Lidia Angeleri, Francesca Mantese
Università di Verona, 2010/11
Corso di Laurea Magistrale in Matematica

Introduzione alla teoria delle rappresentazioni di grafi

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Introduzione alla teoria delle rappresentazioni di grafi
Programma svolto nel Secondo Modulo

<u>CENNI DI ALGEBRA OMOLOGICA</u>

1 Push out e pull back

1. Push out e pull back
2. Proposizione (successioni esatte corte indotte $\beta\epsilon$ e $\epsilon\alpha$)
3. Osservazioni
4. Lemma del serpente
5. Naturalità nel Lemma del serpente

2 Il funtore Ext^1

1. Il gruppo delle estensioni $\mathrm{Ext}^1_R(A, B)$
2. Le applicazioni $\mathrm{Ext}^1_R(A, \beta)$ e $\mathrm{Ext}^1_R(\alpha, B)$
3. Proposizione su $\mathrm{Ext}^1_R(\alpha, \beta)$
4. la somma di Baer
5. Teorema: la struttura di gruppo su $\mathrm{Ext}^1_R(A, B)$

3 Risoluzioni proiettive

1. Richiamo: complesso, mappa di catene, gruppi di omologia
2. Mappe di catene omotopiche
3. Lemma: mappe di catene omotopiche coincidono sui gruppi di omologia
4. Risoluzioni proiettive
5. Teorema: sollevamento di omomorfismi a una risoluzione proiettiva

4 I funtori Ext^n

1. Teorema: Hom e Ext^1 come gruppi di omologia del complesso $\mathrm{Hom}_R(P^\bullet, B)$
2. Esempi
3. Definizione dei funtori Ext^n
4. Richiamo: la successione esatta lunga dei gruppi di omologia
5. Teorema: la successione esatta lunga dei gruppi di estensione
6. Osservazione
7. Corollario: $\mathrm{Ext}^1_R(A, B)$ come conucleo

TEORIA DI AUSLANDER-REITEN

5 Il trasposto di Auslander-Bridger

1. Richiamo sui proiettivi
2. Osservazione: esistenza di una risoluzione proiettiva minimale
3. Definizione del trasposto $\operatorname{Tr} A$
4. Lemma: prime proprietà del trasposto $\operatorname{Tr} A$
5. Osservazione su $\operatorname{Tr} f$
6. $P(M, N)$ e $\underline{\operatorname{Hom}}_R(M, N)$
7. Proposizione: $\underline{\operatorname{Hom}}_R(M, N) \cong \underline{\operatorname{Hom}}_R(\operatorname{Tr} N, \operatorname{Tr} M)$
8. La categoria stabile modulo proiettivi
9. Dualità di Auslander-Bridger

6 Il prodotto tensoriale

1. Richiamo: il prodotto tensoriale
2. Aggiunzione di Hom e $\otimes$
3. Il funtore $A \otimes -$
4. Lemma: $\operatorname{Hom}(P, M) \cong P^* \otimes M$
5. Proposizione: la successione esatta lunga data dal trasposto

7 La formula di Auslander-Reiten

1. Il funtore di Nakayama e τ
2. Esempio
3. La categoria stabile modulo iniettivi
4. La traslazione di Auslander-Reiten
5. La formula di Auslander-Reiten
6. Lemma
7. Esempio

8 Sequenze di Auslander-Reiten

1. Lemma
2. Applicazioni quasi spezzanti
3. Teorema di esistenza
4. Applicazioni minimali a destra e a sinistra
5. Sequenze di Auslander-Reiten
6. Proposizione: unicità di una successione quasi spezzante
7. Teorema di Auslander-Reiten

9 Il quiver di Auslander-Reiten

1. Applicazioni quasi-spezzanti per proiettivi e iniettivi
2. Morfismi irriducibili
3. Morfismi irriducibili come componenti delle applicazioni quasi spezzanti
4. $r^n(M,N)$
5. I morfismi irriducibili sono gli elementi di $r(M,N) \setminus r^2(M,N)$
6. Numero delle componenti delle applicazioni quasi spezzanti
7. Il quiver Γ di Auslander-Reiten
8. Esempio
9. Costruzione di Γ per traslazione e maglie
10. Esempio rivisitato
11. Knitting Procedure per le componenti preproiettive e preiniettive

10 Algebre di tipo finito e di tipo mansueto

1. Algebre di tipo di rappresentazione finito
2. Teorema di classificazione di Gabriel (grafi di Dynkin e grafi Euclidei)
3. Il vettore delle dimensioni
4. Teorema di Gabriel-Riedtmann sulle componenti preproiettive e preiniettive
5. Esempio
6. Teoremi di Auslander sul quiver di Auslander-Reiten per un'algebra di tipo finito
7. Componenti regolari
8. La matrice di Cartan e la trasformazione di Coxeter
9. L'algebra di Kronecker

TAME AND WILD ALGEBRAS

Definition. For each module $A \in \Lambda\,\mathrm{mod}$ denote by $\underline{\dim} A = (m_1, \dots, m_n) \in \mathbb{Z}^n$ the *dimension vector* of A given by the Jordan-Hölder multiplicities, that is, m_i is the number of composition factors of A that are isomorphic to the simple module S_i for aech $1 \leq i \leq n$. We set

$$\underline{e_i} = (0, \dots, 1, 0, \dots, 0) = \underline{\dim} S_i$$

$$\underline{p_i} = \underline{\dim}\, \Lambda e_i = \underline{\dim} P_i$$

$$\underline{q_i} = \underline{\dim}\, D(e_i \Lambda) = \underline{\dim} I_i$$

Remark. (1) For every exact sequence $0 \to A' \to A \to A'' \to 0$ we have

$$\underline{\dim} A = \underline{\dim} A' + \underline{\dim} A''$$

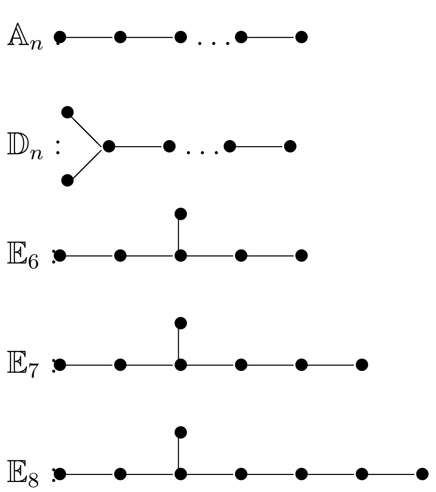
(2) If $\underline{\dim} A = (m_1, \dots, m_n)$, then $l(A) = \sum_{i=1}^{n} m_i$.

(3) Consider the Grothendieck group $K_0(\Lambda)$ defined as the group generated by the isomorphism classes $[A]$ of $\Lambda\,\mathrm{mod}$ with the relations $[A'] + [A''] = [A]$ whenever $0 \to A' \to A \to A'' \to 0$ is exact in $\Lambda\,\mathrm{mod}$. Note that $K_0(\Lambda)$ is a free abelian group with basis $[S_1], \dots, [S_n]$, see [1, I,1.7]. The assignment $[A] \mapsto \underline{\dim} A$ defines an isomorphism between $K_0(\Lambda)$ and $\mathbb{Z}^n$.

Theorem [Gabriel 1972] Let Λ be a finite dimensional hereditary algebra over an algebraically closed field k, and let Q be the Gabriel-quiver of Λ. The following statements are equivalent.

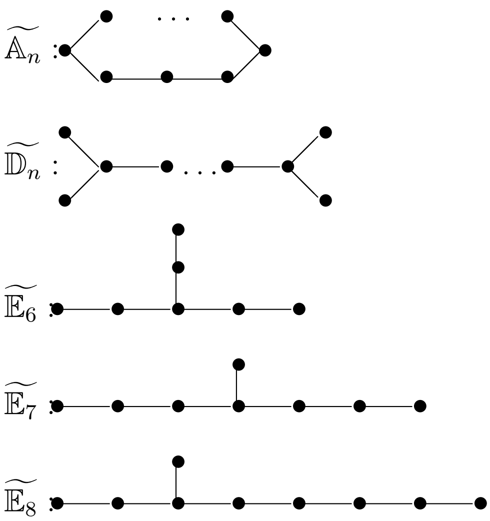
(a) Λ is of finite representations type.

(b) Q is of *Dynkin* type, that is, its underlying graph belongs to the following list.



Moreover, if (a) - (b) are satisfied, the finite dimensional indecomposable modules are uniquely determined by their dimension vector.

Remark. Let Λ be a finite dimensional hereditary algebra over an algebraically closed field k, and let Q be the Gabriel-quiver of Λ. If Q is of *Euclidean* type, that is, its underlying graph belongs to the following list:



then Λ is said to be *tame of infinite representation type.* Also in this case the isomorphism classes of indecomposable finite dimensional modules, though infinite in number, can be classified.

There is a general definiton of tameness for arbitrary finite dimensional algebras. A finite-dimensional k-algebra Λ over an algebraically closed field k is called *tame* if, for each dimension d, there are finitely many Λ-$k[x]$-bimodules $M_1, \cdots, M_n$ which are free of rank d as right $k[x]$-modules, such that every indecomposable Λ-module of dimension d is isomorphic to $M_i \otimes_{k[x]} k[x]/(x-\lambda)$ for some $1 \leq i \leq n$ and $\lambda \in k$. In other words, Λ is tame iff for each dimension d there is a finite number of one-parameter families of indecomposable d-dimensional modules such that all indecomposable modules of dimension d belong (up to isomorphism) to one of these families.

Moreover, Λ is said to be *of wild representation type* if there is a representation embedding from $k < x, y >$mod into Λ mod, where $k < x, y >$ denotes the free associative algebra in two non-commuting variables, see [2]. Observe that in this case there is a representation embedding AMod $\to \Lambda$ Mod for any finite dimensional k-algebra A, and furthermore, any finite dimensional k-algebra A occurs as the endomorphism ring of some Λ-module.

A celebrated theorem of Drozd [3] states that every finite dimensional algebra Λ over an algebraically closed field k is either tame or wild.

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ALGEBRAS OF FINITE REPRESENTATION TYPE

Definition. An Artin algebra Λ is said to be *of finite representation type* if there are only finitely many isomorphism classes of finitely generated indecomposable left Λ-modules. (equivalently: there are only finitely many isomorphism classes of finitely generated indecomposable right Λ-modules; indeed, the transpose Tr yields a bijection between the isomorphism classes of indecomposable non-projective left and right Λ-modules.)

Artin algebras of finite representation type are completely described by their AR-quiver.

Theorem 1 [**Auslander, Ringel-Tachikawa 1974**] *Let Λ be a finite dimensional algebra of finite representation type. Then every module is a direct sum of finitely generated indecomposable modules. Moreover, every non-zero non-isomorphism $f: X \to Y$ between indecomposable modules X, Y is a sum of compositions of irreducible maps between indecomposable modules.*

Remark In [2], Auslander also proved the converse of the first statement in Theorem 1. In other words, a finite dimensional algebra is of finite representation type if and only if every left module is a direct sum of indecomposable left modules. The question whether the same holds true for any left artinian ring is known as the *Pure-Semisimple Conjecture*.

Theorem 2 [**Auslander 1974**] *Let Λ be an indecomposable finite dimensional algebra with AR-quiver Γ. Assume that Γ has a connected component $\mathcal{C}$ such that the lengths of the modules in $\mathcal{C}$ are bounded. Then Λ is of finite representation type, and $\Gamma = \mathcal{C}$.*

In particular, of course, this applies to the case where Γ has a finite component.

Theorem 2 confirms the

First Brauer-Thrall-Conjecture: *An Artin algebra is of finite representation type if and only if the lengths of the indecomposable finitely generated modules are bounded.*

The following conjecture is verified for finite dimensional algebras over perfect fields, but it is still open in general.

Second Brauer-Thrall-Conjecture: *If Λ is a finite dimensional k-algebra where k is an infinite field, and Λ is not of finite representation type, then there are infinitely many $n_1, n_2, n_3, \dots \in \mathbb{N}$ and for each n_k there are infinitely many isomorphism classes of indecomposable Λ-modules of length n_k.*

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