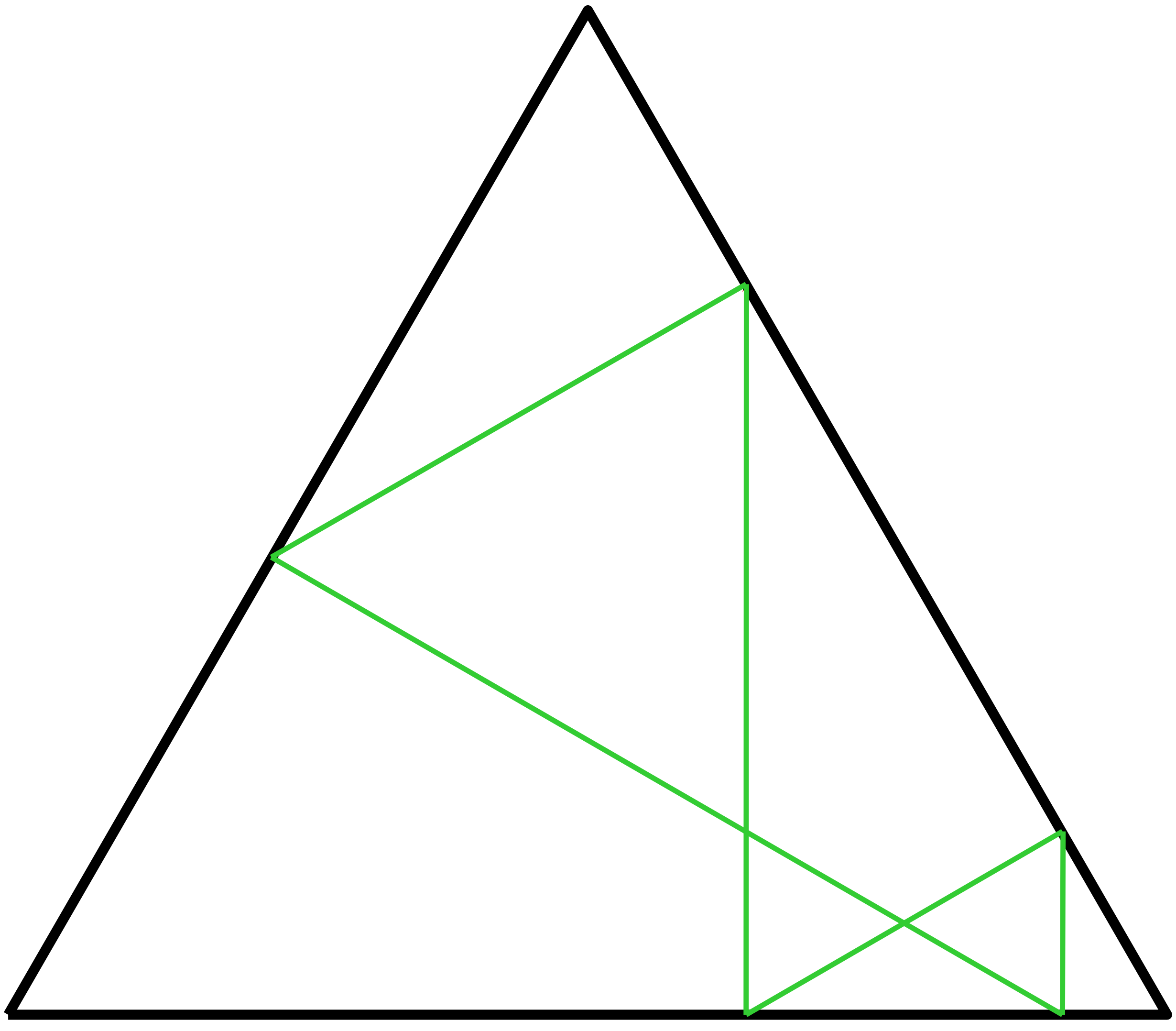
$$\forall t \quad \begin{cases} x_1(t) + x_2(t) + x_3(t) = 1 \\ x_1(t) \geq 0 \\ x_2(t) \geq 0 \\ x_3(t) \geq 0 \end{cases}$$

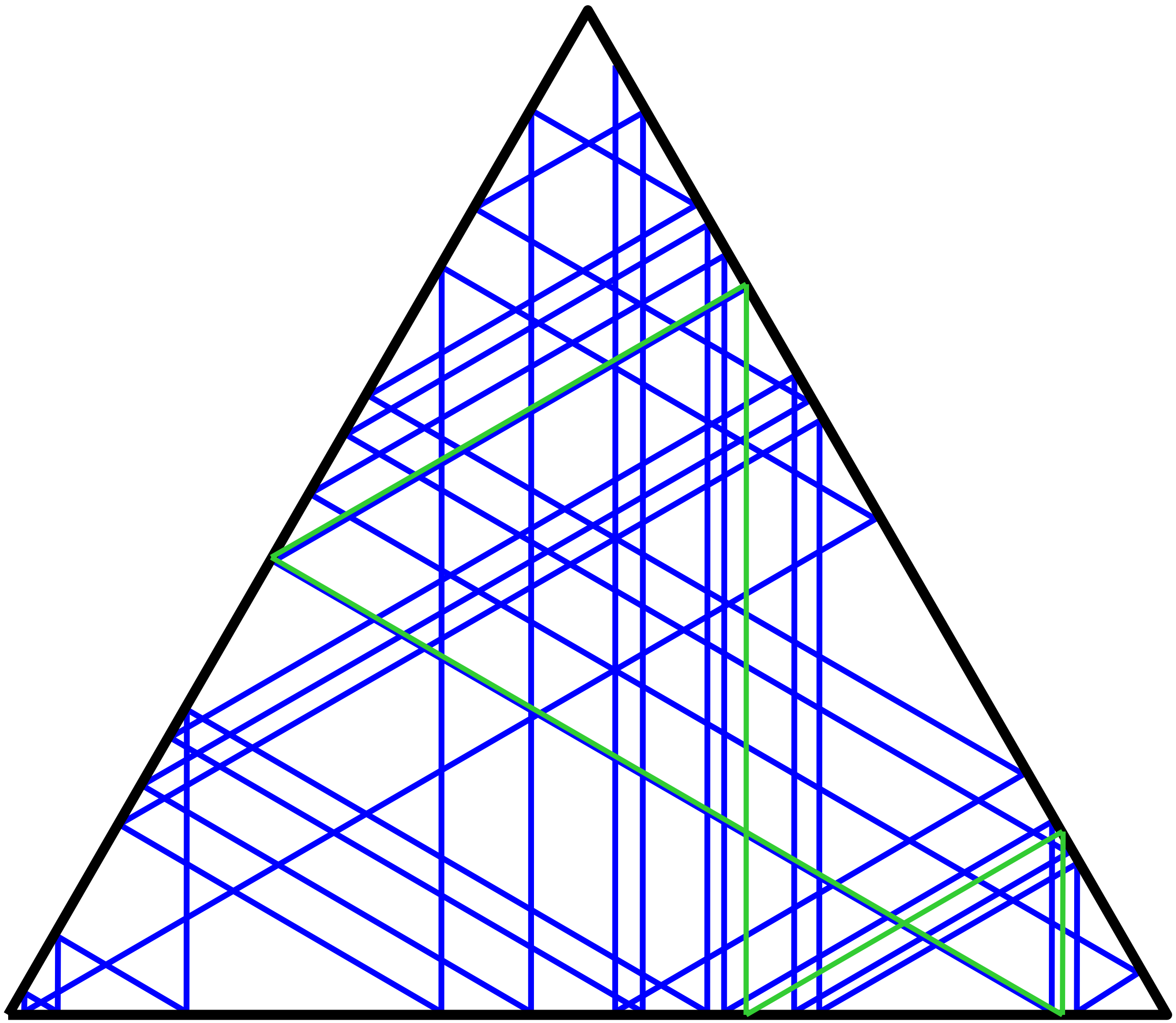
$$x(0) = \begin{pmatrix} 0 \\ 0.37 \\ 0.63 \end{pmatrix} \qquad x(0) = \begin{pmatrix} 0 \\ 0.37 + 0.0001 \\ 0.63 - 0.0001 \end{pmatrix}$$

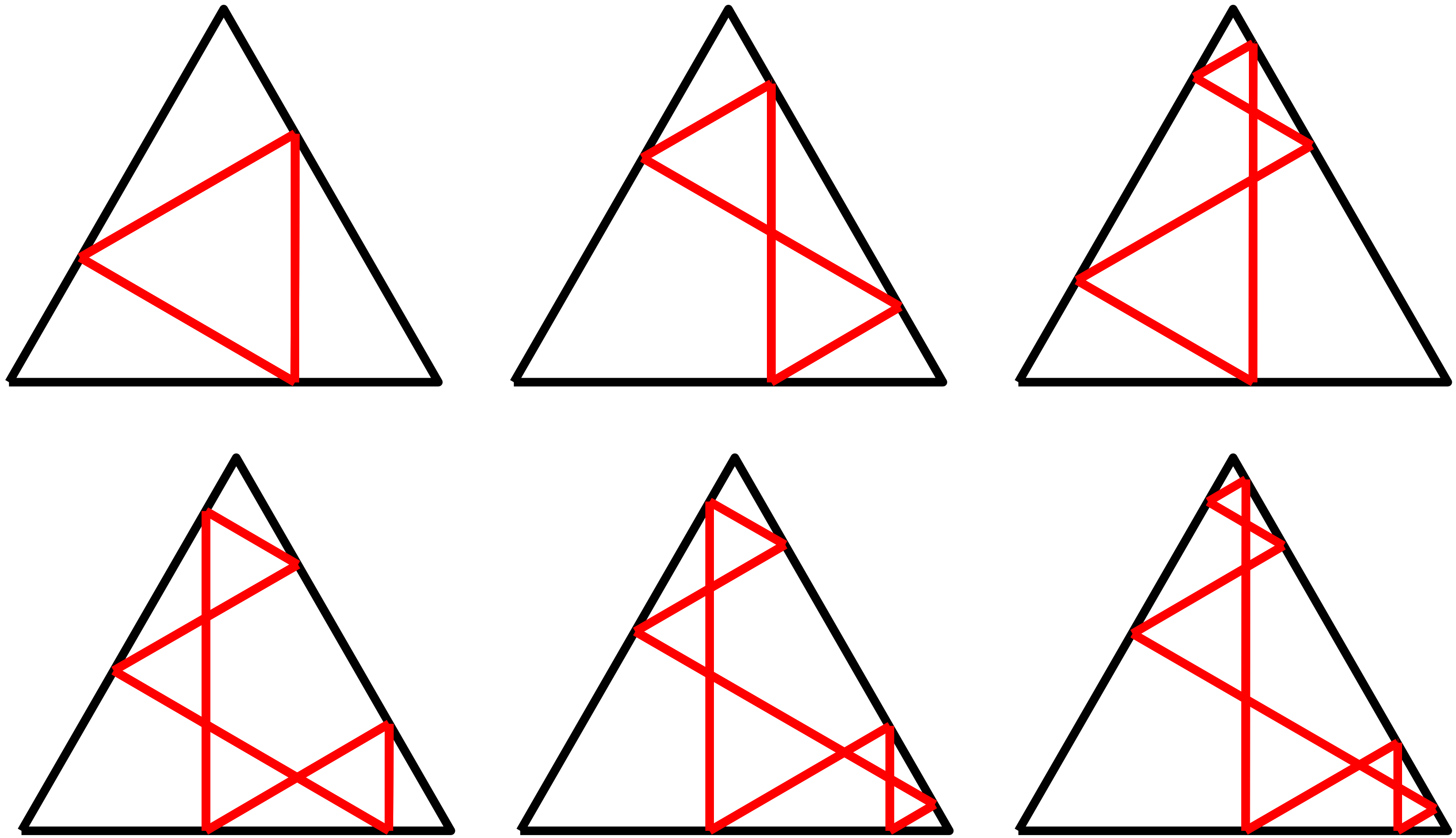


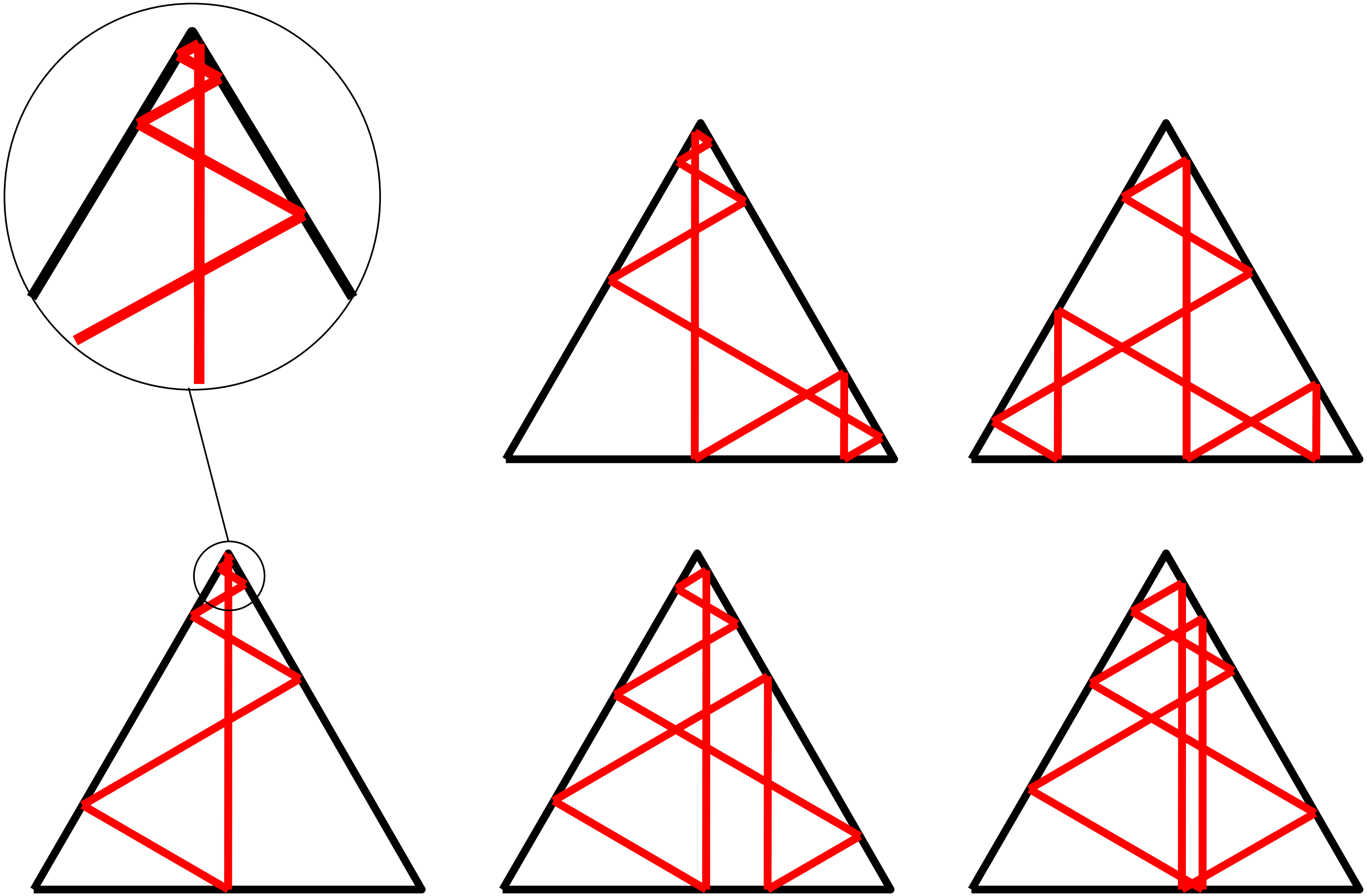



$x(0)$
$x(0) + \epsilon$









Equilibria

$$H = (Q, X, Init, f, Inv, E, G, R)$$

Definition (equilibrium):

$x_e \in X$ is an equilibrium point of H if there exists a nonempty set $\hat{Q} \subseteq Q$ such that

- $f(q, x_e) = 0$ for all $q \in \hat{Q}$
- if $e = (q, q') \in E \wedge x_e \in G(e) \Rightarrow R(e, x_e) = \{x_e\}$

Remarks:

Discrete transitions are allowed out of (q, x_e) but only to (q', x_e)

if $(q, x_e) \in Init$ and $\chi = (\tau, q, x)$ is an execution of H starting from (q, x_e), then $x(t) = x_e$ for all $t \in \tau$

Stability

$$H = (Q, X, Init, f, Inv, E, G, R)$$

Definition (stable equilibrium):
Let $x_e \in X$ be an equilibrium point of H.

x_e is **stable** if

$\forall \epsilon, \exists\, \delta(\epsilon)$: for all executions $\chi = (\tau, q, x)$ of H starting from (q, x_0)

$$|x_0 - x_e| < \delta \Rightarrow |x(t) - x_e| < \epsilon, \forall\, t \in \tau$$

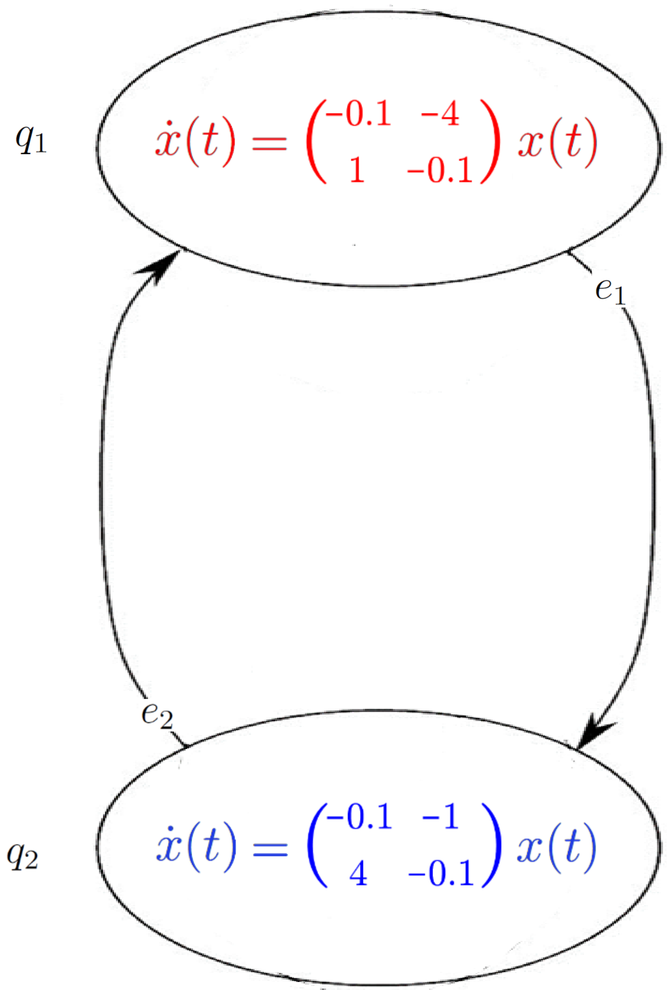
Definition (stable equilibrium):
Let $x_e \in X$ be an equilibrium point of H .

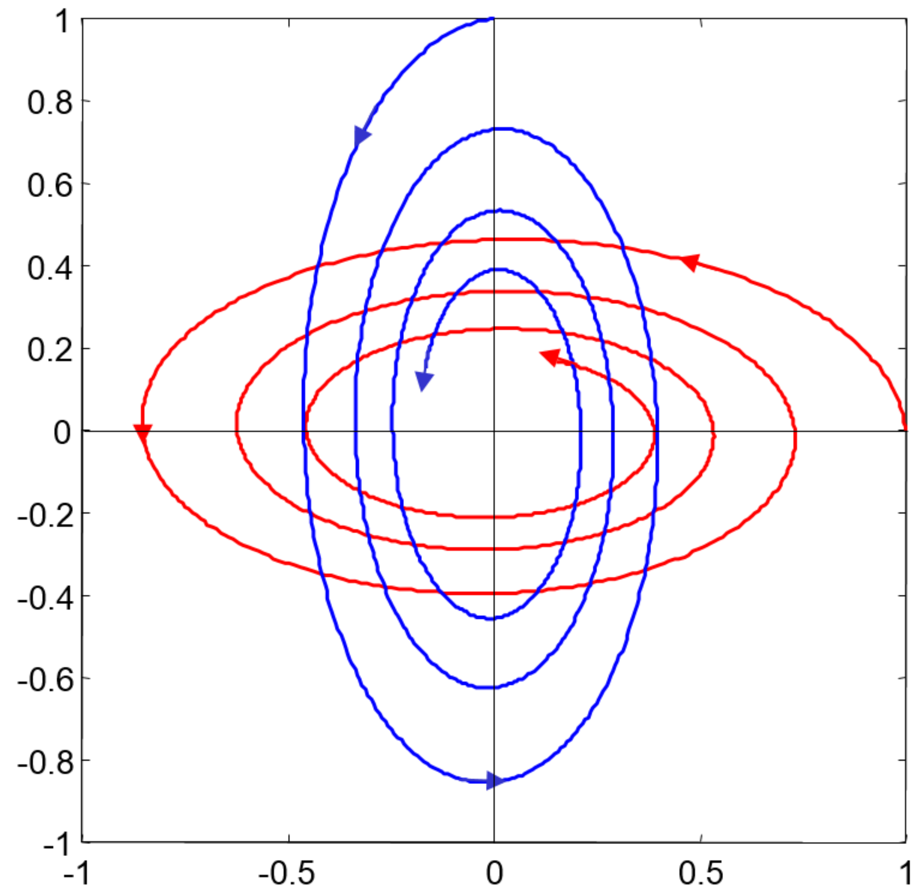
x_e is **asymptotically stable** if

- it is stable
- $\exists\, \delta_a$: for all executions $\chi = (\tau, q, x)$ of H starting from (q, x_0)

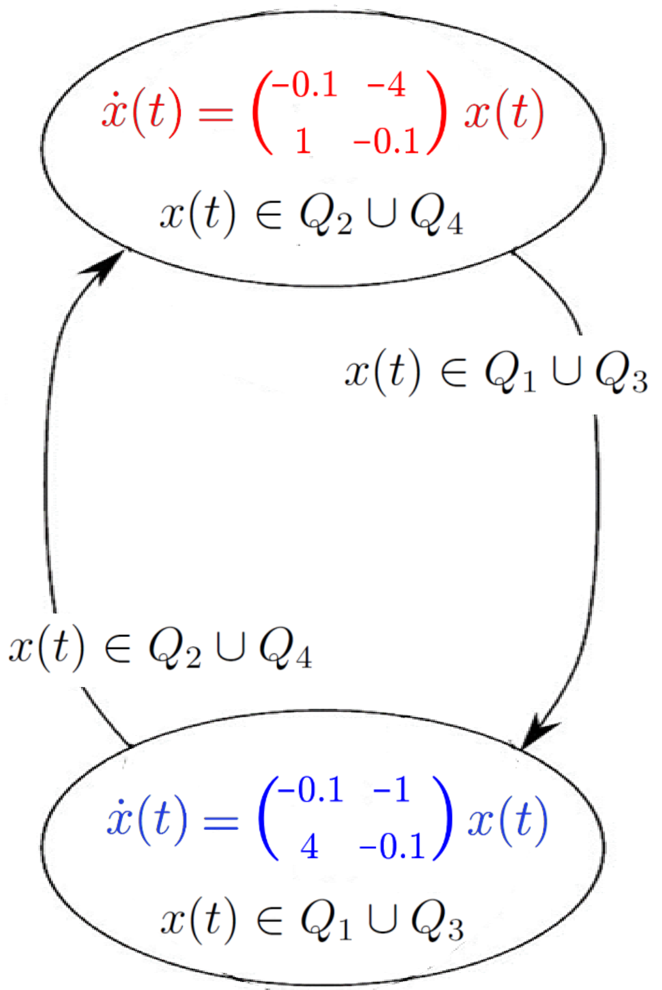
$$|x_0 - x_e| < \delta_a \Rightarrow \lim_{t \to \infty} |x(t) - x_e| = 0$$

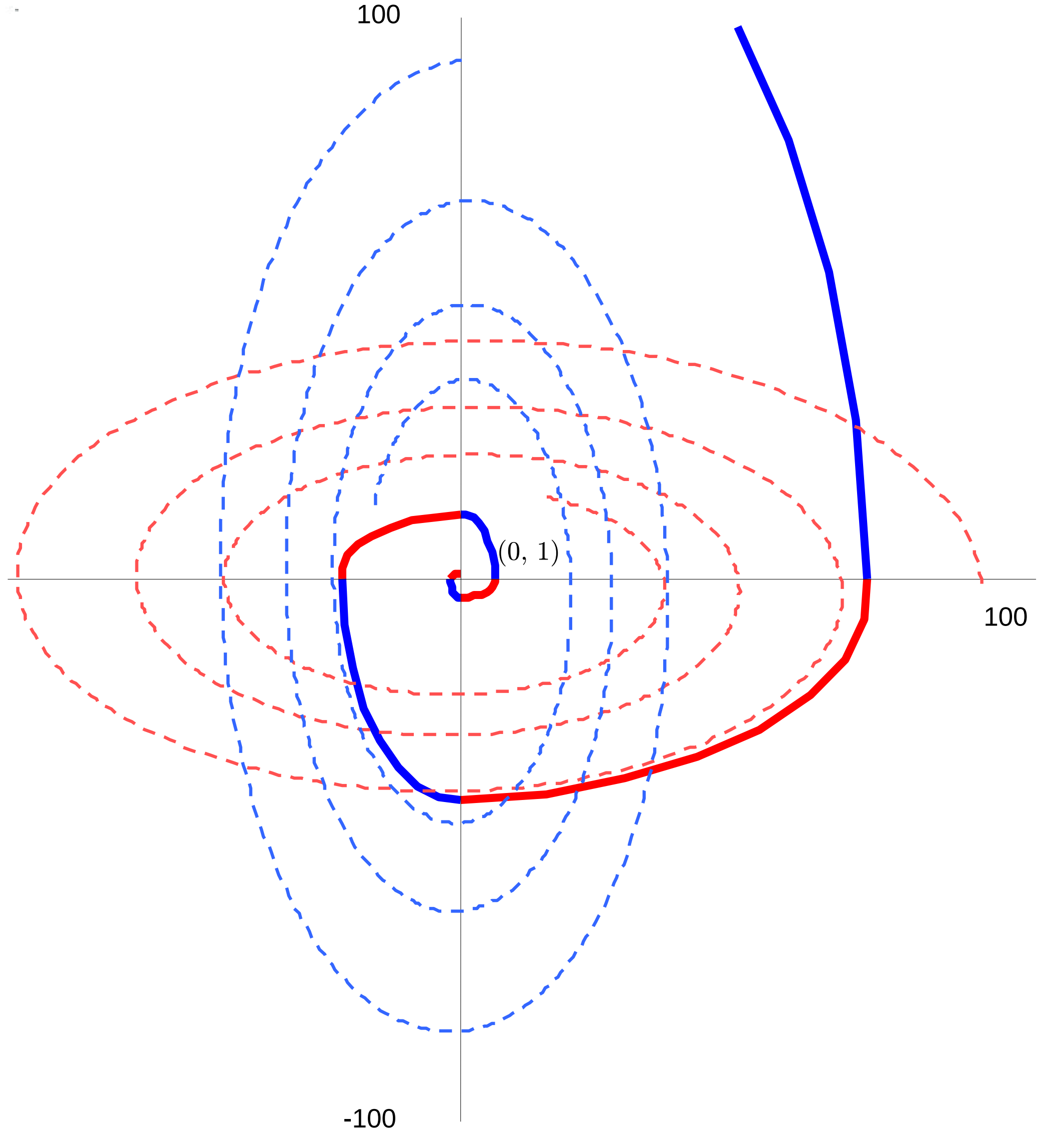
A point which is an asymptotically stable equilibrium point of each location system is an aymptotically stable point of H?





Switching between asymptotically stable linear systems, but $x_e = 0$ is an unstable equilibrium of H !





Switching between asymptotically stable linear systems and $x_e = 0$ is an asymptotically stable equilibrium of H