Introduzione alla proof theory

(i) Equivalence of formulas: in fact what we call a formula is indeed an equivalence class: we identify two formulas which differ only by the names of their bound variables, precisely: $A \sim A$; if $A \sim A'$ and $B \sim B'$, then $\neg A \sim \neg A'$, $A \wedge B \sim A' \wedge B'$, $A \vee B \sim A' \vee B'$, $A \to B \sim A' \to B'$. If $A[x_n]$ and $A'[x_m]$ are formulas, let x_p be a variable occurring neither in A nor in A'; then, if $A[x_p] \sim A'[x_p]$ we have $\forall x_n A[x_n] \sim \forall x_m A'[x_m]$ and $\exists x_n A[x_n] \sim \exists x_m A[x_m]$. An immediate consequence of the definition is that, given C, it is possible to find D such that $C \sim D$ and

- no variable in D is both free and bound
- any bound variable in D occurs in the scope of only one occurrence of a quantifier.

If one wants to substitute a term t in the formula $A[x_n]$ for the variable x_n, one first chooses $A' \sim A$ such that no free variable of t occurs bound in $A'[x_n]$; what we denote by $A[t]$ is indeed (the equivalence class of) $A'[t]$.

$\underbrace{A \to B}$ $\Downarrow$ formula

$\underbrace{A \vdash B}$ esiste una deduzione di B dall' ipotesi A

$$\Downarrow$$

ANALISI FINE DEI COSTRUTTI LOGICI

UNO DEGLI OBIETTIVI:

CONSISTENZA (SENZA USARE I MODELLI)

$\nvdash$ "$\perp$"

PROPR per ogni C esiste D tale che

1) $C \sim D$
2) in D nessuna variabile è libera e legata
3) ogni variabile legata in D è nello scopo di esattamente un quantificatore

SOSTITUZIONE ?

In realtà LAVORIAMO CON CLASSI DI EQUIVAL. ($FORM/_{\sim}$)

Data $A[x]$ e t $A[t]$ è $[A'[t]]$, t.c. $A'[x] \sim A[x]$ e NESSUNA VARIABILE DI t OCCORRE LIBERA IN $A'[x]$

2.1.1. - definition

A **sequent** is a formal expression $\Gamma \vdash \Delta$ where Γ and Δ are finite sequences of formulas of **L**.

IL CALCOLO $\boxed{LK}$

$\Gamma \vdash \Delta$ → ANTECEDENTE / SUCCEDENTE ⇒ INTUIZIONE

$$(\Gamma \vdash \Delta) = \bigwedge \Gamma \to \bigvee \Delta$$

$$(\ \vdash \Delta) = \bigvee \Delta \qquad (\top \to \bigvee \Delta)$$

$$(\Gamma \vdash\ ) = \bigwedge \Gamma \to \bot$$

$$(\ \vdash\ ) = \bot \qquad (A \wedge \neg A)$$

(i) identity axioms

$$A \vdash A$$

(ii) logical rules

conjunction

$$\frac{\Gamma \vdash A^{\dagger}, \Delta \quad \Lambda \vdash B^{\dagger}, \Pi}{\Gamma, \Lambda \vdash A \wedge B^{\dagger}, \Delta, \Pi}\, r\wedge \qquad \frac{\Gamma, A^{\dagger} \vdash \Delta}{\Gamma, A \wedge B^{\dagger} \vdash \Delta}\, l1\wedge \qquad \frac{\Gamma, B^{\dagger} \vdash \Delta}{\Gamma, A \wedge B^{\dagger} \vdash \Delta}\, l2\wedge$$

disjunction

$$\frac{\Gamma \vdash A^{\dagger}, \Delta}{\Gamma \vdash A \vee B^{\dagger}, \Delta}\, r1\vee \qquad \frac{\Gamma \vdash B^{\dagger}, \Delta}{\Gamma \vdash A \vee B^{\dagger}, \Delta}\, r2\vee \qquad \frac{\Gamma, A^{\dagger} \vdash \Delta \quad \Lambda, B^{\dagger} \vdash \Pi}{\Gamma, \Lambda, A \vee B^{\dagger} \vdash \Delta, \Pi}\, l\vee$$

negation

$$\frac{\Gamma, A^{\dagger} \vdash \Delta}{\Gamma \vdash \neg A^{\dagger}, \Delta}\, r\neg \qquad \frac{\Gamma \vdash A^{\dagger}, \Delta}{\Gamma, \neg A^{\dagger} \vdash \Delta}\, l\neg$$

implication

$$\frac{\Gamma, A^{\dagger} \vdash B^{\dagger}, \Delta}{\Gamma \vdash A \to B^{\dagger}, \Delta}\, r\to \qquad \frac{\Gamma \vdash A^{\dagger}, \Delta \quad \Lambda, B^{\dagger} \vdash \Pi}{\Gamma, \Lambda, A \to B^{\dagger} \vdash \Delta, \Pi}\, l\to$$

for all

$$\frac{\Gamma \vdash A^{\dagger}, \Delta}{\Gamma \vdash \forall x\, A^{\dagger}, \Delta}\, r\forall(*) \qquad \frac{\Gamma, A[t]^{\dagger} \vdash \Delta}{\Gamma, \forall x\, A[x]^{\dagger} \vdash \Delta}\, l\forall(**)$$

there is

$$\frac{\Gamma \vdash A[t]^{\dagger}, \Delta}{\Gamma \vdash \exists x\, A[x]^{\dagger}, \Delta}\, r\exists(**) \qquad \frac{\Gamma, A^{\dagger} \vdash \Delta}{\Gamma, \exists x\, A^{\dagger} \vdash \Delta}\, l\exists(*)$$

(iii) structural rules

weakening

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash A^{\dagger}, \Delta}\, rW \qquad \frac{\Gamma \vdash \Delta}{\Gamma, A^{\dagger} \vdash \Delta}\, lW$$

exchange

$$\frac{\Gamma \vdash \Delta', A^{\dagger}, B^{\dagger}, \Delta''}{\Gamma \vdash \Delta', B^{\dagger}, A^{\dagger}, \Delta''}\, rE \qquad \frac{\Gamma', A^{\dagger}, B^{\dagger}, \Gamma'' \vdash \Delta}{\Gamma', B^{\dagger}, A^{\dagger}, \Gamma'' \vdash \Delta}\, lE$$

contraction

$$\frac{\Gamma \vdash A^{\dagger}, A^{\dagger}, \Delta}{\Gamma \vdash A^{\dagger}, \Delta}\, rC \qquad \frac{\Gamma, A^{\dagger}, A^{\dagger} \vdash \Delta}{\Gamma, A^{\dagger} \vdash \Delta}\, lC$$

(iv) Cut

$$\frac{\Gamma \vdash A^{\dagger}, \Delta \quad \Lambda, A^{\dagger} \vdash \Pi}{\Gamma, \Lambda \vdash \Delta, \Pi}\, CUT$$

2.1.5. - terminology

(i) The rules (i)-(ii)-(iii) are the **cut-free rules**; they are sometimes styled the **genetic rules**, because they are closely related to the construction of the formulas of the sequents. A **cut-free proof** is a proof using only the cut-free rules.

(ii) The essential part of the sequent calculus lies in the so-called logical rules 2.1.4. (ii). These rules are naturally divided into *right* and *left* rules; right rules are sometimes styled **introductions** and left rules **eliminations**. A rule is right iff its **main formula** (i.e. the formula labelled with $\dagger$ in the conclusion of the rule)(*) appears in the right part of the sequent; similarly for left rules. The left part of the sequent is the **antecedent** while its right part is the **succedent**.

(iii) The terminology introduced in (ii) for logical rules can also be used for structural rules, except for exchange.

(iv) In the case of a cut, the formula $A^{\dagger}$ is called the **cut-formula**.

We see that, in proofs, eigenvariables are considered as bound variables. The basic property is that given π, one can find $\pi' \sim \pi$ such that all eigenvariables of rules $(r\forall)$ or $(l\exists)$ are distinct and different from the remaining variables occurring in π.

$\mathcal{S}$ ⇒ INSIEME DEI SEQUENTI

$\mathcal{T}$ ⇒ INSIEME DEGLI ALBERI ORDINATI FINITI (2-ari)

$\langle T, \varphi \rangle$ $\quad \varphi : \text{nodi}(T) \to \mathcal{S}$

è DETTO ALBERO ETICHETTATO

$$\begin{array}{ccc} S_1 & & S_2 \\ & \diagdown \quad \diagup & \\ & S_3 & \end{array} \rightsquigarrow \dfrac{S_1 \qquad S_2}{S_3}$$

$$\begin{array}{c} S_1 \\ | \\ S_2 \end{array} \rightsquigarrow \dfrac{S_1}{S_2}$$

Relazione padre figlio

UNA DERIVAZIONE (PROOF-TREE ...)

É UN ALBERO ETICHETTATO

$\Pi = \langle T, \varphi \rangle$

TALE CHE

- LE FOGLIE SONO ETICHETT. CON $A \vdash A$

$$\frac{S_1}{S_2} \in \Pi \Rightarrow \frac{S_1}{S_2}$$ ISTANZA REG. UNARIA

$$\frac{S_1 \quad S_2}{S_3} \in \Pi \Rightarrow \frac{S_1 \quad S_2}{S_3}$$ ISTANZ REG BIN.

$$\left.\begin{array}{c}\Pi \\ \Gamma \vdash \Delta\end{array}\right\}$$ DERIVAZ. CON RADICE (DETTA CONCLUSIONE) ETICHETTATA CON $\Gamma \vdash \Delta$

$\Downarrow$

Π È UNA DERIVAZIONE DI $\Gamma \vdash \Delta$

$\Gamma \vdash \Delta$ È DIMOSTRABILE SE

$\exists\, \Pi$ $\quad \dfrac{\Pi}{\Gamma \vdash \Delta}$

SE $\Gamma \vdash \Delta$ È DIMOSTRABILE

ALLORA È DETTO TEOREMA

Notazione $\quad \Pi \left\{ \begin{array}{c} \vdots \\ \Gamma \vdash \Delta \end{array} \right.$

In practice, we shall very often forget the structural rules, especially the trifling exchange rules. If we want to indicate the use of structural rules, we shall use $=\!=\!=\!=$ instead of $\underline{\qquad}$: $\begin{array}{c}\Gamma \vdash \Delta \\ \hline\hline \Gamma' \vdash \Delta'\end{array}$ means that $\Gamma' \vdash \Delta'$ has been obtained from $\Gamma \vdash \Delta$ by a finite number (possibly zero) of structural rules; $\begin{array}{c}\Gamma \vdash \Delta \\ \hline\hline \Gamma' \vdash \Delta'\end{array}\, r \rightarrow$ means that finitely many structural rules, together with one application of $(r \rightarrow)$, have been used. For instance:

$$
\begin{array}{c}
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A[y] \vdash A[y]}{A[y] \vdash \forall x \neg A[x] \vee A[y]}\, r2\vee}{A[y] \vdash \exists y\, (\forall x \neg A[x] \vee A[y])}\, r\exists}{\vdash \neg A[y], \exists y\, (\forall x \neg A[x] \vee A[y])}\, r\neg}{\vdash \forall x \neg A[x], \exists y\, (\forall x \neg A[x] \vee A[y])}\, r\forall}{\vdash \forall x \neg A[x] \vee A[y], \exists y\, (\forall x \neg A[x] \vee A[y])}\, r1\vee}{\vdash \exists y\, (\forall x \neg A[x] \vee A[y]), \exists y\, (\forall x \neg A[x] \vee A[y])}\, r\exists \\
\hline\hline
\vdash \exists y\, (\forall x \neg A[x] \vee A[y])
\end{array}
$$

$\neg\neg A \to A$

$$\dfrac{\dfrac{\dfrac{A \vdash A}{\vdash \neg A, A}}{\neg\neg A \vdash A}}{\vdash \neg\neg A \to A}$$

ESERCIZIO (OBBLIGATORIO)

LK^* = ASSIOMI ATOMICI

DIMOSTRARE CHE

"Se $\Gamma \vdash \Delta$ è dimostrabile in LK allora è dimostrabile in LK^*"

1)

$$\cfrac{\cfrac{\cfrac{A \vdash A}{A, B \vdash A}}{A \vdash B \to A}}{\vdash A \to (B \to A)}$$

2)

$C \vdash C \quad B \vdash B$

$B, B \to C \vdash C$

$B \to C, B \vdash C \qquad A \vdash A$

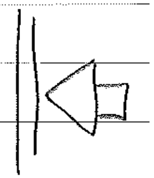
$B \to C, A, A \to B \vdash C$

exc

$A, A \to B, B \to C \vdash C \qquad A \vdash A$

$A, A \to B, A, A \to (B \to C) \vdash C$

$A \to (B \to C), A \to B, A^{\circ}, A^{\circ} \vdash C$



$A \to (B \to C), A \to B, A \vdash C$

$A \to (B \to C), A \to B \vdash A \to C$

$A \to (B \to C) \vdash (A \to B) \to (A \to C)$

$\vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))$

$$B \vdash B \qquad\qquad A \vdash A$$

$$B, \sim B \vdash \qquad\qquad \vdash \sim A, A$$

$$A \vdash A$$

$$B, \sim A \to \sim B \vdash A \qquad\qquad \vdash \sim A, A$$

$$\sim A \to \sim B, \sim A \to B \vdash A, A$$

$$\sim A \to \sim B, \sim A \to B \vdash A$$

$$\sim A \to \sim B \vdash (\sim A \to B) \to A$$

$$\vdash (\sim A \to \sim B) \to ((\sim A \to B) \to A)$$

ALTEZZA DI UNA PROVA

$$h(\Pi) = \begin{cases} 0 & \text{se } \Pi \text{ è un assioma.} \\ \sup\{ h(\Pi_i) + 1 \mid \Pi_i \text{ è sottoprove premesse di } \Pi \} \end{cases}$$

SOTTOFORMULA (A LA GENTZEN)

$SF(A) = \{A\}$ SE A É ATOMICA

$SF(A \circ B) = SF(A) \cup SF(B) \cup \{A \circ B\}$

$SF(\neg A) = SF(A) \cup \{\neg A\}$

$$SF(\forall x\, A(x)) = \{\forall x\, A(x)\} \cup \bigcup_{t \in TERM} SF(A(t))$$

$$SF(\exists x\, A(x)) = \{\exists x\, A(x)\} \cup \bigcup_{t \in TER} SF(A(t))$$

$\# SF(\forall x\, A(x)) = \aleph_0$

poiché $\# VAR = \aleph_0$

$SF(\Gamma \vdash \Delta) = SF(\Gamma) \cup SF(\Delta)$ | $SF(A_1, -, A_n) = \bigcup SF(A_i)$

TEOREMA

SE $$\frac{\Pi}{\Gamma \vdash \Delta}$$

E Π E' CUT-FREE

ALLORA SE $A_1, \ldots, A_n \vdash B_1, \ldots, B_n \in \Pi$

ALLORA TUTTE LE A_i E TUTTE LE B_i SONO SOTTOFORMULE DI Γ, Δ

DIM OVVIA (PER INDUZIONE)

ESERCIZIO

SIA p ATOMICA

LK NON DIMOSTRA $\vdash p$

E LK NON DIMOSTRA $\vdash \neg p$

DIM

SUGGERIMENTO : usare il principio delle sottoformule ...

PROP

I SEGUENTI FATTI SONO EQUIVALENTI

1) ESISTE $\begin{array}{c}\Pi\\ \hline \vdash\end{array}$

2) PER OGNI $\Gamma \vdash \Delta$ ESISTE $\begin{array}{c}\Pi\\ \hline \Gamma \vdash \Delta\end{array}$

3) ESISTE $\begin{array}{c}\Pi_1\\ \hline \vdash A\end{array}$ & ESISTE $\begin{array}{c}\Pi_2\\ \hline \vdash \neg A\end{array}$

DIM

(1) $\Rightarrow$ 2

SIA $\begin{array}{c}\overline{\pi}\\ \vdash\end{array}$

CON REG. STR.

$$\dfrac{\begin{array}{c}\overline{\pi}\\ \vdash\end{array}}{\Gamma \vdash \Delta}$$

PER OGNI Γ, Δ

(2) $\Rightarrow$ (3) IMMED.

(3) $\Rightarrow$ (1)

$$\dfrac{\begin{array}{c}\overline{\pi_1}\\ \vdash A\end{array} \qquad \dfrac{\begin{array}{c}\overline{\pi_2}\\ \vdash \neg A\end{array} \qquad \dfrac{A \vdash A}{A, \neg A \vdash}}{A \vdash}\ \text{CUT}}{\vdash}$$

TEO

LK È CONSISTENTE

DIM

SIA Π
$\vdash$

ALLORA PER IL TEO SOTTOFORM.
~~IN Π NON POSSONO OCCORRERE~~
FORMULE ... IMP.

SIST. A LA HILBERT $\mathcal{H}$

ASSIOMI

$$A \to (B \to A)$$

$$A \to (B \to C) \to ((A \to B) \to (A \to C))$$

$$(\sim A \to B) \to ((\sim A \to \sim B) \to A)$$

$$\forall x\, A[x] \to A[t] \quad (t \text{ libero per } x \text{ in } A)$$

$$\forall x\,(A \to B) \to (A \to \forall x\, B) \quad (x \notin FV(A))$$

REGOLE DI INFERENZA

$$\frac{A \quad A \to B}{B} \qquad \frac{A}{\forall x\, A}$$

TEO

A è DERIVABILE IN $\mathcal{H}$

$\Longleftrightarrow$

$\vdash A$ è DERIVABILE IN LK

Semantics

$$\sigma \models \Gamma \vdash \Delta \iff \left[\forall G (G \in \Gamma \Rightarrow \sigma \models G) \Rightarrow \exists D (D \in \Delta \;\&\; \sigma \models D)\right]$$

NOTA BENE

$$\forall \sigma \quad \sigma \not\models \;\vdash$$

Teorema

Se $\Gamma \vdash \Delta$ è derivabile

allora $\models \Gamma \vdash \Delta$

Vediamo un caso

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash \forall x A, \Delta}$$

vogliamo dimostrare che

$$\models \Gamma \vdash A, \Delta \Rightarrow \models \Gamma \vdash \forall x. A, \Delta$$

$$\models \Gamma \vdash A, \Delta \Leftrightarrow \forall \sigma \left(\forall G \in \Gamma.\ \sigma \models G \Rightarrow \left(\begin{array}{l} \exists D \in \Delta\ \ \sigma \models D \\ \quad \text{o} \quad \sigma \models A \end{array} \right) \right)$$

supponiamo ora che $\exists \bar{\sigma}$ t.c.

$$\forall G \in \Gamma.\ \bar{\sigma} \models G \ \&\ \bar{\sigma} \not\models \forall x. A \ \&\ \forall D \in \Delta\ \bar{\sigma} \not\models D$$

quindi

$$\exists \bar{\sigma} \text{ t.c. } \bar{\sigma} \models \Gamma \ \&\ \exists u\ \bar{\sigma}(x/u) \not\models A \ \&\ \forall D \in \Delta\ \bar{\sigma} \not\models D$$

$\Rightarrow$ (dato che $x \notin FV(\Gamma, \Delta)$)

$$\exists \bar{\sigma}, u \left[\bar{\sigma}(x/u) \models \Gamma \ \&\ \bar{\sigma}(x/\bar{u}) \not\models A \ \&\ \forall D \in \Delta\ \bar{\sigma}(x/u) \not\models D \right]$$

IMPOSSIBILE

TEOREMA (LEMMA)

$\mathcal{H}; A \Rightarrow \exists \Pi \quad \begin{array}{c}\Pi \\ \vdash A\end{array}$

INDUZIONE SULLA DERIV. $A_0, \ldots, A_k (A)$

1) se A è un assioma dimostraz. diretta

$$\dfrac{\dfrac{\dfrac{A \vdash A}{A, B \vdash A}}{A \vdash B \to A}}{\vdash A \to (B \to A)}$$

2) Se A è ottenuto applicando una regola di inferenza (MP)

$$\frac{B \qquad B \to A}{A}$$

$$\dfrac{\begin{matrix}\text{II}\\ \vdash B\end{matrix} \qquad \dfrac{\begin{matrix}\text{II}\\ \vdash B \to A\end{matrix} \qquad \dfrac{A \vdash A \qquad B \vdash B}{B \to A, B \vdash A}}{B \vdash A}}{\vdash A}$$

ecc..

3) $\dfrac{A}{\forall x A}$ $\qquad$ $\dfrac{\begin{matrix}\text{II}\\ \vdash A\end{matrix}}{\vdash \forall x A}$

COROLLARIO

$\mathfrak{A} : A \iff \exists \pi \quad \begin{array}{c} \pi \\ \vdash A \end{array}$

$\Rightarrow$ appena mostrato

$\Leftarrow$

$\begin{array}{c} \pi \\ \vdash A \end{array} \Rightarrow \models \vdash A \Rightarrow \models A \Rightarrow \mathfrak{A} : A$

(ii) Sequent calculus: the distinction between free and bound variables can be naturally extended to sequent calculus. Since the formulas in a sequent are equivalence classes, it follows that the rules ($r\forall$) and ($l\exists$) could be written as well:

$$\frac{\Gamma \vdash A[x_n]^\dagger, \Delta}{\Gamma \vdash \forall x_m A[x_m]^\dagger, \Delta}\, r\forall(*) \qquad \frac{\Gamma, A[x_n]^\dagger \vdash \Delta}{\Gamma, \exists x_m A[x_m]^\dagger \vdash \Delta}\, l\exists(*)$$

(see 2.1.4.), i.e. there is no identification between x_m in $\forall x_m A[x_m]$ and the *eigenvariable* of the rule, x_n.

1) EQUIVALENZA DI DIMOSTRAZIONI

$$\Downarrow$$

EIGENVARIABLE TRATTATE COME VARIABILI QUANTIFICATE

IDEA

$$\dfrac{\begin{matrix}\vdots \\ \Pi \\ \Gamma \vdash A[x_m], \Delta\end{matrix}}{\Gamma \vdash \forall x_n\, A[x_n], \Delta} \quad \sim \quad \dfrac{\begin{matrix}\Pi\,[x_m/x_p] \\ \Gamma \vdash A[x_p], \Delta\end{matrix}}{\Gamma \vdash \forall x_n\, A[x_n], \Delta}$$

DEFINIZIONE

Supponiamo che

$$I = \emptyset, \{1\}, \{1,2\}$$

$$\pi \left\{ \dfrac{\begin{pmatrix} (\pi_i) \\ \Gamma_i \vdash \Delta_i \end{pmatrix}_{i \in I}}{\Gamma \vdash \Delta} R \right. \qquad \pi' \left\{ \dfrac{\begin{pmatrix} \pi'_i \\ \Gamma'_i \vdash \Delta'_i \end{pmatrix}_{i \in I}}{\Gamma' \vdash \Delta'} R \right.$$

e che $\Gamma \vdash \Delta \sim \Gamma' \vdash \Delta'$

1) se $R \neq \vdash\forall, \exists\vdash$

allora se $\forall i \in I$, $\pi_i \sim \pi'_i$

allora $\pi \sim \pi'$

$$r = \vdash\neg \qquad r' = \neg\vdash \qquad A = \neg B$$

$$\dfrac{\begin{array}{c}\Pi_1\\ \Gamma, B \vdash \Delta_1\end{array}}{\Gamma \vdash \neg B, \Delta_1} \qquad \dfrac{\begin{array}{c}\Pi'_1\\ \Gamma'_1 \vdash B, \Delta'\end{array}}{\Gamma'_1, \neg B \vdash \Delta'}$$

$$\dfrac{\begin{array}{c}\mathrm{MIX}[\Pi_1, \Pi']\\ \Gamma, B, \Gamma'_1 - \neg B \vdash \Delta_1 - \neg B, \Delta'\end{array} \qquad \begin{array}{c}\mathrm{MIX}[\Pi, \Pi'_1]\\ \Gamma, \Gamma'_1 - \neg B \vdash \Delta_1 - \neg B, B, \Delta'\end{array}}{\overline{\Gamma, \Gamma'_1 - \neg B \vdash \Delta_1 - \neg B, \Delta'}}$$

2) Se $r = \vdash\forall$

(ovvero

$$\Pi \left\{ \dfrac{\begin{matrix}\Pi_1 \\ \Gamma \vdash A[x_n], \Delta\end{matrix}}{\Gamma \vdash \forall x_n A[x_n], \Delta} \qquad \dfrac{\begin{matrix}\Pi'_1 \\ \Gamma' \vdash A'[x_m], \Delta'\end{matrix}}{\Gamma' \vdash \forall x_m A'[x_m], \Delta'} \right\} \Pi'$$

se esistono $\check{\Pi}_1$ e $\check{\Pi}'_1$ t.c. $\check{\Pi}_1$ non usa x_n come eigenvar.
e $\check{\Pi}'_1$ non usa x_m come eigenvar

e t.c. $\Pi_1 \sim \check{\Pi}_1$ $\quad \Pi'_1 \sim \check{\Pi}'_1$

e se per x_p FRESH

$$\check{\Pi}_1[x_n/x_p] \sim \check{\Pi}'_1[x_m/x_p]$$

ALLORA $\Pi \sim \Pi'$

CONSEGUENZA

DATA Π $\exists\ \Pi'$ $\Pi \sim \Pi'$ t.c.

- Tutte le eig.v. di $\vdash\forall$ $\exists\vdash$ sono distinte e diverse da tutte le altre variabili.

GRADO DI UNA FORMULA

$d(P) = 0$ se P è atomica

$d(A \circ B) = \sup(d(A)+1, (B)+1)$

$d(\neg A) = d(\forall x A) = d(\exists x A) = d(A)+1$

GRADO DI UN CUT

È IL GRADO DELLA CUT-FORMULA

GRADO DI UNA PROVA

$$\delta(\Pi) = \begin{cases} 0 & \text{se } \Pi \text{ è } \cancel{\text{CUT-FREE}} \\ \sup\{ d(A)+1 \mid A \text{ cut form in } \Pi \} \end{cases}$$

ALTEZZA DI UNA PROVA

$$h(\Pi) = \begin{cases} 0 & \text{se } \Pi \text{ è un assioma.} \\ \sup\{ h(\Pi_i)+1 \mid \Pi_i \text{ è sottoprove premesse di } \Pi \} \end{cases}$$

$$\Pi \left\{ \dfrac{\begin{matrix}\Pi_1 \\ \Gamma_1 \vdash \Delta_1\end{matrix}}{\Gamma \vdash \Delta} R \right. \qquad \Pi \left\{ \dfrac{\begin{matrix}\Pi_1 \\ \Gamma_1 \vdash \Delta_1\end{matrix} \quad \begin{matrix}\Pi_2 \\ \Gamma_2 \vdash \Delta_2\end{matrix}}{\Gamma \vdash \Delta} R \right.$$

$$\Pi \left\{ A \vdash A \right.$$

NOTA

SE IN Π TUTTI I CUT SONO TRA FORMULE ATOMICHE ALLORA

$$\delta(\Pi) = 1$$

Sia Γ una sequenza

con $\Gamma - A$ denoto

la NUOVA SEQUENZA OTTENUTA

ELIMINANDO TUTTE LE OCCORRENZE

DI A

$$\underbrace{B, A, C, A, A, D}_{\Gamma} - A = B, C, D$$

LEMMA

SIA A UNA FORMULA DI GRADO n $(d(A)=n)$
E SIANO

$$\frac{\Pi}{\Gamma \vdash \Delta}, \quad \frac{\Pi'}{\Gamma' \vdash \Delta'}$$

t.c.
$$\delta(\Pi), \delta(\Pi') \leq n$$

ALLORA È POSSIBILE OTTENERE
(IN MODO EFFETTIVO)

UNA DIMOSTRAZIONE $\dfrac{Mix[\Pi, \Pi']}{\Gamma, \Gamma' - A \vdash \Delta - A, \Delta'}$

t.c. $\delta(Mix[\Pi, \Pi']) \leq n$

OSSERVAZIONE: NON POSSO USARE CUT

$$\tilde{\Pi}\left\{\dfrac{\dfrac{\Pi}{\Gamma \vdash \Delta, A} \qquad \dfrac{\Pi'}{\Gamma', A \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\text{CUT}\right.$$

se $\delta(A) = n$

$\delta(\Pi), \delta(\Pi') \leq n$

$\delta(\tilde{\Pi}) = n+1$!!!

DIMOSTRAZIONE

Per induzione su $h(\Pi) + h(\Pi')$

Siano

$$\Pi \left\{ \frac{\left\{ \begin{array}{c} \Pi_i \\ \Gamma_i \vdash \Delta_i \end{array} \right\}_{i \in I}}{\Gamma \vdash \Delta} r \right. \qquad \Pi' \left\{ \frac{\left\{ \begin{array}{c} \Pi'_j \\ \Gamma_j \vdash \Delta_j \end{array} \right\}_{j \in I'}}{\Gamma' \vdash \Delta'} r' \right.$$

$(I, I' \in \{\emptyset, \{1\}, \{1,2\}\})$

PROCEDIAMO PER CASI (da esaminare in modo sequenziale ...):

1) r ē Ax

i) $\Gamma \vdash \Delta \equiv A \vdash A$

$\Gamma, \Gamma' - A \vdash \Delta - A, \Delta' \equiv \underbrace{A, \Gamma' - A \vdash \Delta'}$

regole strutt. (contrarre tutte le occorrenze di A in Γ' o oppure se non ci sono fare un weak di A)

ii) $\Gamma \vdash \Delta \equiv B \vdash B$

$\Gamma, \Gamma' - A \vdash \Delta - A, \Delta' \equiv$

$B, \Gamma' - A \vdash B, \Delta'$ regole strutt.

2) r' è Ax [come (1)

3) r è una regola strutturale

$$\dfrac{\dfrac{\mathrm{II}(\Pi_1, \Pi') : \mathrm{MIX}[\Pi_1, \Pi']}{}}{\Gamma, \Gamma' - A \vdash \Delta - A, \Delta'}$$

4) r' è una regola strutturale
simmetrica a (3)

5) r è cut OPPURE REGOLA
LOGICA CHE NON INTRODUCE A
A DESTRA

$$\dfrac{\left\{\begin{array}{c} \underline{\Pi_i : \mathrm{MIX}[\Pi_i, \Pi']} \\ \Gamma_i, \Gamma' - A \vdash \Delta_i - A, \Delta' \end{array}\right\}_{i \in I}}{\Gamma, \Gamma' - A \vdash \Delta - A, \Delta'} \; r$$

NOTA r non può essere
UN CUT DI A
altrimenti $\delta(\Pi) \not\geq n+1$

poiché $d(A) = n$

6) r' È UN CUT O UNA REG.
LOGICA CHE NON INTRODUCE A A SINISTR.
SIMMETRICA A 5

7) r INTRODUCE A A DESTRA
r' INTRODUCE A A SINISTRA

CI SONO VARI CASI

7-1) $r = \vdash\to$ $r' = \to\vdash$ $A \equiv B\to C$

$$\Pi \left\{ \frac{\begin{matrix}\Pi_1 \\ \Gamma, B \vdash C, \Delta_1\end{matrix}}{\Gamma \vdash B\to C, \Delta_1} \qquad \frac{\begin{matrix}\Pi'_1 & & \Pi'_2 \\ \Gamma'_1, C \vdash \Delta'_1 & ; & \Gamma'_2 \vdash B, \Delta'_2\end{matrix}}{\Gamma'_1, \Gamma'_2, B\to C \vdash \Delta'_1, \Delta'_2} \right\} \Pi'$$

L'IDEA INTUITIVA : $\underbrace{\text{CUT}(B) : \Pi'_2 , \Pi_1}$
$\Downarrow$ METTERE IN
CUT CON Π'_1

$\text{MIX}[\Pi, \Pi'_2]$ PAG $\Longrightarrow$ $\dashrightarrow$

$\Gamma, \Gamma'_2 - A \vdash B, \Delta_1 - A, \Delta'_2$ $\Gamma, \Gamma'_1 - A, \Gamma'_2 - A, B \vdash C, \Delta_1 - A,$

$$
\dfrac{
\dfrac{\dfrac{\mathrm{MIX}[\Pi, \Pi'_2]}{\Gamma, \Gamma'_2 - A \vdash B, \Delta_1 - A, \Delta'_2}
\qquad
\dfrac{\mathrm{MIX}[\Pi_2, \Pi']}{\Gamma, \Gamma'_1 - A, \Gamma'_2 - A, B \vdash C, \Delta_1 - A, \Delta'_1, \Delta'_2}}
{\Gamma, \Gamma'_1 - A, \Gamma'_2 - A \vdash C, \Delta_1 - A, \Delta'_1, \Delta'_2}
\qquad
\dfrac{\mathrm{MIX}[\Pi, \Pi'_1]}{\Gamma, \Gamma'_1 - A, C \vdash \Delta_1 - A, \Delta'_1}
}
{\Gamma, \Gamma'_1 - A, \Gamma'_2 - A \vdash \Delta_1 - A, \Delta'_1, \Delta'_2}
$$

7-2) $\quad r = \vdash\wedge \quad r' = \underline{\underline{\wedge 1 \vdash}} \quad A = B \wedge C$

$$\Pi \left\{ \dfrac{\begin{array}{c}\Pi_1\\ \Gamma_1 \vdash B, \Delta_1\end{array} \qquad \begin{array}{c}\Pi_2\\ \Gamma_2 \vdash C, \Delta_2\end{array}}{\Gamma_1, \Gamma_2 \vdash B \wedge C, \Delta_1, \Delta_2} \right. \qquad \left. \dfrac{\begin{array}{c}\Pi_2'\\ \Gamma'_2, B \vdash \Delta'\end{array}}{\Gamma'_2, B \wedge C \vdash \Delta'} \right\} \Pi'$$

$$\dfrac{\begin{array}{c}MIX[\Pi_1, \Pi']\\ \Gamma_1, \Gamma'_2 - A \vdash B, \Delta_1 - A, \Delta'\end{array} \qquad \begin{array}{c}MIX[\Pi, \Pi'_1]\\ \Gamma_1, \Gamma_2, \Gamma'_2 - A, B \vdash \Delta_1 - A, \Delta_2 - A, \Delta'\end{array}}{\Gamma_1, \Gamma_2, \Gamma'_2 - A \vdash \Delta_1 - A, \Delta_1 - A, \Delta'}$$

7-3) $\quad \ldots \quad r' = \wedge 2 \vdash \quad$ ANALOGA

7-4) $r = \vdash \vee 1$, $r' = \vee\vdash$. $A = B \vee C$

7-5) $r = \vdash \vee 2$ $r' = \vee\vdash$ $A = B \vee C$

SIMMETRICHE

7-6) $r = \vdash \forall$ $r' = \forall\vdash$ $A = \forall x B$

$$\Pi \left\{ \dfrac{\begin{array}{c}\Pi_1(x)\\ \Gamma \vdash B(x), \Delta_1\end{array}}{\Gamma \vdash \forall x B(x), \Delta_1} \right. \qquad \left. \dfrac{\begin{array}{c}\Pi'_1\\ \Gamma'_1, B(t) \vdash \Delta'\end{array}}{\Gamma'_1, \forall x B(x) \vdash \Delta'} \right\} \Pi'$$

SOST.

$$\dfrac{\begin{array}{cc} \mathrm{MIX}[\Pi_1(t), \Pi'] & \mathrm{MIX}[\Pi, \Pi'_1] \\ \Gamma, \Gamma'_1 - A \vdash B(t), \Delta_1 - A, \Delta' & \Gamma, \Gamma'_1 - A, B(t) \vdash \Delta_1 - A, \Delta' \end{array}}{\Gamma, \Gamma'_1 - A \vdash \Delta_1 - A, \Delta'}$$

7-7) $r = \vdash \exists$ $r' = \exists \vdash$ $A = \exists \times B$

SIMM.

TEOREMA CUT-ELIM.

SIA Π É UNA DIM DI $\Gamma \vdash \Delta$,

ALLORA $\exists \Pi^*$ CUT FREE DI $\Gamma \vdash \Delta$

<u>DIMOSTRAZIONE</u>

PER INDUZIONE SU $\langle \delta(\Pi), h(\Pi) \rangle$

* SUPPONIAMO CHE Π <u>NON</u> SIA CUT-FREE

<u>CASO 1</u> L'ULTIMA REGOLA r DI Π <u>NON</u> È UN CUT

$$\dfrac{\left\{\begin{array}{c}\Pi_i \\ \Gamma_i \vdash \Delta_i\end{array}\right\}_{i \in I}}{\Gamma \vdash \Delta}\, r$$

APPLICHIAMO II AD OGNI Π_i

NOTA
$\langle \delta(\Pi_i), h(\Pi_i) \rangle <$
$\langle \delta(\Pi), h(\Pi) \rangle$

$\delta(\bar{\bar{\Pi}})$

$$\dfrac{\left\{\begin{array}{c}\mathrm{II}: \Pi^{*} \\ \Gamma_i \vdash \Delta_i\end{array}\right\}}{\Gamma \vdash \Delta}\, r$$

CASO 2

r È UN CUT

se $d(A) = n$

$\Downarrow$

$d(A) < \delta(\Pi)$
per def
di $\delta(\Pi)$

$$\Pi \left\{ \dfrac{\begin{array}{cc} \Pi_1 & \Pi_2 \\ \Gamma_1 \vdash A, \Delta_1 & \Gamma_2, A \vdash \Delta_2 \end{array}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \,\text{CUT} \right.$$

PER II ABBIAMO LE PROVE CUT-FREE

$$\begin{array}{ccc} \Pi_1^* & & \Pi_2^* \\ \Gamma_1 \vdash A, \Delta_1 & & \Gamma_2, A \vdash \Delta_2 \end{array}$$

$|\delta(\Pi_1^*) = \delta(\Pi_2^*) = \emptyset$

hanno grado $\emptyset$

APPLICHIAMO IL LEMMA A $\langle \Pi_1^*, \Pi_2^* \rangle$

ED OTTENIAMO

$\Pi_0 = \mathrm{MIX}[\Pi_1^*, \Pi_2^*]$

$$\begin{array}{c} \Pi_0 \\ \Gamma_1, \Gamma_2 - A \vdash \Delta_1 - A, \Delta_2, \end{array}$$

t.c. $\delta(\Pi_0) \leq d(A) < \delta(\Pi)$.

ABBIAMO CHE

$$\langle \delta(\Pi_0), h(\Pi_0) \rangle < \langle \delta(\Pi), h(\Pi) \rangle$$

per II $\quad \Pi_0^*$ $\quad$ [Π_0^* cut free)

$$\Gamma_1, \Gamma_2 \neg A \vdash \Delta_1, \neg A, \Delta^2$$

E CON REG. STRUTT.

$$\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2$$

TEO

$\mathcal{H}$ È CONSISTENTE

DIM. SUPPONIAMO CHE

$\mathcal{H}$ NON SIA CONSISTENTE

$\vdash_{\mathcal{H}} A$ e $\vdash_{\mathcal{H}} \neg A$

e QUINDI IN LK $\exists\ \Pi_1, \Pi_2$

$$\begin{array}{cc} \Pi_1 & \Pi_2 \\ \vdash A & \vdash \neg A \end{array}$$

$\boxed{\text{NOTAZIONE}}$

D'ORA IN AVANTI CON

$LK : \Gamma \vdash \Delta$

DENOTERÒ CHE $\exists\ \Pi$ IN LK

di $\Gamma \vdash \Delta$

LEMMA

Se $\dfrac{\Pi}{\Gamma \vdash \Delta}$ allora $\exists$ $\dfrac{\Pi^{\circ}}{\Gamma \vdash \Delta}$

t.c.

1) tutti gli assiomi sono atomici

2) le formule introdotte da $w\vdash$ $\vdash w$ sono atomiche

GENERALIZZAZIONI

Φ INSIEME DI SEQUENTI CHIUSI PER SOSTITUZIONE

$$\left(\Gamma(x) \vdash \Delta(x) \in \Phi \Rightarrow \Gamma(t) \vdash \Delta(t) \in \Phi \right)$$

LE DERIVAZIONI SONO "AMPLIATE" CON GLI ASSIOMI OVVERO NEGLI ALBERI DI DERIVAZIONE È AMMESSO ETICHETTARE LE FOGLIE CON $S \in \Phi$

$$\delta(\Pi) = \begin{cases} 0 & \text{se } \Pi \text{ è cut-free oppure ogni cut-formula occorre in } S \in \Phi \\ \sup\{d(A)+1 \mid A \text{ è una cut-formula e } A \text{ non occorre in nessun } S \in \Phi\} \end{cases}$$

LK_{Φ} IL CALCOLO CON Φ COME ASSIOMI (TEORIA)

LEMMA SIA Φ chiuso per sost.

e sia A una formula che

non occorre in nessun $S \in \Phi$.

t.c. $d(A)=n$

SIANO Π, Π' DUE DERIV

IN LK_{Φ} DI $\Gamma \vdash \Delta$, $\Gamma' \vdash \Delta'$

t.c. $\delta(\Pi), \delta(\Pi') \leq n$

ALLORA ESISTE UNA DERIV.

$MIX[\Pi, \Pi']$

$\Gamma, \Gamma' - A \vdash \Delta - A, \Delta'$

t.c. $\delta(MIX[\Pi, \Pi']) \leq n$

DIMOSTR

LA DIM. PROCEDE

COME NEL CASO "PURO" ...

TEOREMA

SIA $\dfrac{\Pi}{\Gamma \vdash \Delta}$ IN LK_{Φ}

ALLORA ESISTE $\dfrac{\Pi^{*}}{\Gamma \vdash \Delta}$

t.c. OGNI CUT FORMULA OCCORRE IN ~~B~~ UN $S \in \Phi$

<u>DIM</u> BASTA MODIFICARE LA DIMOSTR. ORIGINALE INCLUDENDO NEL I° CASO LA REGOLA DI CUT CON CUT-FORMULA ~~OCCOR~~ CHE OCCORRE IN UN $S \in \Phi$

2.7.2. - definition

A system of **Post rules** (or **Post system**) is a set of axioms in sequent calculus, of the form

$$P_1, \ldots, P_k \vdash Q_1, \ldots, Q_m$$

with $P_1, \ldots, P_k$, $Q_1, \ldots, Q_m$ atomic, and which is closed under substitution.

2.7.3. - examples

(i) The axioms for equality can be expressed by means of a Post system, namely: all axioms

$$\vdash t = t \quad \text{and}$$

$$t = u, P[t] \vdash P[u] \quad \text{with } P \text{ atomic.}$$

(ii) The elementary axioms for arithmetic can be expressed by means of a Post system, namely the axioms (i) for equality, and $\vdash t + \bar{o} = t$; $\vdash t + Su = S(t + u)$; $\vdash t \cdot \bar{o} = \bar{o}$; $\vdash t \cdot Su = t \cdot u + t$; $t < \bar{o} \vdash$; $u < t \vdash u < St$; $u = t \vdash u < St$; $u < St \vdash u < t, u = t$; $\vdash u < t, u = t, t < u$; $St = \bar{o} \vdash$; $St = Su \vdash t = u$.

TEOREMA

Sia Φ un sistema di POST.

Ogni sequente dimostrabile in LK_Φ

ha una dim. di grado ≤ 1