

Lectures on  
DIFFERENTIAL GEOMETRY AND TOPOLOGY

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Lecture II

Dual homomorphism p. 2

Covariance & contravariance p. 4

Example from analysis p. 8

\* Dual (or adjoint) homomorphism

Let  $T \in \text{Hom}(V, W)$

$\uparrow$   
homomorphism  
linear map  
linear transformation  
linear operator

$\dim_K V = n \quad \dim_K W = m$

$$\begin{aligned} & \text{operations in } V \\ & [ T(\alpha \cdot v_1 + \beta \cdot v_2) \\ & = \alpha \cdot T v_1 + \beta \cdot T v_2 ] \\ & \text{operations in } W \end{aligned}$$

Let  $e = (e_1, \dots, e_n)$ ,  $f = (f_1, \dots, f_m)$  bases in  $V$  and  $W$ , respectively

The  $m \times n$  matrix  $m_{fe}(T) = (t_{ik})_{\substack{i=1..m \\ k=1..n}}$  representing  $T$

with respect to the given bases is

$$m \text{ rows} \left\{ \left( \begin{array}{c} \\ \vdots \\ t_{ik} \\ \vdots \\ t_{1k} \end{array} \right) \right\}_{i=1}^m \leftarrow i \quad \text{that is} \quad T e_k = \sum_{i=1}^m t_{ik} f_i$$

$t_{ik}$

$m$  columns

and, clearly

$$t_{ik} = f_i^*(T e_k) \quad (\diamond)$$

$i$ -th vector of the dual basis  
 $f^* = (f_1^*, \dots, f_m^*)$  of  $W^*$

The adjoint (or dual) homomorphism  $T'$  (of  $T$ )

is an element in  $\text{Hom}(W^*, V^*)$

$\uparrow$   
caveat!

$$T: V \rightarrow W$$

$$T': W^* \rightarrow V^* \quad \text{defined as}$$

$$\boxed{(T' l)(v) := l(Tv)} \quad \begin{array}{l} \forall v \in V \\ \forall l \in W^* \end{array}$$

$\underbrace{\qquad}_{\begin{array}{c} \cap \\ W^* \end{array}}$        $\underbrace{\qquad}_{\begin{array}{c} \cap \\ V \end{array}}$        $\underbrace{\qquad}_{\begin{array}{c} \cap \\ W^* \end{array}}$        $\underbrace{\qquad}_{\begin{array}{c} \cap \\ W \end{array}}$

Any matrix representation of  $T'$  will be of type  $m \times m$ ; in particular, if  $e^* = (e_1^* \dots e_n^*)$ ,  $f^* = (f_1^* \dots f_m^*)$  denote the dual bases of  $e$ ,  $f$ , respectively, one finds

$$\boxed{m_{e^* f^*}(T') = m_{fe}(T)^t}$$

$\uparrow$  final     $\uparrow$  initial

(This provides an intrinsic meaning to the notion of transpose of a matrix)

$$\text{Pf. } [m_{e^* f^*}(T')]_{i,k} \stackrel{\text{recall } (\diamond)}{=} e_i^{**} (T' f_k^*) = (T' f_k^*)(e_i)$$

recall  $\alpha^{**}(\gamma^*) = \gamma^*(\alpha)$

$$= f_k^*(Te_i) = t_{k,i}$$

by definition  
of  $T'$

( $\diamond$ ) again

This concludes the proof  $\square$

Example

Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$

columns

$$T \equiv m_{f,e}(T) = \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 0 & 5 \end{pmatrix}$$

canonical  
bases

we show that  $T' = T^t$  using the very definition

$$T': \mathbb{R}^3 \rightarrow \mathbb{R}^2$$

rows

$$l: \mathbb{R}^3 \rightarrow \mathbb{R} \quad l \mapsto (\alpha, y, z)$$

$$(T'l)(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}) := l(Te_1) = (\alpha, y, z) \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$= \alpha + 2y$$

$$(T'l)(\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}) := l(Te_2) = (\alpha, y, z) \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}$$

$$= \alpha + 4y + 5z$$

Therefore

$$(\alpha, y, z) \xrightarrow{T'} (\alpha + 2y, \alpha + 4y + 5z)$$

||

$$\underbrace{\begin{pmatrix} 1 & 2 & 0 \\ 1 & 4 & 5 \end{pmatrix}}_{T^t} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

# \* Transformation laws (contravariance & covariance)

(I) Let  $T \in \text{GL}(V)$        $T: V \rightarrow V$   
 ↗ invertible  
 endomorphisms

notation: general linear group  
 associated to  $V$

The problem is the following: find  $S': V^* \rightarrow V^*$ ,  
 (stemming from  $S: V \rightarrow V$ ,  
 $S \in \text{GL}(V)$ )

such that,  $\forall \ell \in V^*, \forall v \in V,$   

$$\boxed{(S' \ell)(Tv) = \ell(v)}$$

that is, if vectors in  $V$  are transformed via  $T$ , how  
 should vectors in  $V$  transform, in order that the  
 corresponding evaluations do not change?

[ "Gelfand's Principle" ]

One immediately finds, given  $\ell \in V^*$   
 $(S' \ell)(Tv) = \ell(Sv) = \ell(v) \quad \forall v \in V$   
 def.

if and only if     $S T = I \Rightarrow S = T^{-1}$

$$\Rightarrow S' = (T^{-1})' \quad (\text{which is easily seen to equal } (T')^{-1})$$

The upshot is that

to  $T: V \rightarrow V$  "contravariance"

there corresponds  $(T')^{-1} = (T^{-1})'$  "covariance"

vectors in  $V$ : contravariant vectors      "vectors"  
 $V^*$ : covariant vectors      "covectors"

Vectors from  $V$  and  $V^*$  can be distinguished  
 by their behaviour under linear transformations.  
 "vectors"      "covectors"

② Let us deal with the same problem from  
 a slightly different, more concrete  
 stand point.

Again consider the scalar  $\ell(v)$

$$\ell(v) = \sum_{i=1}^n v_i e_i$$

fix bases  $e = (e_1, \dots, e_n)$  and  $e' = (e'_1, \dots, e'_n)$ .

together with the corresponding dual bases.

$$\text{Then } \ell = \sum_{i=1}^n x_i e_i = \sum_{i=1}^n x'_i e'_i$$

and one has

$$x' = M_{e' e}^{-1}(I) x$$

change of basis matrix

$$A = \underbrace{M_{e' e}^{-1}(I)}_{\text{non singular}} \quad A \in \text{GL}(n, K)$$

Similarly, one has

$$1 = \sum_{i=1}^n y_i e_i^* = \sum_{i=1}^n y'_i e'^*$$

and, obviously  $y' = B y$

for some  $B \in \text{GL}(n, K)$

what is then  $B$ ?

we have, successively

$$l(v) = y^t x = y'^t x'$$

$$v \leftrightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x$$

$$l \leftrightarrow y^t = (y_1 \dots y_n)$$

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$\text{But } y^t x = y^t A^{-1} A x \\ = (A^{-t} y)^t x'$$

$$\Rightarrow (A^{-t} y)^t x' = y'^t x'$$

$$\forall x' \in \mathbb{R}^n$$

$$\Rightarrow y'^t = A^{-t} y \quad \text{i.e. } B = A^{-t}$$

Therefore

$$v \mapsto B^{-t} v$$

$$l \mapsto \underline{\underline{B} \cdot l}$$

This is of course consistent with the previous

discussion.

III

Still another point of view

"vintage"  
definition

Let  $e = \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$  vectors.

$$x = \sum_i x_i e_i \equiv x^t e \quad x^t = (x_1 \dots x_n)$$

$$= \sum_i x'_i e'_i \equiv x'^t e' \quad \boxed{x^t} \quad \boxed{e}$$

Then, if  $x' = Ax$

$$\boxed{x^t} = \boxed{\begin{matrix} & & \\ A & & \\ & & \end{matrix}} \boxed{x'^t} \quad A \quad \boxed{x^t}$$

Covariance, i.e. coordinate-like transformation  
coordinates transform covariantly

$$\text{Then } e' = A^{-t} e$$

i.e. bases transform contravariantly

Indeed:

$$x'^t e' = (Ax)^t e' = x^t A^t e' \quad \forall x \in \mathbb{R}^n$$

~~$x^t e$~~

$$\Rightarrow e = A^t e' \quad , \quad e' = A^{-t} e$$

Covariance: coordinate type transformation

Contravariance: basis type transformation

## \* Important example

Let  $r = r(u, v)$   $(u, v) \in U \subset \mathbb{R}^2$   $r \in \mathcal{C}^2$   
 parametric surface

(This concept is invariant under regular parameter changes)

$$\begin{cases} u' = u'(u, v) \\ v' = v'(u, v) \end{cases} \quad (u, v) \in U$$

f

the differential of  $(u, v) \mapsto (u', v')$  reads  
 at a generic point:

$$\begin{cases} du' = \frac{\partial u'}{\partial u} du + \frac{\partial u'}{\partial v} dv \\ dv' = \frac{\partial v'}{\partial u} du + \frac{\partial v'}{\partial v} dv \end{cases} \quad \text{i.e. } \begin{pmatrix} du' \\ dv' \end{pmatrix} = \begin{pmatrix} \frac{\partial u'}{\partial u} & \frac{\partial u'}{\partial v} \\ \frac{\partial v'}{\partial u} & \frac{\partial v'}{\partial v} \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}$$

One has a corresponding transformation (basis change)  
 on the tangent plane (at any point) J: Jacobian matrix

$$\begin{cases} \frac{\partial r}{\partial u'} = \frac{\partial r}{\partial u} \frac{\partial u}{\partial u'} + \frac{\partial r}{\partial v} \frac{\partial v}{\partial u'} \\ \frac{\partial r}{\partial v'} = \frac{\partial r}{\partial u} \frac{\partial u}{\partial v'} + \frac{\partial r}{\partial v} \frac{\partial v}{\partial v'} \end{cases} \quad \begin{pmatrix} \frac{\partial r}{\partial u'} \\ \frac{\partial r}{\partial v'} \end{pmatrix} = \begin{pmatrix} \frac{\partial r}{\partial u} & \frac{\partial r}{\partial v} \\ \frac{\partial r}{\partial u} & \frac{\partial r}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial u'} \\ \frac{\partial v}{\partial v'} \end{pmatrix}$$

abstractly:

$$\begin{pmatrix} \frac{\partial}{\partial u'} \\ \frac{\partial}{\partial v'} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial u'} & \frac{\partial v}{\partial u'} \\ \frac{\partial u}{\partial v'} & \frac{\partial v}{\partial v'} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial v} \end{pmatrix}$$

remember  $J_{f^{-1}} = J_f^{-1}$

J<sup>-t</sup>

directional derivatives along coordinates

From  $\frac{\partial \chi}{\partial u} = \frac{\partial v}{\partial v} = 1$ ,  $\frac{\partial \chi}{\partial v} = \frac{\partial v}{\partial u} = 0$

We conclude that, setting

$$e = \left( \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right)$$

then

$$e^* = (du, dv)$$

We shall resume this discussion later on.



## 2-dimensional case

Let's start with the following case:



This is the simplest one with no boundary vertex.

Proposition 6. Two parts of a 2-dimensional  $\mathbb{R}$ -surface are 2-dimensional

Example 10. If  $[CO]$  is a boundary part of a 2-dimensional

dimensional torus up to codimension. This is the case if

$\dim_{\mathbb{R}} \text{Bisectors}(\text{torus}) - \dim_{\mathbb{R}} \text{torus} = 1$ .

Proposition 6. Bisection 2-dimensional follows by a construction of Mather

and Palais [MP]. It follows that the 2-dimensional

we see a result of Mather and Smale [MS] (see also Mather and Palais

[MP]) that the boundary of a 2-dimensional surface is a 1-dimensional

surface. This means that the boundary of some 2-dimensional surface

which is a 2-dimensional surface, it is easy to construct an  $\mathbb{R}$ -surface

such that the boundary of this surface is a 2-dimensional surface. We

can choose points on the boundary of this surface to be vertices of a

triangle. Now if we take three 2-dimensional surfaces

such that we can connect the boundaries of these three surfaces by